Alessia Naccarato (Italy), Andrea Pierini (Italy)

Element-by-element estimation of a volatility matrix. An Italian portfolio simulation

Abstract

The authors propose a procedure for a mean-variance Markovitz type portfolio selection based on estimates of average returns on shares and volatility of share prices. In other words, the authors address the problem of estimating average returns and the associated risk on the basis of the prices of a certain number of shares over time. The estimate is then used to identify the assets offering the best performance and hence constituting the best investments. The use of VAR (1) models is common practice in the literature; here instead we suggest the CVAR models, which take into account cointegration between the series employed and the market trend as measured by means of the Equity Italy Index. The use of BEKK is then applied to the residuals obtained before in a multiple bi-dimensional way so that the computational is made feasible while retaining a complex structure representation. The model put forward is applied to a series of data regarding the prices of 150 best capitalized shares traded on the Italian stock market (BIT) between 1 January 1975 and 31 August 2011; it takes into account the intrinsic value of the stocks to select the best performing ones. Eventually the authors find the efficient portfolio by minimizing the risk. The proposed methodology allows for the inclusion of more information and has very appealing strengths when compared to established models.

Keywords: Markowitz portfolio, cointegrated vector autoregressive models, multivariate volatility models.
JEL Classification: C580.

Introduction

The selection of a stock portfolio is broadly discussed in the literature, generally with reference to heteroskedastic regression models (Bollerslev et al., 1994). The model used in the case of multiple time series is of the vector autoregressive (VAR) type and rests on the predictability of the average return on shares (Brown and Reily, 2008; Hamilton, 1994). As shown by the Markowitz theory (Markowitz, 1952), the selection of a share portfolio involves estimating not only risk but also average return.

As far as the average return is concerned, the application of multivariate time series to model financial returns has been debated in the literature with modellers taking mainly two different approaches: the first one (Fama, 1965) has a priori belief that returns should be uncorrelated; the second approach (Campbell, 2003) allows for the returns to be correlated. Following the second approach and on the basis of empirical evidence provided by statistical tests (see (6) and par. 4), we believe sensible to consider the possibility that financial returns can be integrated and cointegrated in the short term, while stationary in the long term. We hence propose a modelling approach that allows for this eventuality.

In particular this paper suggests the combined use of cointegrated vector autoregressive models (CVAR) and, as far as the risk is concerned, Baba-Engle-Kraft-Kroner models (BEKK) (Engle and Kroner, 1995) for the selection of a stock portfolio.

In other words, it addresses the problem of estimating average returns and the associated risk on the basis of the prices of a certain number of shares over time. The possibility of combining the two different models, in order to estimate the magnitudes required for portfolio selection, is supported by asymptotic theory. It can in fact be shown (Lutkepohl, 2007, p. 571) that estimates of parameters obtained by combining the two models (applying the CVAR model first, then using the residuals thus obtained to estimate the diagonal elements of the volatility matrix by means of the BEKK model) coincide with those that would be obtained if the log-likelihood function to be maximized contained all the parameters simultaneously: both those of the CVAR and of the BEKK models. These estimates are then used to identify the assets offering the best performance and hence constituting the best investments.

While Campbell et al. (2003) propose the use of a VAR (1) model, we suggest here using vector error correction (VEC) models, as they take into account the cointegration between the series employed and the market trend. Moreover, while Bollerslev, Engle and Wooldridge (1988) employ diagonal vectorization (DVEC) models to estimate share volatility, the use of a BEKK model, as proposed here, allows to extend the estimation procedure based on DVECs to take also into account the correlation between the volatility of the series and the volatility of the market trend. We present application to the Italian stock market (BIT): specifically the monthly figures for the top 150 shares in terms of capitalization, from 1 January 1975 to 31 August 2011.

The estimation procedure proposed for portfolio selection involves two steps. In the first step, a two-dimensional CVAR model is developed for all of the 150 shares considered in order of capitalization

The paper has been presented at the conference Maf2014 (maf2014@unisa.it).

to obtain an estimate of the average stock market return. A BEKK model is then applied to the series of residuals in order to estimate the volatility of the series. The BEKK model appears particularly suitable because it does not entail the condition of normality for the innovations of the model (Hamilton, 1994). The second step regards the selection of shares for inclusion in the portfolio. Only those identified as presenting positive average returns during the first phase are considered eligible. For the purpose of selecting the most suitable of these, a new endogenous variable is constructed as the product of two further elements, namely the price-to-earnings ratio (P/E) and earnings per share (EPS). This variable indicates the *intrinsic value* of the share.

The variable P/E · EPS (Nicholson, 1960) is estimated for each industrial sector and for each share. The CVAR-BEKK model is applied once again to this new series in order to estimate the intrinsic value of the shares and the best are selected for inclusion in the portfolio on the basis of the difference between this intrinsic value and the price estimated in the first phase (Brown and Reily, 2008). For every fixed number of assets included in the portfolio (n), a quadratic programing model is used to determine which of the different combinations of quantities of each share involves the least risk. In other words, as the quantity to be bought varies for each of the n shares chosen, different portfolios are identified. Different minimum-risk portfolios are thus obtained for variation of the dimension n. The portfolio to be invested in will ultimately be the one that involves the least risk out of all the minima identified.

