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A survey of simulation-based methods for pricing complex American type options

Abstract

This paper gives an overview of simulation-based methods which have been developed for the valuation of "complex" American type options that cannot be valued analytically. The authors focus on one especially promising approach which we call "bounded recursive stochastic simulation" (BRSS) after having stripped off some time consuming but dispensable working steps. Then, we test the BRSS-approach by comparing it with three other simulation-based methods. Because of its superiority with regard to accuracy, computational costs, and flexibility, the paper describes the BRSS-approach in detail, thus, providing scientific know-how for efficient numerical option pricing.

Keywords: numerical option pricing, real options, Monte Carlo simulation, recursive stochastic simulation, early-exercise frontier.

Introduction

Options are traded not only on stocks but also on indices, currencies, bonds, natural resources (e.g., copper, oil) or agricultural commodities. The owner of an option is – independent of the development of the underlying asset – entitled to buy or sell that asset at a fixed price in the future. With American type option contracts, option owners are allowed to exercise their contractual rights at any date during the contracted time interval. Due to the great practical relevance of option pricing, and fostered by the bestowal of the Nobel Prize to Black, Scholes and Merton for their development of modern option pricing theory, the determination of the value of an option in accordance with the optimal early-exercise strategy has attracted considerable scientific efforts.

The main difficulty in practical option pricing is caused by the fact that closed form analytical solutions are only available for very simple valuation problems (e.g., European type options) or special cases. Even for valuing simple American style options one has to resort to numerical procedures such as lattice approaches (cf. Hull, 2000, ch. 16). But lattice approaches (e.g., binomial trees) themselves are restricted to a limited category of valuation problems. For one thing, they lack sufficient flexibility with regard to the form of stochastic processes and the number of stochastic variables. Furthermore, they become extremely cumbersome with larger problems. Referring to these valuation difficulties, we hereafter call options "complex" if they can neither be priced analytically nor through lattice approaches, nor through the standard stochastic simulation procedure known from European type option pricing.

To give a few examples: (1) lattice approaches cannot be used to value a bond option whose underlying follows a complex stochastic process, i.e., non-Markov process; (2) they are equally unsuited to value path-dependent (Asian) options, or stock options in the case of a stochastic variance (GARCH-process). The latter makes the problem a complex multiple stochastic variable problem even though the underlying may follow a Geometric Brownian Motion (GBM); (3) another example comes from the sphere of entrepreneurial investment opportunities which are often labelled "real options" (see Dixit and Pindyck, 1994)[1]. With regard to flexible investment decisions, not only multiple (stochastic) state variables and correlations between state variables but also non-GBM processes for these state variables are common problems which need to be considered (see Lund, 1993)[2].

Stochastic simulation procedures can handle alternative stochastic processes, multiple stochastic variables, correlations, etc. easily (cf. Boyle, 1977; Broadie and Glasserman, 1996). While being thus applicable to European type option pricing, stochastic simulation is not directly applicable to American type option pricing. The problem with American options is that with a simple forward moving simulation for the path of the underlying asset, it is not clear at potential early-exercise dates whether waiting or exercising represents the optimal strategy. That is, prior to option valuation, a stochastic dynamic decision problem of determining the optimal early-exercise strategy needs to be solved. This is

[1] Since investments are partly irreversible as well as characterized by uncertain future returns, potential investors run a risk and enjoy flexibility at the same time. They have, for instance, the choice to carry out risky investments at different dates. Hence, an analogy is claimed between an investment opportunity ("real option") and an American type call option. Without needing to discuss the adequateness of theoretical assumptions, one can state that the "real options" approach has generated "added value" by making financial option pricing methods available for the valuation of flexible investment opportunities which represent stochastic dynamic optimization problems as well.

[2] Recent research shows that the kind of stochastic process has a decisive influence both on option prices and critical early-exercise values (see Odening et al., 2005). The identification of stochastic processes can be based on statistical test procedures. For example, unit-root tests can be used for testing whether a stochastic variable follows a random-walk (e.g., GBM) or a stationary process (e.g., mean-reversion process; see Pindyck and Rubinfeld, 1998). Time series models (e.g., ARIMA-processes) can be identified using a Box-Jenkins test procedure (Box and Jenkins, 1976).

the reason why, for a long time, stochastic simulation was not believed to be feasible for the valuation of American options (see Hull, 1993, p. 363; Briys et al., 1998, p. 62). Accordingly, it was as well deemed unsuitable for the valuation of real options (see Trigeorgis, 1996).

However, due to the great flexibility of simulation-based approaches with regard to the representation of stochastic processes, many successful attempts have been made, in the last one and a half decade, to embed stochastic simulation(s) in a more sophisticated methodical framework to value American type options. The various methods are different with regard to their ways of determining the optimal early-exercise strategy before actually valuing the option. Their common feature is that they all simulate the stochastic development of one or several state variables. Some of these valuation methods provide accurate results and are relatively simple to use. Yet, the economic literature on simulation based option pricing remains highly fragmented and hard to access, and what it lacks is synthesis. Thus, the potential of stochastic simulation for the valuation of American options – be they complex financial options or real options – is still not fully acknowledged (see Hull, 2000, p. 408). While Glasserman (2004, ch. 8) tries to mitigate this problem by a brief description of different simulation based methods, an exhaustive and systematic overview as well as guidance regarding the selection of practical valuation procedures for different situations is still missing.

With a view to this fragmented literature background, the aim of the paper is to systematize the most meaningful simulation-based valuation methods. This provides scientific guidance with regard to successful numerical option pricing that is economical with resources such as programing effort and computational time. A result of the classification is that the integration of a stochastic simulation of the state variable(s) into a backward-recursive framework of option pricing is a very flexible and intuitive way to price complex American type options. We are able to improve the most promising representative of this class (Grant et al., 1997a) by stripping some time consuming but dispensable steps from it. Aiming at providing scientific know-how for successful and efficient numerical option pricing and anticipating the favourable results with regard to accuracy and computational costs, the simplified approach which could be called "bounded recursive stochastic simulation" (BRSS) is demonstrated in detail. This includes a description of its capacities in situations with multiple stochastic variables. Furthermore, four promising simulation-based approaches, including BRSS, are selected and tested. We use a straightforward Bermudan type option pricing problem (here: a stylised real option example with only few early-exercise dates) as a testbed in order to demonstrate that BRSS – compared to the other approaches with similar flexibility – provides correct results in a very efficient way.

The paper is organized as follows. Section 1 outlines the overall valuation problem for American type options and introduces the notation. Given the important role stochastic simulation plays in more sophisticated methods of option pricing, Section 2 describes and classifies the different methods which have been developed up to now and which integrate – in one way or another – stochastic simulation into a more complex framework to value American type options. Section 3 describes the simple and straightforward BRSS method in greater detail, thus, providing the scientific knowledge for its practical application. In Section 4, we compare and validate the BRSS and three alternative simulation based methods against the reference solution provided by the binomial tree method. The paper closes with an outlook emphasizing the fact that more complex valuation methods may be needed in some circumstances (the last Section).

1. Problems in valuing American style options

In the case of European type options there is only one question to be answered: what is the value of the option? The exercise strategy is known: the option should be exercised at the expiry date if the difference between the (market) value of the underlying asset and the strike price is positive. In the case of American type options, which can also be exercised before maturity, there is an additional question to be answered: at a given point in time, at which price of the underlying asset (i.e., critical early-exercise value) should the option be exercised? Due to its American type character we have to answer both questions for the real option which we use as demonstration example.