1. The cointegrated vector autoregressive models

This paragraph provides a concise outline of the phases involved in the selection of shares to be included in the portfolio as well as their respective quantity. The starting point is the K = 150 series, regarding the average returns $R_{k,t}$ on the shares, and the average return of the market $R_{M,t}$ where $t = t_k,\ldots, T$ with = 1,…, K. It should be noted that the length of the series considered is not homogeneous because not all of the joint-stock companies are quoted from the same point in time. This aspect involves further complications in the estimation procedure.

For each series, the model CVAR(p) is defined for the random vector:

$$y_t = [y_{1,t}, y_{2,t}]' = [R_{k,t}, R_{M,t}]',$$

$$y_t = \mu_t + A_1 y_{t-1} + \ldots + A_p y_{t-p} + u_t, \qquad (1)$$

with $\mu_t = \mu_0 + \mu_1 \cdot t$, $A_i (i = 1,\ldots,p)$, a matrix of unknown coefficients and $u_t = [u_{1,t}, u_{2,t}]'$ a vector of errors such that $u_t \sim N(0, \Sigma_u)$. It can be rewritten as

$$\Delta y_t = \mu_t + \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \ldots + \Gamma_{p-1} \Delta y_{t-p+1} + u_t, \qquad (2)$$

where $\Gamma_i = -(A_{i+1} + \ldots + A_p)$, (3)

and $\Pi = -(I - A_1 - \ldots - A_p)$. (4)

If Π is singular, $y_1 = [y_{1,1},\ldots,y_{1,T}]'$ and $y_2 = [y_{2,1},\ldots,y_{1,T}]'$ are cointegrated (Johansen, 1995; Lutkepohl, 2007).

In specifying the CVAR model, the lag order and the cointegration rank have to be determined. We start by determining a suitable lag length because this task does not require knowledge of the cointegration rank. The AIC criterion is used to estimate the lag $\hat{p}$, with reference to model (1):

$$\hat{p} = arg\, min_m\, C(m) =$$

$$= \min_m \left[\ln\left(det\left(\tilde{\Sigma}_u(m) \right) \right) + \frac{mc_T}{T},\ m = 0,\ldots,p_{max} \right], \qquad (5)$$

where $\tilde{\Sigma}_u(m)$ is the maximum likelihood estimate (MLE) of $\Sigma_u(m)$ for a $VAR(m)$ of type (1) with a sample of breadth $T - t_k + m$ and m values of initialization, with $c_T = 8$, $p_{\max} = 10$. Note that, while we have considered a $VAR(p)$, the criterion is also applicable for choosing the number of lagged differences in a VEC model (2) because $p - 1$ lagged differences in a VEC correspond to a VAR order p (Lutkepohl, 2007).

In practice it is common to use statistical tests in specifying the cointegration rank.

In this framework to ascertain the presence of cointegration in model (2), the likelihood ratio test (LR) is used:

$$LR(r_0) = -T \sum_{j=r_0+1}^{2} \log(1 - \lambda_i), \qquad (6)$$

where $r_0 = rank(\Pi) = 0,1$ and λ_j are the eigenvalues of the matrix:

$$S_{11}^{-1/2} S_{10} S_{00}^{-1/2} S_{01} S_{11}^{-1/2},$$

with $$S_{00} = \frac{1}{T-p} \sum_{t=p+1}^{T} \hat{u}_t \hat{u}_t',$$

$$S_{01} = \frac{1}{T-p} \sum_{t=p+1}^{T} \hat{u}_t \hat{v}_t',$$

$$S_{11} = \frac{1}{T-p} \sum_{t=p+1}^{T} \hat{v}_t \hat{v}_t'.$$

The quantities $\hat{u}_t \hat{v}_t'$ are the residuals of the regressions of Δy_t and y_{t-1} estimated by maximum likelihood (Johansen, 1995). However, the asymptotic distribution of the LR statistic is nonstandard, in particular it is not a chi-squared distribution. In fact the limiting distribution is a functional of a standard Wiener process. Quantiles of the asymptotic distribution, thus, critical values for the LR test can be generated considering multivariate random walks (Johansen et al., 1990).