Firstly, we briefly present the principal problem of the pricing of American type (real) options: let I be the constant (investment) costs or purchase price of a real asset which generates a stochastic expected present value of investment cash flows V. The option to realize the investment is given in a period [0, T] at discrete potential exercise dates τ, $\tau = 0 \cdot \Delta\tau, 1 \cdot \Delta\tau, ..., \Gamma \cdot \Delta\tau$. The number $\Gamma + 1$ of potential exercise dates is determined by the time between two potential exercise dates $\Delta\tau$ ($\Gamma = T / \Delta\tau$). Note that for the sake of more convenient formulation we define for the rest of this paper $\Delta\tau = 1$. The investment costs are sunk, once, the investment is carried out. We seek both the value of the investment option F_0 and the critical early-exercise values V_τ^* which would trigger an immediate investment at any given exercise date τ. According to traditional investment theory, the value of this investment opportunity at any one exercise date equals the positive net present value i_τ :

$$i_\tau = \max(0, V_\tau - I) . \quad (1)$$

Traditional investment theory recommends the exercise strategy $V_\tau^* = I$, i.e., the investment should be carried out as soon as there is a positive net present value (NPV).

However, using the theory of option pricing, we know that the NPV-calculus only takes into account one part of the option value, namely the intrinsic value. But an option does not have to be exercised at date τ. Therefore, the option also has a continuation value f_τ which represents the discounted expectation value of the option, if it is not exercised at time τ, but the decision is put off until the next early-exercise date $\tau + 1$:

$$f_\tau = \hat{E}(F_{\tau+1}) \cdot \exp(-r) . \quad (2)$$

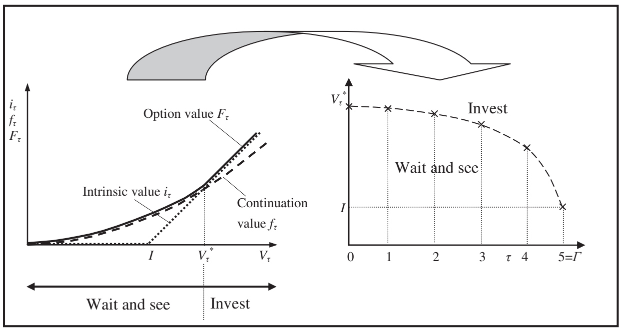
In this equation r denotes the continuously compounded risk free interest rate, and $\hat{E}$ the risk neutral expectation operator. The use of the risk neutral expectation operator and the risk-free interest rate follows the risk-neutral-valuation principle (see Cox and Ross, 1976), or – if this principle isn't applicable – this procedure implies a risk neutral decision maker. The fact that we take into account both the intrinsic value and the continuation value expresses the fact that the decision is regarded as a choice between two alternatives: (1) immediate investment; and (2) delaying the investment decision. As a normative rule the investment option should be exercised if the intrinsic value equals or exceeds the continuation value. Therefore, the value of the option is calculated as follows:

$$F_\tau = \max(i_\tau , f_\tau) . \quad (3)$$

The binary decision problem between exercising and waiting can be understood as a specific stopping problem. Equation (3) is equivalent to the Bellman-equation (see Bellman, 1957; Dixit and Pindyck, 1994, p. 109). It can be shown that under certain regularity conditions[1], there is an optimal exercise path which separates the stopping region from the continuation region. This exercise path or frontier consists of boundary values for the underlying V_τ^* which indicate the values in time, where a decision-maker is indifferent with regard to early-exercise or continuation. In this case, the intrinsic value $i_\tau (V_\tau^*) = i_\tau^*$ must equal the continuation value $f_\tau (V_\tau^*) = f_\tau^*$ (identity- or value-matching condition):

$$i_\tau^* = f_\tau^* . \quad (4)$$

Figure 1 (left) shows the graphs of the intrinsic value and the continuation value as functions of the expected present value of the investment cash flows for one potential exercise date τ (equivalent to a non-dividend-protected American type call option on a dividend paying underlying[2]). A profit maximizing decision-maker should exercise an option immediately if $V_\tau \geq V_\tau^*$.



Note: Depicted for a non-dividend-protected American call option with six potential exercise dates on a dividend paying underlying: $\Delta\tau = 1$.

Fig. 1. Relationship between intrinsic value and continuation value (on the left) and exercise frontier (on the right)

Figure 1 (right) shows the entire critical early-exercise path over time for an American call option with six potential exercise dates. One characteristic of the critical early-exercise path is its negative exponential slope which expresses the reduction of flexibility in time. At the last possible exercise date Γ there is no more temporal flexibility to further delay the investment. Then the classical investment theory is valid, and $V_\Gamma^* = I$.

2. Overview of simulation-based methods for valuing American type options

There are many advantages of stochastic simulation in comparison with other valuation procedures. In the framework of a stochastic simulation, for example, various stochastic processes can be handled. Furthermore, it is relatively easy to consider several sto-

[1] The regularity-conditions demand that: (1) the intrinsic value and the continuation value are monotone functions of the value of the underlying; and that (2) the distribution function of the underlying in $\tau + 1$ will shift to the right (left) side, if the value in τ increases (decreases), i.e., a positive persistence of the stochastic process (see Dixit and Pindyck, 1994, p. 129).

[2] Note that the early-exercise of an American type call option without dividends of the underlying is never optimal. Therefore, the option value corresponds to the value of an equivalent European style call option (see Merton, 1973).

chastic variables simultaneously[1]. The most relevant methods which use simulation procedures to determine the early-exercise strategy and the price of American type options may be grouped as follows:

1. Simulation of one finite sample of price paths starting at time 0 and subsequent stratification of the state space:

- Tilley (1993) uses a so-called bundling algorithm. Starting from time 0, a large number of price paths are simulated. At each potential early-exercise date the paths are ordered according to the stock price level at that date and divided into "bundles" of the same size. Next, starting from the expiry date of the option, the average of the continuation values of prices in a bundle is taken as the single continuation value for that bundle. This backwards-recursive procedure yields an early-exercise strategy for each simulated stock price path. The ordering of the prices enables the critical early-exercise value to be determined through the "identity condition". That is, the early-exercise value lies, where the intrinsic value coincides with the continuation value. However, this is based on the assumption that the above-described average is an accurate estimate of the continuation value of the prices in a bundle, and also that there is a small distance between the bundles. In other words, one needs an extremely high number of paths in each bundle and a high number of bundles as well. In fact, there is a transition zone between holding and exercising in which the early-exercise strategy is determined by a pragmatic rule. Finally, the option price is obtained as the average of the discounted payoffs for the initially simulated stock price paths according to the early-exercise strategy.
- Barraquand and Martineau (1995) also reduce the dimensionality of the valuation problem by grouping simulated paths at any point in time into a limited number of "bins". They determine transition frequencies between successive bins by another simulation and, finally, solve backwards like in a multinomial tree.
- Raymar and Zwecher (1997) similarly design a grid of "bucket" regions and simulate paths through that grid. Therefore, at any point in time, the realized outcome will be assigned to one bucket. Then, they determine: (1) transition frequencies into/out-of each bucket at every date τ to each bucket at the next point in time $\tau + 1$; and (2) average realized values in each bucket. Eventually, they determine the average payoff in each bucket at the date of expiration and iterate as in a multinomial tree to compute the current value of the American option.

All three methods[2] are mimicking the standard binomial tree by stratifying the state space and putting the simulated paths into groups which are called: "bundles" by Tilley; "bins" by Barraquand/Martineau; and "buckets" by Raymar/Zwecher. Indeed, Garcia (2000, p. 9) also puts all three methods in one group: "The papers by [Tilley, Barraquand/Martineau, Ramar/Zwecher…] incorporate different aspects of the usual backwards induction algorithm by stratifying the state space and finding the optimal exercise decision in each subset of the state variables". However, unlike Barraquand/Martineau or Raymar/Zwecher, Tilley does not calculate transitions probabilities between successive bundles and solve as in a multinomial tree. Instead, he uses a path-wise determination of the exercise strategy.

2. Simulation of one finite sample of price paths starting at time 0 and subsequent backward-recursive estimation of a continuation value function:

- In a discussion of the Tilley-paper, Carriere (1996) describes Tilley's bundling algorithm as a regression method, albeit crude. In a publication of his own (Carriere, 1996) he develops the regression method. Like Tilley, he simulates the stock price movement a large number of times. Subsequently, however, assuming that the option has not been exercised before, he determines the value functions which describe at any given date the value of the option, depending on the basis values in a backward recursive fashion. At the expiry date this value function is just the intrinsic value. After having determined the value function at a certain exercise date, he approximates a continuation value function at the previous exercise date by carrying out a piecewise polynomial regression of the already determined values against the basis values at the previous exercise point. Then, he takes the value function as the maximum of the continuation

[1] Stochastic simulation is considerably more efficient than other numerical procedures, such as lattice approaches, if two or more stochastic variables (or a great number of potential exercise dates) must be considered. With stochastic simulation the computational requirements increase only linearly with the number of the variables, whereas with other methods it increases exponentially (cf. Hull, 2000).

[2] A variation of Tilley's method, not explicitly included in our description of different subgroups, is given by Broadie and Glasserman (1997). They construct a multinomial tree of simulated paths by simulating z paths from V_0 to V_1. Then they simulate z paths from every of the z values for V_1 to V_2 and so on. It must be noted, that the computational requirements of the method proposed by Broadie and Glasserman (1997) explode, i.e., grow exponentially in the number of exercise dates (similar to a non-recombining tree). Therefore, the application of this method is, according to the authors themselves, limited to pricing options with up to five numbers of exercise opportunities. Nonetheless, the feasible number of simulation runs is necessarily lower than required for a reasonable level of accuracy (cf. Haug, 1998). Thence, they resort to a calculation of two option value estimates, one biased high and one biased low. The estimators are combined to give a valid confidence interval for the option price. Trying to get rid of this problem, Broadie and Glasserman (2004) further develop their 1997-procedure and call it "stochastic mesh method" (see also, Boyle et al., 2000).

value function obtained by this regression and the intrinsic value function. Using the continuation value function, its intersection with the intrinsic value function can be calculated in order to determine the critical early-exercise value. In sum, by different regression methods he arrives at a comparable performance to Tilley's method. It should be noted that, although the critical exercise value is calculated, it is not used for a downstream simulation to determine the option price. The option price is rather determined as the average of the discounted cash flows of all the price paths according to their respective early-exercise strategies.

- The method, proposed by Longstaff and Schwartz (2001), also proceeds in backward-recursive fashion to obtain at each discrete exercise date the continuation value function, depending on the basis value (see also, Tsitsiklis and van Roy, 2001; Clément et al., 2002; Moreno and Navas, 2003). This is achieved through use of the ordinary least squares method. They, subsequently, determine for each basis value whether exercising or holding of the option leads to the higher value. This results in a certain exercise strategy for each price path. However, the critical exercise value is not explicitly calculated and again, no downstream simulation to determine the option price takes place. The option price is rather found as the average of the discounted payoffs of all the paths simulated at the beginning, according to their respective early-exercise strategies.

3. Multiple simulations and determination of the early-exercise strategy through maximization of the option value with regard to parameters of an exercise function over time:

- Bossaerts (1989) estimates American option prices via simulation by parameterising the exercise function over time and then solving for the parameters that maximize the option value.
- Fu and Hu (1995), Fu and Hill (1997) and Fu et al. (2000) likewise parameterise the exercise boundary, and maximize the expected discounted payoff with respect to the parameters. "[...] no dynamic programing is involved, i.e., the procedure simultaneously optimises all parameters by iteration instead of sequentially by backward recursion. [...] It is mimicking steepest-descent algorithms from the deterministic domain of non-linear programing" (see Fu et al., 2000, p. 13).
- Garcia (2000) also tries to find a suitable parameterisation of the exercise boundary by using an optimisation algorithm to determine those values of the parameters which yield the maximum option value.

4. Backward-recursive determination of critical exercise values, using sequential simulation of price paths starting from the respective exercise dates:

- Grant et al. (1997a) suggest a backward-recursive procedure whereby at each possible early-exercise date, the stochastic development of the basis value (state variable) is simulated starting from an arbitrary initial test-value. For this test-value, the intrinsic value is directly derived. Knowing the future exercise strategy, the respective stochastic continuation value is computed by using the expectation operator over all corresponding sample paths. If one finds that the intrinsic value is lower (higher) than the continuation value, one chooses, subsequently, a higher (lower) test-value. The above-described procedure is repeated until two test-values are found between which the "identity function" $f_{id} = i(V_\tau) - f(V_\tau)$ has opposite signs. The zeros of the identity function can be approximated by using a root finding algorithm such as the bisection or secant method, then bracketing the root of f_{id} to a required tolerance and interpolating linearly to obtain an estimate for the root. Grant et al. predefine the bracket of the root by the set interval between the different discrete test-values, where simulations are started.
- Ibanez and Zapatero (2004) or Ibanez (2004) also suggest a backwards-recursive procedure, whereby at each possible early-exercise time the stochastic development of the basis value and the continuation value is simulated. However, the intersection of the easily calculated intrinsic value i_τ and the simulated continuation value f_τ (value-matching condition) is determined by several iterations of Newton's method, starting from an arbitrary basis value.

The approaches of Grant et al. (1997a) and Ibanez and Zapatero (2004) or Ibanez (2004) are quite similar. There is one main difference, Grant et al. find two values of the asset at which the "identity function" has opposite signs and then use linear interpolation to approximate the boundary value (this is actually just one step of the secant method). Ibanez and Zapatero, in contrast, use several iterations of Newton's method to find the zero of the identity function. It would seem that a lot of work could be saved here by using the secant method as a root finding algorithm rather than Newton's. The secant method converges almost as fast as Newton's method but with the advantage that one avoids the rather cumbersome evaluation of the derivative of the identity function.

5. Multiple simulations and determination of the critical exercise path through maximization of the option value with regard to a heuristically varied set of exercise path values:

- Dias (2001) or Balmann and Musshoff (2002) maximize the option value with regard to a complete exercise path, containing a set of critical values which is gradually optimised by means of genetic algorithms. First, S simulation runs, each starting from a different initial price in time 0, are carried out. Then, average option prices are computed for a number of test early-exercise paths (genomes) which had been randomly selected. These test exercise paths are ordered by the level of the option value (fitness) they generate, respectively. The application of the genetic algorithm (selection, recombination, mutation) determines the composition of the test exercise paths in the next test run (the next generation). This process, which generates increasingly fitter exercise paths by mimicking natural evolution, is repeated until all exercise paths of a generation are nearly identical. At this point, they also hardly differ from those of the previous generation and an improvement of the fitness is no longer possible (maximal option value).