Assessment of the presence of cointegration between the series by means of the LR test is followed by estimation of the parameters of the model. The result of the LR test is considered in deciding whether to adopt the model in form (1) or (2). In particular, if the test shows that the rank of matrix Π is equal to 0 (hypothesis of stationarity of Δy_t), then $\Pi = 0$ and the method of maximum likelihood is applied directly to (2) in order to estimate the parameters μ_0, μ_1 and $\Gamma_1,..., \Gamma_{p-1}$. If instead there is evidence that the rank of Π is equal to 1 (hypothesis of cointegration of y_1 and y_2), then $\Pi = \alpha\beta$. In this case, it is necessary to estimate model (2) in two stages. First, an MLE of β is obtained by concentrating the log-likelihood with respect to β. Second, this estimate is inserted into (2) in order to obtain the MLE of the other parameters (Johansen, 1995). If *rank* (Π) = 2, the method of maximum likelihood is applied directly to (1) in order to obtain estimates of the parameters μ_0, μ_1 and $A_1,\dots, A_p$. A portmanteau test is used to ascertain the presence of correlation of residuals, the generalized Lomnicki-Jarque-Bera test for the normality of residuals, and the ARCH test to investigate heteroskedasticity. In order to forecast it is convenient, in this framework, to use the levels VAR representation. Therefore we consider the model (1) with integrated and possibly cointegrated variables replacing the coefficients with their estimates as calculated before:

$$\hat{y}_{T+1|T} = \hat{\mu}_T + \hat{A}_1 y_{T-1} + \dots + \hat{A}_p y_{T+1-p}. \tag{7}$$

2. The BEKK model for heteroskedasticity

In the event of the Lomnicki-Jarque-Bera test revealing the presence of heteroskedasticity, the *BEKK* (1,1) model is used to estimate the conditional variance-covariance matrix

$$\Sigma_{t|t-1} = cov(u_t \,|past) = \sigma_{i,j}(t), i, j = 1,\dots,n, \tag{8}$$

which has the following structure:

$$\Sigma_{t|t-1} = \begin{pmatrix} \sigma_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t} \end{pmatrix} = \begin{pmatrix} c_{0,11} & 0 \\ c_{0,21} & c_{0,22} \end{pmatrix} \cdot \begin{pmatrix} c_{0,11} & c_{0,12} \\ 0 & c_{0,22} \end{pmatrix} + \\ + \begin{pmatrix} c_{1,11} & c_{1,12} \\ c_{1,21} & c_{1,22} \end{pmatrix} \cdot \begin{pmatrix} u_{1,t-1}^2 & u_{1,t-1}\, u_{2,t-1} \\ u_{2,t-1}^2 & u_{1,t-1}\, u_{2,t-1}^2 \end{pmatrix} \cdot \begin{pmatrix} c_{1,11} & c_{1,12} \\ c_{1,21} & c_{1,22} \end{pmatrix} + \\ + \begin{pmatrix} c_{2,11} & c_{2,12} \\ c_{2,21} & c_{2,22} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11,t-1}^2 & \sigma_{12,t-1} \\ \sigma_{21,t-1} & \sigma_{22,t-1}^2 \end{pmatrix} \cdot \begin{pmatrix} c_{2,11} & c_{2,12} \\ c_{2,21} & c_{2,22} \end{pmatrix}. \tag{9}$$

Equation (9) is equivalent to a multivariate (MV) GARCH(1,1) model. This allows (Tsay, 2010) an effective maximum likelihood estimation even when return are highly non Normal. In such a case the parameters of equation (9) are estimated by quasi maximum likelihood. The estimate of parameters $c_{k,i,j}$ at time $t = T$ (Lutkephol, 2010, p. 569) is then obtained by maximizing the log-likelihood function:

$$-\ln(2\pi) - \frac{\ln\left|\Sigma_{t|t-1}\right|}{2} - u_t{}'\Sigma_{t|t-1}^{-1}u_t. \tag{10}$$

Once the parameters have been estimated, the generalized portmanteau test (Hosking, 1980) is applied to ascertain that the model BEKK(1,1) has effectively eliminated the ARCH effects in the residuals of the CVAR model:

$$\tilde{Q}_m = n^2 \sum_{k=1}^{m} (n-k)^{-1} \hat{g}_k{}' (\hat{G}_0^{-1} \otimes \hat{G}_0^{-1}) \hat{g}_k \sim \chi^2_{m-p}, \tag{11}$$

where p is the CVAR order, $n = T - t_{max}$, $t_{max} = \max\{t_i, t_j\}$, $\hat{g}_k = vec(\hat{G}_k)$, $\hat{G}_k = L'\hat{C}_k L$, $\hat{G}_k = n^{-1} \sum_{t=k+1}^{m} \hat{u}_t \hat{u}_{t-k}$, $\hat{G}_k = n^{-1} \sum_{t=k+1}^{m} \hat{u}_t \hat{u}_{t-k}$, $L'L = \hat{C}_0$, $m = 1,\dots, 10$.

We define $\delta = vec(C_0, C_1, C_2)$ the array of unknown parameters, the estimator $\tilde{\delta}$ of the array d has an asymptotic normal distribution with variance-covariance matrix given by the inverse of the asymptotic information matrix (Comte and Lieberman, 2001).

It follows that $\tilde{\delta}$ is the asymptomatically most efficient estimator under the normality hypothesis of u_t.

Even if the conditional distribution of u_t underlying an ARCH(q) model is normal, the unconditional distribution generated will generally be non-normal. In particular, it is leptokurtic, that is, it has more mass around zero and in the tails than the normal distribution and, hence, it can produce occasional outliers.