An overview of the basic characteristics of the above-described simulation-based methods to value American type option is given in Table 1.

Table 1. Classification of different simulation-based methods

	Nonrecurring simulation	Stratification of state space	Backward-recursive determination of exercise strategy within known paths	Combination with lattice method	Estimation of continuation function	Explicit backward-recursive determination of critical values	Multiple simulations for determination of free boundary	Downstream simulation for determination of option value	Maximization of option value with regard to parameters of exercise function	Maximization of option value with regard to heuristically varied set of critical values
Tilley (1993)	X	X	X			X				
Barraquand and Martineau (1995)	X	X		X						
Raymar and Zwecher (1997)	X	X		X						
Carriere (1996)	X		X		X	X				
Longstaff and Schwartz (2001)	X		X		X					
Bossaerts (1989)							X	X	X	
Fu and Hu (1995); Fu and Hill (1997); Fu et al. (2000)							X	X	X	
Garcia (2000)							X	X	X	
Grant et al. (1997)						X	X	X		
Ibanez and Zapatero (2004); Ibanez (2004)						X	X	X		
Dias (2001); Balmann and Musshoff (2002)							X	X	X*	X*

Note: * Genetic algorithms can be used to determine an optimal set of critical values or to optimise the parameters of an exercise function over time.

Many, albeit not all, of these procedures are either not satisfying with regard to the precision of the option valuation or require so much programing effort and computational time that they hardly appear to be practical. This may or may not explain why they are, in general and without much differentiation, regarded by many experts as difficult to implement (see Hull, 2000, p. 408). Another obstacle for widespread applications seems to be that some of the procedures offer little intuition or require a lot of deep mathematics. Indeed, some procedures may be unnecessarily complicated compared to others for their intended application. At the same time, some of the more complex and cumbersome methods are justified because they permit the modelling of special complexities inherent to some options. Agent-based simulations (genetic algorithms), for example, are sometimes necessary because they allow for the endogenous modelling of price dynamics within the framework of a real option pricing model (see Balmann and Musshoff, 2002). It should be noted that some financial options and all real options share the common feature of complexity. A suitable valuation method has to be flexible enough to take account of all real world qualities of an option. They may, for instance, arise from non-Markov processes, multiple stochastic variables, correlations of stochastic variables, interactions of different options etc. The following discussion focuses on the potential of the valuation procedure suggested by Grant et al. (1997a) the performance of which can be significantly improved by some modifications. Through some modifications its performance can be significantly improved. The improved method combines high accuracy and good intuition with simple implementation (see Section 4.2). According to our classification, it belongs to the group of methods which integrate sequential simulations of basis values for respective exercise dates into a backward-recursive procedure which in turn determines the critical exercise path.

3. A simple recursive stochastic simulation approach

The literature usually refers to all procedures that determine the critical early-exercise frontier by a backward-moving recursion (backward induction) as "dynamic programing approaches" (see Fu et al., 2000). This is due to the fact that the binary decision problem between exercising and waiting is regarded as a specific stopping problem (see Bellman, 1957). In this sense, lattice approaches also represent discrete approximations of the dynamic programing principle. Accepting this terminology, those simulation-based procedures which determine the critical early-exercise frontier by a backward-moving recursion have to be subsumed under "dynamic programing" as well. The approach of Grant et al. (1997a) belongs to this group. It will be shown that it represents a straightforward and fast way to determine the critical early-exercise path and the option price of an investment option or an American style call option on a dividend paying underlying. Its advantages can be expanded by adopting some slight but effective modifications. This is already true if we consider a single stochastic variable. For the sake of clarity we choose this case for a detailed description of operational procedures in Section 3.1. It is, however, equally valid if we consider multiple stochastic variables. The extensions, which are necessary in this case, are shown in Section 3.2. In both cases, we could label this modified method "bounded recursive stochastic simulation" (BRSS) due to the specific characteristics of the process used to determine the critical early-exercise values.

3.1. Valuing a call option with a single stochastic state variable. Before describing the BRSS-procedure step by step for the case of a single stochastic variable, we summarize the modifications which we have made to improve the efficiency of the approach of Grant et al. (1997a):

- In all simulations which start from the different test-values and which each comprise *S* paths, we use the same sequence of random numbers. We save a lot of time by simultaneously simulating all price paths starting from different values.
- For the determination of the critical early-exercise value as a free boundary we always use the (already known) critical exercise-value of the subsequent period as a lower bound for the test-values we start simulating from. We make an educated guess at an upper bound, obtaining, thereby, an interval, which is divided into equal subintervals of a length, which is already deemed sufficiently small for interpolation. Using the exercise strategy of the subsequent period as a lower bound, turns out to be even more advantageous when dealing with multiple stochastic variables and, therefore, more cumbersome exercise functions at each time instant. The procedure of using a predefined sequence of test-values has an additional advantage because it facilitates the automation of consecutive work steps. We do not use a manual criterion (i.e., "stop, if intrinsic value exceeds the continuation value for the first time") which would tell us when to stop simulating paths, starting from still another test-value. Instead, we predefine an interval, where we expect the identity function to be zero and program the determination of the critical value (see Figure 2).

Step 1: *Determination of the critical exercise value* V_{Γ}^{*}.

The critical exercise value V_{Γ}^{*} at the expiry date Γ of the option is the starting point of any backward-recursive valuation. Since there is no temporal flexibility at the last potential exercise date, the investment should be carried out as soon as the investment payoff V_{Γ} covers the investment costs I. Therefore, V_{Γ}^{*} equals I. The knowledge of V_{Γ}^{*} is the precondition for calculating $V_{\Gamma-1}^{*}$.

Step 2: *Determination of the critical early-exercise value* $V^*_{\Gamma-1}$.

The critical early-exercise value $V^*_{\Gamma-1}$ is the present value of the investment which yields an identical intrinsic value and continuation value. We calculate the intrinsic value $i_{\Gamma-1}(V_{\Gamma-1})$ and the continuation value $f_{\Gamma-1}(V_{\Gamma-1})$ for a set of different test-values ${}_nV_{\Gamma-1}$, with $n=1,2,...,N$. For each test-value the intrinsic value can be directly derived. The corresponding continuation value is estimated after running a stochastic simulation with S runs, starting from the given test-value. We proceed as follows (see Figure 2):

Step 2.1: *Definition of test-values* (*test present values of the investment*).

The lowest test-value ${}_1V_{\Gamma-1}$, we start simulating from is the theoretically known lower bound for date $\Gamma-1$ which is given by the critical exercise-value of the subsequent period, i.e., V^*_Γ. Then we make an educated guess at a preliminary upper bound. The interval between the lower and upper bound (parameterization interval) is divided into $N-1$ equal subintervals. The endpoints of these subintervals give us a total of N test-values ${}_nV_{\Gamma-1}$ to start test-simulations from. The critical value $V^*_{\Gamma-1}$ falls between those two test-values which yield a change of sign of the difference between intrinsic value and continuation value.

Step 2.2: *Determination of continuation values for each test-value by means of stochastic simulation.*

Given the stochastic process for the underlying asset of the option, S runs or paths starting from the lower bound ${}_1V_{\Gamma-1} = V^*_\Gamma$ are simulated resulting in S values of the investment ${}^s_nV_\Gamma$ at date Γ (s denotes a simulation run out of a total of S runs). Simultaneously, S paths are simulated starting from the other test-values ${}_nV_{\Gamma-1}$. For the simulations starting from different test-values we use the same sequence of random numbers for all S runs[1]. This reduces computation time significantly. Knowing V^*_Γ, we calculate the continuation values ${}^s_nf_{\Gamma-1}$ for all paths starting from any given test-value ${}_nV_{\Gamma-1}$ as the discounted payoff of the option:

$$ {}^s_nf_{\Gamma-1} = \max(0,\ {}^s_nV_\Gamma - I)\cdot\exp(-r)\,. \tag{5}$$

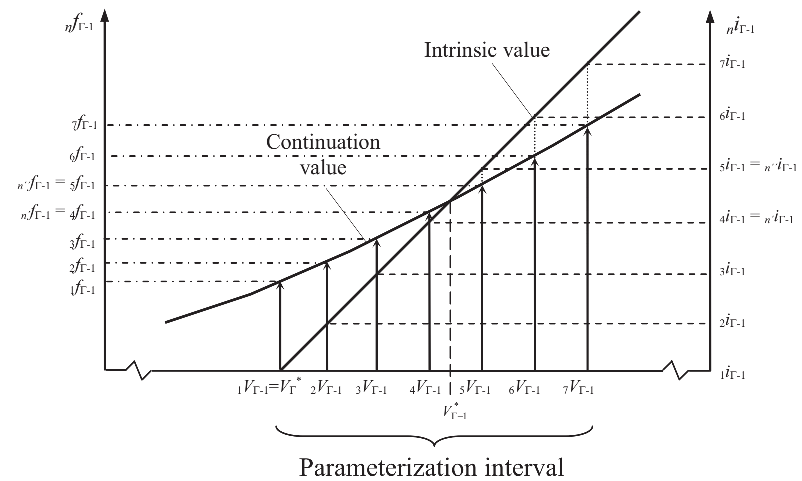
The expected continuation value ${}_nf_{\Gamma-1}$ is the average value of all ${}^s_nf_{\Gamma-1}$:

$$ {}_nf_{\Gamma-1} = \sum_{s=1}^{S} {}^s_nf_{\Gamma-1}\cdot\frac{1}{S}\,. \tag{6}$$

Step 2.3: *Calculation of intrinsic values for each test-value.*

In order to compare the possible strategies "invest" and "wait", we must also calculate the intrinsic value. The intrinsic value ${}_ni_{\Gamma-1}$ for each test-value ${}_nV_{\Gamma-1}$ can be directly derived as:

$$ {}_ni_{\Gamma-1} = \max(0,\ {}_nV_{\Gamma-1} - I)\,. \tag{7}$$



Note: Depicted for $N = 7$ test values.

Fig. 2. Determination of a critical early-exercise value of a (dividend paying) American call option using BRSS

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[1] For technical details of the application of stochastic simulation to a wide variety of stochastic processes with established software packages see Winston (1998, p. 325). For example, Haug (1998, p. 40) stipulates carrying out at least 10 000 simulation runs. Fortunately, with any given number of simulation runs, one can improve the stability of the solution by using so called variance reduction methods without a great increase of computational time. An overview of various variance reduction procedures can be found in Glasserman (2004, ch. 4 and 5), Hull (2000, ch. 16.7) or Morokoff (1998).

Step 2.4: Approximation of the critical early-exercise value $V^*_{\Gamma-1}$ by means of linear interpolation.

It is very unlikely that the intrinsic value and continuation value will coincide at one of the predefined test-values. In most cases, the critical value will fall between those two test-values, where a change of sign for the difference of intrinsic value and continuation value occurs. They will be denoted by n' and n'', where it does not matter which one is the smaller. The respective intrinsic values (${}_{n'}i_{\Gamma-1}$ and ${}_{n''}i_{\Gamma-1}$) and continuation values (${}_{n'}f_{\Gamma-1}$ and ${}_{n''}f_{\Gamma-1}$) are used for linear interpolation (equivalent to one step of the secant method) through which the critical early-exercise value is determined at any one point in time:

$$V^*_{\Gamma-1} = {}_{n''}V_{\Gamma-1} + \frac{{}_{n'}V_{\Gamma-1} - {}_{n''}V_{\Gamma-1}}{({}_{n'}i_{\Gamma-1} - {}_{n'}f_{\Gamma-1}) - ({}_{n''}i_{\Gamma-1} - {}_{n''}f_{\Gamma-1})} \cdot [-({}_{n''}i_{\Gamma-1} - {}_{n''}f_{\Gamma-1})] \quad (8)$$

where $i^*_{\Gamma-1}$ and $f^*_{\Gamma-1}$ denote the intrinsic value and the continuation value of the critical exercise-value $V^*_{\Gamma-1}$. Note that $i^*_{\Gamma-1} - f^*_{\Gamma-1} = 0$ (identity condition).

In the example presented in Figure 2, one must interpolate between the values ${}_{n'}V_{\Gamma-1} = {}_4V_{\Gamma-1}$ and ${}_{n''}V_{\Gamma-1} = {}_5V_{\Gamma-1}$.

Reducing the length of the initial interval (see Step 2.1) improves the approximation, because it shortens the subinterval on which one needs to interpolate after rerunning steps 2.2 and 2.3. The initial interval must be enlarged if it did not include a test-value, yielding an intrinsic value higher than the continuation value (see Table 3).

Step 3: Determination of the critical early-exercise value $V^*_{\Gamma-2}$.

In order to determine the critical early-exercise value $V^*_{\Gamma-2}$, one has to take into account the fact that the option may be exercised both at $\Gamma-1$ and at Γ. We can again use stochastic simulation to determine continuation values for a given set of test-values ${}_nV_{\Gamma-2}$, because we already know $V^*_{\Gamma-1}$ and V^*_{Γ} and, therefore, the future exercise strategy. The procedure to determine $V^*_{\Gamma-2}$ is analogous to the one described in Step 2. Only the computation of the continuation value for each path has to be modified according to the optimality of exercising either at Γ or $\Gamma-1$:

$$ {}^s_n f_{\Gamma-2} = \max(0, {}^s_nV_\kappa - I)\cdot \exp(-r\cdot[\kappa-(\Gamma-2)]) \text{ with } \kappa = \begin{cases} \Gamma-1, \text{ if } {}^s_nV_{\Gamma-1} \geq V^*_{\Gamma-1} \\ \Gamma, \quad \text{otherwise} \end{cases}. \quad (9)$$

Step 4: Definition of the critical early-exercise values $V^*_{\Gamma-3}, V^*_{\Gamma-4}, \ldots, V^*_0$.

The procedure described above is applied backwards until all critical early-exercise values are known. An increasing "length" of future exercise strategy increases the complexity of determining ${}^s_nf_\tau$ according to the optimality of exercising at future dates. Equation (9) has, therefore, to be generalized as follows:

$$ {}^s_n f_\tau = \max(0, {}^s_nV_\kappa - I)\cdot \exp(-r\cdot[\kappa-\tau]) \text{ with } \kappa = \begin{cases} \tau+1, \text{ if } {}^s_nV_{\tau+1} \geq V^*_{\tau+1} \\ \tau+2, \text{ if } ({}^s_nV_{\tau+2} \geq V^*_{\tau+2}) \wedge ({}^s_nV_{\tau+1} < V^*_{\tau+1}) \\ \vdots \\ \Gamma, \text{ otherwise} \end{cases}. \quad (10)$$

Figure 3 gives a graphical representation of the basic procedure to determine the critical early-exercise path.