Moreover the asymptotic information matrix of the VAR parameters and the GARCH parameters is block diagonal so that the estimators of the VAR coefficients are asymptotically independent of the GARCH parameter estimators (Lutkephol, 2010, p. 571). This result suggests that the two step estimation procedure here applied, in which the VAR coefficients are estimated and then a GARCH model is fitted to the residuals, is asymptotically equivalent to the overall estimator.

The procedure, concerning the construction of the CVAR(p) and BEKK(1,1) models, is repeated for each pair i, j of series defined by the random vector

$$y_t = [y_{1,t}, y_{2,t}]' = [R_{k,t}, R_{M,t}].$$

The estimates obtained in precedent phase are used to select the shares for which positive average returns are predicted. For the shares thus selected and for each industrial sector (IS), the model CVAR(p)-BEKK(1,1) is estimated for the random vector y:

$$y_t = [y_{1,t}, y_{2,t}]' = [(P/E)\cdot(EPS)_{h,t}, (P/E)\cdot \cdot(EPS)_{IS(h),t}]',$$

where $h = 1,\ldots, H$ is the index that identifies only the series with positive returns selected out of the initial 150. On the basis of the $(P/E)\cdot(EPS)_{h,T+1}$ and $R_{h,T+1}$ forecasts obtained in phase two, the shares are listed for each industrial sector in decreasing order with respect to the values of the difference between intrinsic value and expected price.

The first $n = 10$ shares are thus selected to make up the portfolio. The choice of $n = 10$ is made on the basis of the assertion by Evans and Archer (1968) that this quantity is sufficient for diversification of portfolio choices. The dimension n of the portfolio is increased by one unit at a time until n_{max}, where n_{max} is the maximum number of shares with positive predicted returns.

In order to determine the quantities to be bought of each of the n shares selected, it is necessary to solve the Markowitz problem (Markowitz, 1952) by estimating the matrix of share volatilities. To this end, let $\hat{V}_t$ be the estimator of the $n \times n$ volatility matrix V_t for $= T + 1$, the elements of which are:

$$v_{i,j}(t) = cov(R_{i,t}, R_{j,t} \,|\, past),\ i, j = 1,\ldots,n.$$

The elements of V_t for $t = T + 1$ are given by:

$$\hat{v}_{i,j}(t) = \begin{cases} \hat{\sigma}_{11,t}^{(i,M)} \text{ if } i = j \\ \hat{\sigma}_{12,t}^{(i,j)} \text{ if } i \neq j \end{cases}, \qquad (12)$$

where $\hat{\sigma}_{11,t}^{(i,M)}$ is the corresponding estimated element of the matrix $\Sigma_{t|t-1}$ defined in equation (9) for the random vector $y_t = [y_{1,t}, y_{2,t}]' = [R_{k,t}, R_{M,t}]'$, whereas $\hat{\sigma}_{12,t}^{(i,j)}$ is the corresponding estimated element of the matrix $\Sigma_{t|t-1}$ defined in equation (9) for the random vector $y_t = [y_{1,t}, y_{2,t}]' = [R_{k,t}, R_{M,t}]'$.

On the basis of (12), a quadratic Markowitz type problem, called global minimum variance portfolio, for the future time $T + 1$ can be obtained with the approximation given by the dual method (Goldfarb, 1983; Higham, 2002) as:

$$min_{\omega\in\mathbb{R}}\{\omega'\hat{V}_{T+1}\omega : \omega'\underline{1} = 1, \omega'\hat{P}_{T+1} = R_{p,T+1}\},$$

where $\hat{R}_{T+1} = [\hat{R}_{1,T+1},\ldots,\hat{R}_{K,T+1}]'$ and $R_{p,T+1} \in \mathbb{R}^+$.

We omit the constraint of a fixed value for the expected return to eliminate the sensitiveness of allocation optimization to errors in predicted returns (Hlouskova et al., 2002). In cases where the matrix $\hat{V}_{T+1}$ is positive neither in (12) nor in (13), we approximate it with the closest matrix, in the sense of Frobenius, possessing the same diagonal given by the elements estimated with the BEKK model.

3. Application to the Italian stock market

As an illustration, the methodology proposed in the previous paragraphs is now applied to the monthly values of the 150 BIT shares with the highest level of capitalization.

The market trend considered for this application is the Thomson Reuters Datastream Global Equity Italy Index (Datastream 2008). On the basis of the minimum AIC, the optimal lag selected for the different shares is between 2 and 9 months. Figure 1 (see Appendix) shows the empirical distribution of the lag. In particular, while the optimal lag is 2 months for 77% of the entire set of 150 shares, it is at most 2 months for 88% of the shares in the portfolio and more than 2 months for the remaining 12%. This means that 2 months of observations are sufficient to predict the average returns on the vast majority of the shares considered, instead of a random walk model. It was therefore decided to set lag 9 as the maximum lag.

Such a high value is probably unrealistic and possibly reflects a multiple testing effect beyond the AIC penalization for extra parameters. A heavier penalization could be adopted.