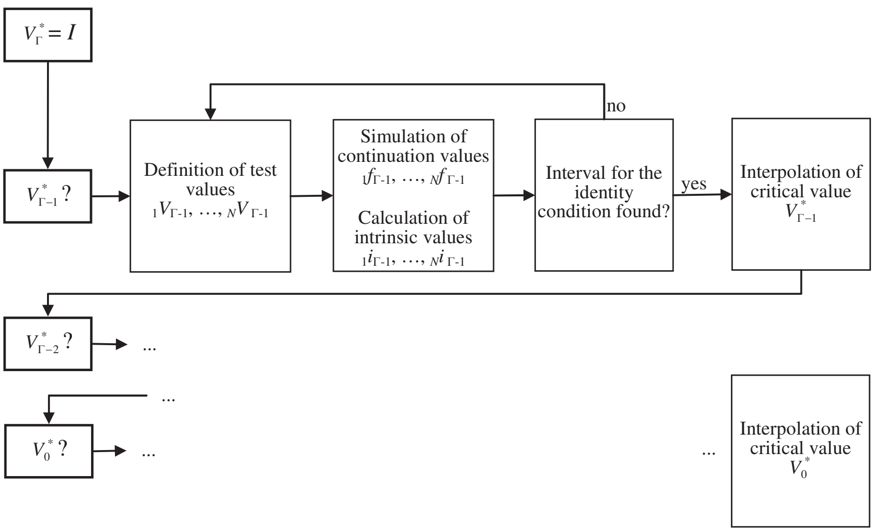


Fig. 3. Basic procedure to determine the critical early-exercise path using BRSS

Step 5: *Determination of the option value.*

F_0 is the maximum of intrinsic value i_0 and continuation value f_0. After having determined the optimal strategy as a free boundary, one has to initiate one last simulation, starting from the actual present value of investment cash flows V_0. Then, the option value F_0 can be determined as the expected value of all simulation runs S by determining i_0 and f_0 analogous to Steps 2.2 and 2.3. Straightforward, stochastic simulation can be applied because the optimal future strategy (i.e., the early-exercise path as a whole) is already known.

3.2. Valuing an option with multiple stochastic state variables. When pricing financial options, we often assume that the value of the underlying is the only stochastic state variable. However, some financial option pricing models consider additional stochastic variables, such as a stochastic variance and/or a stochastic risk-free interest rate. With real options there are still more sources of uncertainty to be taken into account. This is, in part, due to the fact that real options do not represent contractual rights. A good example is the investment costs which are a stochastic variable, even though, they are analogous to the (contractually fixed) strike price of financial options. Another necessity for integrating additional stochastic variables may arise from a disaggregation of the state variable. A disaggregation is necessary if we value compound options with one or several follow-up options, such as options to switch use or to switch operating mode. The modelling of a choice between different outputs and/or inputs requires the use of revenues and variable costs (or even more disaggregated variables such as input or output prices) instead of the aggregated value of the underlying. In brief, we can summarize three reasons why we have to take account of multiple variables in real option pricing:

- several factors which can be considered as being stochastic in the case of financial options may also be stochastic in the case of real options (e.g., variance of the underlying);
- several factors which are contractually fixed in the case of financial options may represent additional stochastic variables in the case of real options (e.g., strike price);
- several factors which arise from a disaggregation of the state variable may replace this stochastic state variable in the case of real options.

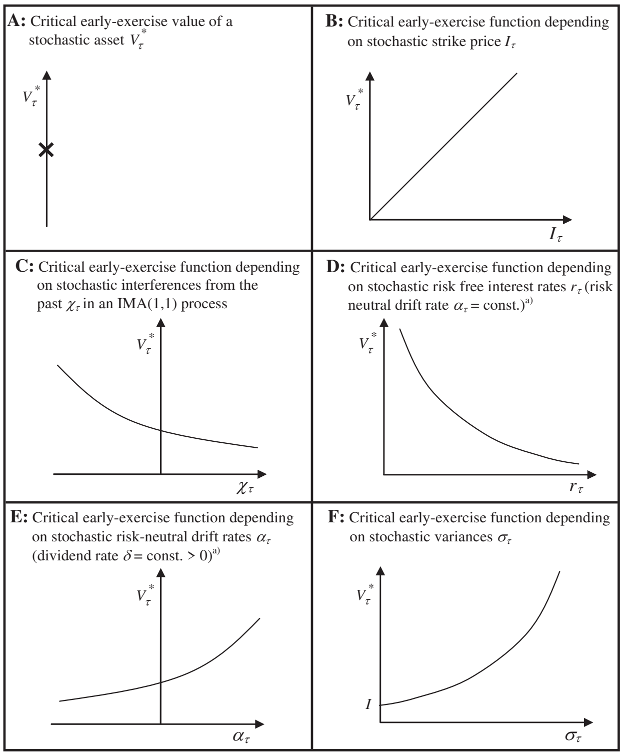
Option pricing, based on more than one stochastic variable, needs to take account of correlations. Correlations between stochastic variables alter the stochastic development of these variables and cause a change of the option value. The modelling of correlations is straightforward in simulation-based option pricing procedures (cf. Hull, 2000, p. 409). Stochastic simulation procedures can generally handle multiple stochastic variables quite easily. Therefore, not many adjustments to option pricing procedures have to be made when determining the value of European options depending on multiple stochastic variables. It should be noted, however, that time discrete versions of stochastic processes cannot be used in the case of stochastic variances or a stochastic risk free interest rate, etc. because they require a constant (i.e., non-stochastic) value of these parameters over time. That is why we have, even in the case of European options, to resort to a sufficiently fine discretisation of time when we simulate the price path of the underlying.

Contrary to European options, it is quite cumbersome to determine the early-exercise strategy and the value of American options which depend on multiple stochastic variables. With only one stochastic variable, one needs, at each time instant, to determine one critical exercise value. This results in a two-dimensional early-exercise strategy over time (see Figure 1 and Figure 4 A). Now, one needs, at each time instant, to determine critical combinations of values for different stochastic variables (early-exercise functions) which form a multi-dimensional early-exercise strategy over time. However, for the sake of clarity and feasibility of graphical representation we subsequently consider only the stochastic underlying and one additional stochastic variable at a time: first, a stochastic strike price (see Figure 4 B), second, an additional interference from the past emanating from an IMA(1,1) process (see Figure 4 C), third, a stochastic risk free interest rate (see Figure 4 D), forth, a stochastic risk neutral drift rate (see Figure 4 E), and fifth, a stochastic standard deviation (see Figure 4 F).

In the case of stochastic investment costs, the problem of multiple stochastic variables can be reduced to determining one critical value m_τ^*, defining the constant positive gradient of the linear early-exercise function (see Figure 4 B).

$$V_\tau^* = m_\tau^* \cdot I_\tau \,. \tag{11}$$

In contrast, a stochastic interference term from the past χ_τ emanating from an IMA(1,1) process or a stochastic risk free interest rate generate early-exercise functions with negative and exponentially decreasing gradients (see Figure 4 C and D).



Note: According to the risk-neutral valuation principle the following applies: $r = \alpha + \delta$.

Fig. 4. Schematic representation of the critical early-exercise function at any point in time depending on two stochastic variables

Stochastic risk-neutral drift rates and stochastic variances, in turn, generate early-exercise functions with positive and exponentially increasing gradients (see Figure 4 E and F).

It has already been shown in the previous part of this paper that the powerful stochastic simulation procedures can be used for pricing American options by embedding them in a backward-recursive option valuation framework. Now, dealing with multiple stochastic variables and, therefore, early-exercise functions instead of early-exercise values, we are again able to integrate operational procedures solving this problem into the broader methodical framework of simulation-based option pricing. The following steps specify the additional operational procedures needed within a framework of option pricing which is based on a backward recursive stochastic simulation of price paths:

- At expiration, the critical exercise function is reduced to the well-known critical exercise value I.
- In the last early-exercise date we use the free boundary approach. Instead of determining one critical value, however, we determine a discrete critical value of the underlying for a given value of the second stochastic variable, using the value matching condition. By a subsequent systematic variation of the second variable, we find critical values of the underlying, depending on the second stochastic variable (critical combinations).
- In most cases subsequently simulated price paths will not coincide exactly with these selected combinations of discrete values. We could therefore use linear interpolation in order to determine the (piecewise) exercise strategy. We could even avoid having to store selected discrete values to interpolate between by estimating, at each time instant, an explicit critical early-exercise function via regression.
- By proceeding recursively backwards, all the other critical combinations (respectively early-exercise functions) can be determined because at any single exercise point the future strategy is known. In order to save time, we always use the (already known) critical combination of values of the subsequent period as a lower bound for the test-values we start simulating from.
- After the determination of this multidimensional early-exercise strategy the value of the option can again be calculated by one simple Monte Carlo simulation, starting from the presently observed value.

A major problem with dynamic programing type algorithms is the so called "curse of dimensionality": except for perfect correlations, the number of samples per variable grows exponentially with the number of stochastic variables if a given level of accuracy is to be maintained. This increases computational costs and in particular storage consumption. Compared to pure dynamic programming methods, which model the development of the stochastic variable over time, the main advantage of the above-described approach for the consideration of multiple stochastic variables is that, at each time instant, only a discrete approximation of the hyperplane of the critical early-exercise values (or even only a critical early-exercise function) needs to be stored.

4. Option pricing and validation of different approaches

In this Section we compare, test and validate four different methods to determine the critical early-exercise paths and the option value: (1) the BRSS-method derived above, (2) the original approach by Grant et al. (1997a), (3) the approach by Ibanez and Zapatero (2004) or Ibanez (2004), and (4) genetic algorithms. We apply these methods to a straightforward valuation problem. This enables us to validate the results of the simulation based approaches against the reference solution provided by the binomial tree method.

The context is that of a Bermudan type investment option which can only be exercised at certain points in time. This assumption is plausible for real options because the realization of an investment is often restricted by seasonal conditions, the legal framework, etc. We assume that there is only one source of uncertainty, namely the present value of future cash flows (the value of the underlying asset) which follows GBM. In order to generate an adequate validation benchmark, we assume the same Bermudan option (i.e., the same number of early-exercise opportunities) and use a very fine discretisation of time in the binomial tree (cf. Cox et al., 1979).

4.1. Model assumptions. A present value of investment cash flows V_0 of 110 T€ is expected. The investment costs I are 100 T€. They are completely sunk once the investment is carried out. Investment opportunities are given at dates τ, $\tau = 0,1,...,\Gamma$ with $\Gamma = 5$. The lifetime of the option is $T = 5$ years. The length of a time period between two potential exercise dates is $\Delta\tau = T/\Gamma = 1$ year. Therefore, there are $\Gamma + 1 = 6$ potential exercise opportunities. The standard deviation of the stochastic process for the expected present value of the investment cash flows σ is 20% p.a. Additionally, there is a continuous convenience yield (dividend payment) δ of 5.83%. The continuous risk-free interest rate r is also 5.83 %. This is equivalent to a discrete interest payment of 6% p.a. It is assumed that σ, r and δ are constant. The state variable V follows GBM (see Hull, 2000, p. 407):

$$dV = (r - \delta) \cdot V \cdot dt + \sigma \cdot V \cdot dz, \quad (12)$$

where dz describes a Wiener process.

We use the discrete-time version of a GBM for the simulation:

$$V_{\tau+1} = V_{\tau} \cdot \exp[(r - \delta - \sigma^2/2) + \sigma \cdot \varepsilon_{\tau+1}], \quad (13)$$

where ε is a standard normally distributed random number. In order to improve the stability of the solution of simulation based option valuation methods, we use the antithetic variables technique.

We are to answer the following questions: (1) What is the value of the investment option? and (2) Which present value of investment cash flows would trigger an immediate investment?

4.2. Results and validation. The results for both the early-exercise path and the value of the option are shown in Table 2. According to the BRSS-method, the investment option should be immediately exercised if the expected present value of investment cash flows exceeds 145.47 T€. Looking at the critical values at subsequent dates, one sees very easily that the critical exercise path decreases exponentially with the reduction of the lifetime of the option. That was expected from theoretical insight (see right illustration in Figure 1). The value of the investment option according to the BRSS-method is 19.91 T€.

Table 2. Comparison of different valuation procedures

τ	Binomial tree method * (reference)	Bounded recursive stochastic simulation (BRSS)	Approach by Grant et al.	Approach by Ibanez and Zapatero	Genetic algorithm **
Critical early-exercise value V_τ^*					
0	145.27	145.47	145.43	145.39	144.57
1	142.50	142.43	141.53	142.66	140.69
2	138.96	138.55	137.78	138.55	134.24
3	133.56	132.53	133.73	133.42	128.59
4	125.11	124.69	124.48	124.94	121.94
5	100.00	100.00	100.00	100.00	100.00
American style option value F_0					
	19.86	19.91	19.88	19.77	19.67
Confidence interval for the "true" option value $\tilde{F}_0$ (with 5% error probability)					
	-	$19.70 < \tilde{F}_0 < 20.12$	$19.67 < \tilde{F}_0 < 20.09$	$19.56 < \tilde{F}_0 < 19.97$	$19.47 < \tilde{F}_0 < 19.88$
Time required for programing the model					
	small	small	small	high	very high
Time required for computation of V_τ^* and F_0 ***					
	approx. 5 min	approx. 30 min	approx. 2 h	approx. 8 h	approx. 12 – 24 ****

Note: * The discretisation of the development of the state variable (value of investment cash flows) is 0.05 years;
** 100 generations with in each case 50 000 simulation runs;
*** computing time with direct programing in MS-EXCEL. For the simulation-based methods, 50 000 simulation runs are carried out; antithetic variables technique is used; 1 400 MHz-processor;
**** highly dependent on random numbers.

The reference solution from the binomial tree and the results of alternative simulation based methods are also given in Table 2. It is apparent that all procedures yield almost identical values for the critical early-exercise path. Only the early-exercise values found by the genetic algorithms approach deviate a little bit from those found by the other methods. But the option prices F_0 found by the different methods, including genetic algorithms, are virtually the same. Confidence intervals for the option price are used to describe the quality of the different simulation based procedures. They are quite small and very similar for all simulation procedures.