The results of the LR test for all the shares considered, show an order of cointegration equal to 2 for 91% of the 150 shares, 1 for 7% and 0 for the remaining the VEC model we used is estimable (Johansen, 1995) and stationary. Under these conditions, the alternative VAR model is neither directly estimable nor stationary.

The coefficients of the model estimated in both steps of the procedure are significant for almost all of the series considered. The p value of the F statistic to test the joint significance of the coefficients is in fact below 0.10 in 83% of cases and the model is therefore significant for nearly all of the series. Figures 2, 3 and 4 (see Appendix) show diagnostic tests for models (1) and (8). The results of the portmanteau test on the residuals of the CVAR and BEKK models are given in the form of histograms for the p-values. As regards the presence of ARCH effects in the residuals of the CVAR model (Figure 2), the null hypothesis of no presence of a heteroskedastic component in the CVAR model estimated, at a confidence level of

95%, cannot be ruled out for 79% of the shares considered. The results of the Hosking test carried out in order to ascertain the presence of ARCH effects in the residuals of the BEKK model (Figure 3), lead to the conclusion that the hypothesis of the presence of a heteroskedastic component in the model is rejected in 76% of cases. In other words, it can be concluded that the combination of the CVAR and BEKK models captures the information regarding the heteroskedastic component to a satisfactory degree.

Figure 4 shows the results of the portmanteau test carried out in order to ascertain the presence of autocorrelation of residuals in the BEKK model. They suggest that the null hypothesis of no correlation in the BEKK residuals cannot be ruled out for 90% of the shares considered and that the maximum lag considered is sufficient.

An OLS-based CUSUM test for stability of the market index was also carried out, and the results suggest that the hypothesis of stability of the series over the period considered is acceptable.

The BEKK estimate of volatility for each share is between 0.001 and 0.01 for 93% of the series and never above 0.031. Hence it can be concluded that for most of the series considered, the estimated value of the share does not differ from its real value, at a confidence level of 95%, by more than 0.2. In actual fact, the value at risk calculation put forward by J.P. Morgan (Longerstaey et al., 1995; Duffie et al., 1997) could be used in order to include the information deriving from the presence of correlation between the series considered and hence to assess the overall risk rather than the risk of the individual shares.

Further confirmation of the adequacy of the CVAR-BEKK model with respect to the series observed was sought before selecting the shares to be included in the portfolio. Specifically, the confidence interval, at the level of significance of 95%, contains the actual value $T + 1$ in 94% of the series.

The CVAR-BEKK model can therefore be considered reliable for most of the series for the purpose of prediction. The next step after verification of the suitability of the model was the prediction of the prices of the shares as well as their intrinsic values.

Figure 5 shows the values predicted on the basis of model (7) together with the actual values and the estimated confidence interval for each time series considered.

Figure 6 presents the estimates of the elements of the volatility matrix and shows that the risk is mostly due to the variances of the shares, to which the highest peaks correspond. It is evident, however, that the values of variance and covariance are comparable for some subsets of shares. This suggests that it could prove useful, in order to reduce computational complexity, to take covariance into consideration only for specific subgroups of shares and variance alone for the others. It therefore becomes necessary to develop a criterion, based for example on the Granger principle of causality or on analysis of cross-correlation, in order to identify the groups of shares to be addressed in a different way.

On the basis of potential, expressed as the difference between share price and intrinsic value, the ten shares with the highest potential returns were then selected.

The criterion of partial ranking was used. It involves arranging the values of potential of all the shares considered in decreasing order for every industrial sector (Goodman and Peavy III, 1983) and selecting the first share in each. The use of the partial criterion is connected with the relationship between P/E and share performance manifested most strongly in each industrial sector. As Goodman and Peavy III write, '*firms in the same industry tend to cluster in the same relative P/E ranking, detected return differences between P/E groups may be attributable to industry performances rather than P/E level. This bias is eliminated by using P/E relative to its industry*'.

Figure 7 shows the efficient frontiers obtained by solving the optimization problem (12) for variation of the expected return $R_{p,T+1}$ and the dimension n of the portfolio (n = 10, 11,…, n_{max}), where $n_{max} = 25$ is the maximum number of shares with positive forecasted returns.

Overall the portfolio risk tends to decrease as n increases. The optimal risk from a risk-averse standpoint (i.e. the least of all those calculated) corresponds to $n = 25$; this is indicated with the letter X appearing furthest to the left in Figure 6. This portfolio has a monthly average return of 0.00993, a monthly standard deviation of 0.0630, and a Sharpe index of 0.15771. We therefore select the portfolio made up of $n = 25$ shares with the optimal allocations shown in Figure 8.

Conclusions

The selection of a share portfolio has historically constituted a complex problem that has no single solution but depends both on market conditions and on the information available to investors. In other words, the choice of shares to invest in must be based on objective criteria making it possible to assess risk and return without ignoring investors' opinions. To this end, the paper suggests the use of a model for the analysis of multiple historical series with a view to the prediction of share return and associated risk but also taking the indications of the market into account at the same time in the specification of the model itself. Variables obtained as functions of P/E and EPS have thus been used together with the market index as regressors of the combined model (1) and (8). The innovative choices in the construction of a portfolio selection model put forward here regard two distinct aspects. The first multiple historical series (Campbell, 2003) to one of the CVAR (p) type, which makes it possible to take into consideration any cointegration of the series considered and therefore constitutes an improvement on the inclusion of the information available for estimation purposes. The subsequent use of a combination of CVAR and BEKK models, which extends the results of Bollerslev, Engle and Wooldrige (1988), makes it possible to consider also the temporally variable correlation between the volatility of the series and the volatility of the market index within the estimation procedure. The second aspect concerns the choice of the criterion for the selection of shares, which is addressed here by seeking to insert a typical financial concept such as intrinsic value into the primarily statistical context of the prediction of multiple historical series. An intrinsic value estimated by means of the combined CVAR-BEKK model is used to obtain a potential value serving as a basis to rank the different shares and then select the top ten. The method put forward was applied to return time series of 150 shares with highest capitalization quoted on the Italian stock exchange and led to the selection of 10 shares constituting a portfolio with an average monthly return of 0.00993 and a risk of 0.0630. Comparison of the results of the CVAR-BEKK model proposed here and those obtained by means of VAR (1) and DVEC models, established in the literature, was carried out on the basis of the values of their log-likelihood. In other words, since one model could produce a higher but less reliable value of return than another, it was decided to assess the models performance in terms of correspondence to the series observed. The log-likelihood of the CVAR-BEKK model always proves greater than that of the other models, thus indicating more accurate representation of the series observed and hence better predictions. A further element of innovation of this work regards the method of volatility matrix estimation. In particular, an individual element on the diagonal of the volatility matrix is estimated by applying the model to the series of log returns both of the share *i* to which it refers and of the market index. An extra-diagonal element is instead estimated by using in the model the covariances between the series of log returns of the two shares *i* and *j* to which the element of the volatility matrix corresponds. This procedure eliminates the typical computation problem of BEKK models, namely failure to converge on a solution if there are more than five series (Ding and Engle, 1994). As the number of shares considered in order to define a portfolio is generally higher than five, this problem would appear somewhat important. As stated in Francq and Zakoian (2010), '... the specification should be parsimonious enough to enable feasible estimation. However the model should not be too simple to be able to capture the, possibly sophisticated, dynamics in the covariance structure'.

Hence the need to harness all the potential of the BEKK model also in the case of a large number of shares and the proposal of a BEKK model estimated element by element for each of the elements of the volatility matrix.

Moreover we find a decrease in the overall risk as the number *n* of shares in the portfolio increases as expected (Nyholm, 2008). Finally, a further development could regard the study of *value at risk*, understood as assessment of the greatest loss possible, as well as identification of possible structural breaks of the individual series of share returns with a view to making the model more adaptable.

References

1. Bollerslev, T., Engle, R.F., Nelson, D.B. (1994). ARCH Model, Handbook of Econometrics, Elsevier Science, Amsterdam, Vol. IV, pp. 2959-3038.
2. Bollerslev, T., Engle, R.F., Wooldridge, J.M. (1988). A Capital Asset Pricing Model with Time-Varying Covariance, *Journal of Political Economy*, 96 (1), pp. 116-131.
3. Brandt, M.W., Goyal, A., Santa-Clara, P., Stroud, H.R. (2005). A simulation approach to dynamic portfolio choice with an application to learning out return predictability, *The Review of Financial Studies*, 18 (3), pp. 831-873.
4. Brown, K., Reily, F. (2008). *Investment Analysis and Portfolio Management*, South-Western College Publishing.
5. Campbell, J.Y., Chan, Y.L., Viceira, L.M. (2003). A Multivariate Model of Strategic Asset Allocation, *Journal of Financial Economics*, 67 (1), pp. 41-80.
6. F. Comte and O. Lieberman (2001). Asymptotic Theory for Multivariate GARCH (2003). Processes, *Journal of Multivariate Analysis*, 84, pp. 61-84.
7. Datastream Global Equity Indices (2008). User guide, issue 5, Thomson Reuters Ltd.
8. Duffie, D., Pan, J. (1997). An overview of value at risk, *Journal of Derivatives*, pp. 7-48.
9. Engle, R.F., Kroner, K.F. (1995). Multivariate Simultaneous Generalized ARCH, Econometric Theory, 11 (1), pp. 122-150.