The performance of the different numerical methods from the practical point of view (i.e., programing effort and computational cost) is also summarized in Table 2. Of all simulation procedures, BRSS causes the least computational costs. Naturally, for this simple problem, the binomial tree is the least cumbersome, both with regard to programing effort and computational time. It must be emphasised, however, that lattice methods do not show enough flexibility to consider complex stochastic processes (e.g., non-Markov processes like autoregressive integrated moving-average processes), multiple stochastic variables, correlations, path-dependent Asian options, etc. In contrast, all these complexities can be easily implemented within the framework of simulation-based procedures. This has been shown in a number of publications which, while not systematically comparing different methods, use simulation-based methods to solve different types of complex option problems. To name a few examples: Odening et al. (2005) consider different stochastic processes in the real options context (mean reverting process, arithmetic Brownian Motion, Poisson process etc.). Grant et al. (1997b) analyze price path-dependent Asian options, and Ibanez and Zapatero (2004) concern themselves with multiple stochastic variables. Genetic algorithms feature the highest programing and computation requirements and represent the most flexible option pricing method (cf. Balmann and Musshoff, 2002). They are needed, for instance, if real option pricing is carried out in a framework of game theory, where we have to consider different "players", and where we must look for an equilibrium-strategy (Nash-equilibrium). However, since genetic algorithms require significantly higher programing efforts and more computational time compared to other simulation approaches, they should be used only if they are really required. In all circumstances in which we do not have to deal with competition, but only with complex (American type) options, the BRSS-method is to be preferred. Focussing on the BRSS-method, one can state that it yields highly accurate results the quality of which is comparable to the quality of other simulation procedures (see confidence interval). It furthermore provides the fastest solution of all simulation-based procedures used in the test bed. It also requires the smallest programming effort.

One might assume that, within the BRSS-method, the length of the parameterization interval for the test values (cf. Step 2.1 in Section 3.1) plays a pivotal role for the method's accuracy and resource requirement. Using the last early-exercise date as an example and ignoring the already known lower bound, we show in Table 3 that the critical value is quite robust regarding a variation of the length of the parameterization interval. However, as shown in the last two rows of Table 3, the interpolation error increases significantly if the intersection of the continuation value function and the intrinsic value function is situated outside the parameterization interval. While the option value is not very sensitive to minor errors in the early-exercise strategy, we recommend that, in this case, the parameterization interval should be redefined.

Table 3. Sensitivity of the last early-exercise value with respect to the parameterization interval

Parameterization interval	Length of the interval	Last early-exercise value ($\tau = 4$)
100 to 172	8.00	124.69
120 to 130	1.11	124.60
122 to 126	0.44	124.59
123 to 125	0.22	124.59
100 to 120	2.22	123.95
130 to 150	2.22	123.66

Note: We always use ten test-values. Thus a reduction of the parameterization interval translates into an equivalent reduction of the interpolation interval; 50 000 simulation runs are carried out.

Leaving the assumption of a Bermudan option, we turn to the discretisation problem inherent to numerical option pricing approaches if applied to continuous time problems. For financial options with continuous exercising opportunities, option pricing based on discrete exercise points only approximates the true value of the option and leads to a low bias because it underestimates the flexibility. It is clear that both the effort for and the accuracy of the numerical pricing of an American option increases with the number of discrete time intervals and exercise opportunities that are considered. However, while effort increases linearly with the number of critical early exercise values to be determined, the marginal contribution to accuracy resulting from additional effort is not known a priori. That is why we generate solutions for different time intervals. Referring to the results in Table 4 we can derive the following conclusions:

1. Cox et al. (1979) show that, with a decreasing length of time steps, the results of the binomial method converge to the results of analytical solutions. Comparing the binomial tree solution for the European option with the analytical solution (see last row in Table 4) demonstrates that the chosen time interval of 0.05 years is already an acceptable

representation of a continuous stochastic process since it produces only a marginally differing option price. The same applies for the stochastic simulation with 50 000 runs and combined with the antithetic variables technique.

2. Each row in Table 4 can be understood as a Bermudan option. For each one of these Bermudan options, the BRSS produces results are comparable to those derived from the binomial tree method. With regard to an American option (with continuous exercise opportunities), it should be noted that the bias caused by the inevitable discretisation of the exercise opportunities within numerical approaches arises in both methods.
3. There is very little additional flexibility and thence little increase of the option value if we increase the number of early-exercise opportunities from, let's say, 20 to 50. We have only a very marginal contribution to accuracy by a lot of additional effort. Again this applies to both methods likewise.

Table 4. Values for options with different numbers of exercise opportunities

	Bounded recursive stochastic simulation (BRSS) *		Binomial tree method **	Black-Scholes-Merton equation ***
Γ	Option value F_0	Confidence interval (5% error probability)	Option value F_0	Option value F_0
100	20.20	$20.01 < \tilde{F}_0 < 20.40$	20.18	n.a.
50	20.17	$19.97 < \tilde{F}_0 < 20.36$	20.17	n.a.
20	20.15	$19.95 < \tilde{F}_0 < 20.35$	20.11	n.a.
10	19.97	$19.77 < \tilde{F}_0 < 20.18$	20.02	n.a.
5	19.88	$19.67 < \tilde{F}_0 < 20.09$	19.86	n.a.
2	19.22	$18.99 < \tilde{F}_0 < 19.45$	19.29	n.a.
0	17.95	$17.61 < \tilde{F}_0 < 18.29$	17.91	17.93

Note: * 50 000 simulation runs are carried out; antithetic variables technique is used;
** the discretisation of the development of the state variable (value of investment cash flows) is 0.05 years;
*** cf. Black and Scholes (1973) as well as Merton (1973).

Summary and conclusions

Quite contrary to a belief still prevailing even in the "professional world" (see Hull, 2000, p. 408; Trigeorgis, 1996), American type options can be quite easily priced by methods using Monte Carlo simulation. In the last decade, a great number of simulation-based procedures have been proposed. They are flexible enough to value even complex options and, in particular, real options which are often characterized by complex stochastic processes, multiple stochastic factors, correlation etc. Some of the procedures proposed so far suffer from an unsatisfactory flexibility or accuracy and/or a high programing and computational demand. Others are particularly appealing because of their accurateness, simplicity, flexibility and intuition. This is particularly true for the modification of the approach of Grant et al. (1997a) which we propose in this paper and which we call "bounded recursive stochastic simulation" (BRSS). Although, this modification appears to be rather marginal at first view, it allows for a significant reduction of computational time without loss of applicability.

In accordance with Grant et al. (1997a), the BRSS integrates a sequential stochastic simulation of price paths in a backward recursive programing approach to determine the critical early-exercise path. Then it values the option by initiating a simple Monte Carlo simulation from the valuation date of the option. The determination of the critical early-exercise values is straightforward: starting from the end and moving backward, for every exercise date, the critical value is determined by systematically simulating sample paths for the underlying asset emanating from different test-values at the respective date. The critical value falls between those test-values which yield a change of sign in the difference of intrinsic value and continuation value. It can be estimated by linear interpolation.

The BRSS is to be preferred to many other simulation-based valuation procedures due to its flexibility, especially with regard to the consideration of multiple stochastic variables. It is also to be preferred due to its simpicity, intuition and ease of implementation. Comparing its results to those of the binomial tree method shows that it yields highly accurate results. However, it must be recognized that there are real options problems which require even more flexible valuation methods. This is so, for instance, if we need to consider competition and, therefore, price dynamics endogenously instead of using a given stochastic price process as an input for a real option pricing model. In this case, option pricing will have to take account of decisions of agents or "players" and will be set in a framework of game theory where we have to look for an equilibrium strategy (Nash-equilibrium). This can be implemented by integrating genetic algorithms into simulation procedures (agent-based simulation procedures). Yet, it should be stated that these methods are more complex and require much more computational time. That is, the choice of the option pricing procedure should follow the economic rationale to use the least costly one needed for the option pricing situation under consideration.

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