10. Evans, J.L., Archer, S.H. (1968). Diversification and the Reduction of Dispersion: an Empirical Analysis, *The Journal of Finance*, 23, pp. 761-767.
11. Ding, Z., Engle, R.F. (1994). Large scale conditional covariance matrix modeling, estimation and testing, mimeo, UCSD.
12. Fama, E.F. (1995). Random Walks In Stock Market Prices, *Financial Analysts Journal*, 21 (5), pp. 55-59.
13. Francq, C., Zakoian J.M. (2010). *GARCH Models*, Wiley and Sons.
14. Goldfarb, D., Idnani, A. (1983). A numerically stable dual method for solving strictly convex quadratic programs, *Mathematical Programming*, 27, pp. 1-33.
15. Goodman, D.A., Peavy III, J.W. (1983). Industry Relative Price-Earnings Ratios as Indicators of Investment Returns, *Financial Analysts Journal*, 39 (4), pp. 60-66.
16. Hamilton, J.D. (1994). Time Series Analysis, Princeton University Press.
17. Higham, N. (2002). Computing the nearest correlation matrix - a problem from finance, *IMA Journalof Numerical Analysis*, 22, pp. 329-343.
18. Hlouskova, J., Schmidheiny, K., Wagner, M. (2002). *Multistep Predictions from Multivariate ARMA-GARCH Models and their Value for Portfolio Management*, Diskussionsschriften, UniversittBern.
19. Hosking, J.R.M. (1980). The multivariate portmanteau statistic, *Journal of American Statitical Association*, 75, pp. 602-608.
20. Johansen, S., Juselius, K. (1990). Maximum likelihood estimation and inference on cointegration with applications to the demand for money, *Oxford Bulletin of Economics and Statistics*, 52, pp. 169-210.
21. Johansen, S. (1995). *Likelihood-Based Inference in Cointegrated Vector Autoregressive models*, Oxford University Press, Oxford.
22. Longerstaey, J., More, L. (1995). *Introduction to Risk Metrics*, 4th ed., Morgan Guaranty Trust Company, New York.
23. Lutkepohl, H. (2007). *New Introduction to Multiple Time Series Analysis*, Springer.
24. Markowitz, H. (1952). Portfolio Selection, *Journal of Finance*, 7, pp. 77-91.
25. Nicholson, S.F. (1960). Price-earnings ratios, *Financial Analysts Journal*, 16 (4), pp. 43-50.
26. Nyholm, K. (2008). Strategic asset allocation in fixed-income markets: a MATLAB-based user's guide, J. Wiley and Sons.

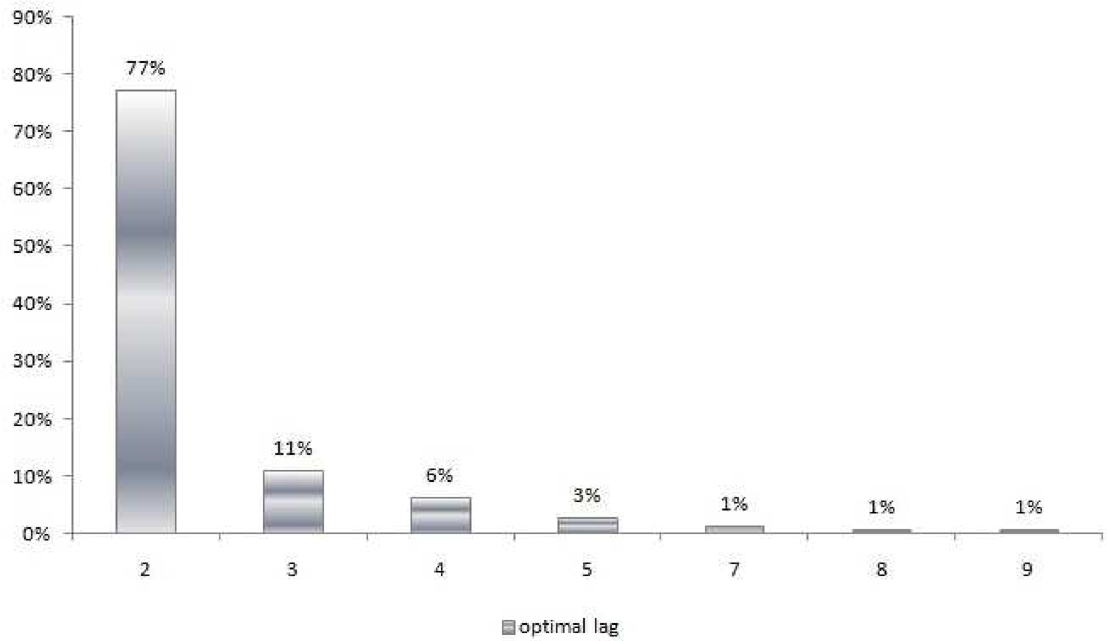

Fig. 1. Optimal lag

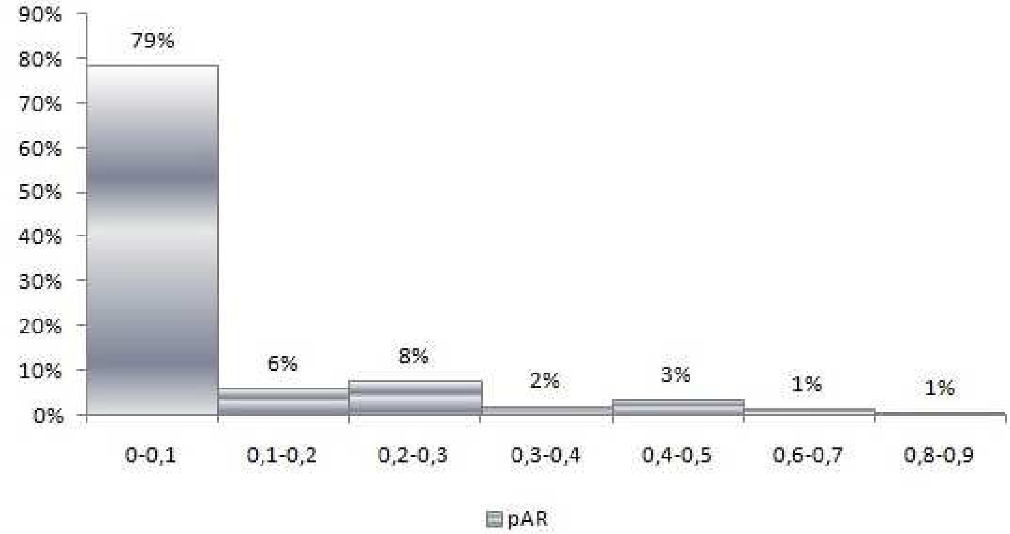

Fig. 2. ARCH effects test *p*-values (pAR) for CVAR residuals

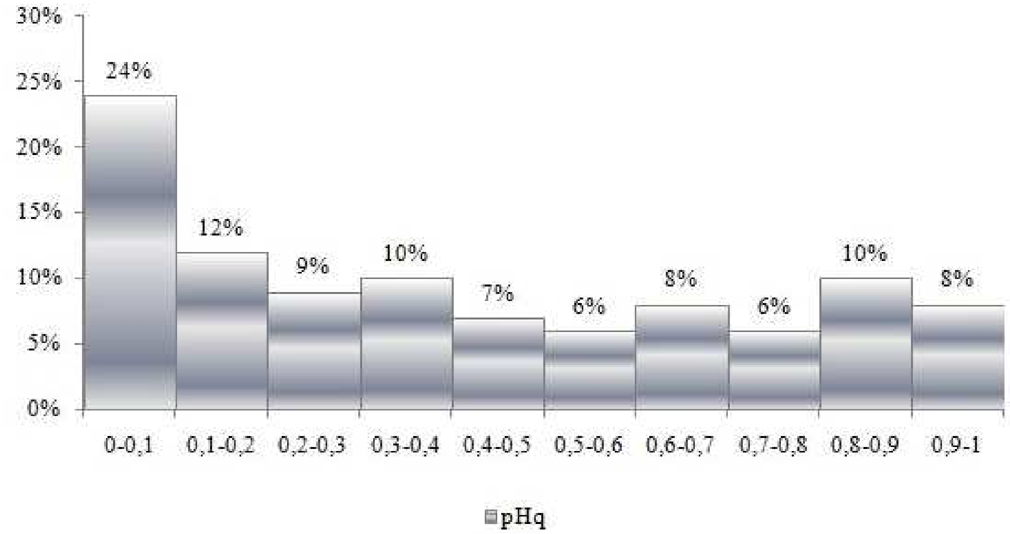

Fig. 3. ARCH effects test *p*-values (pHq) for BEKK residuals

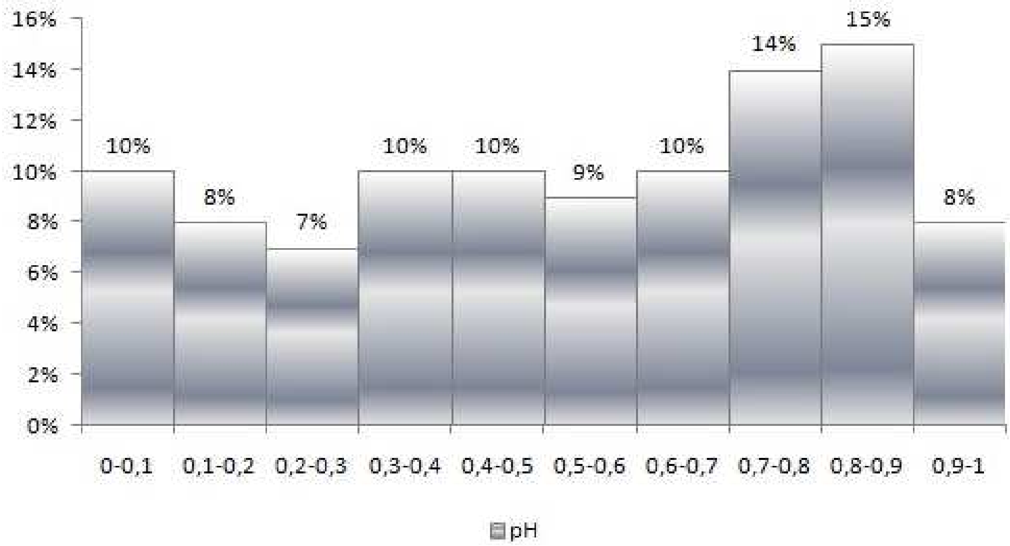

Fig. 4. No correlation test *p*-values (pH) for BEKK residuals

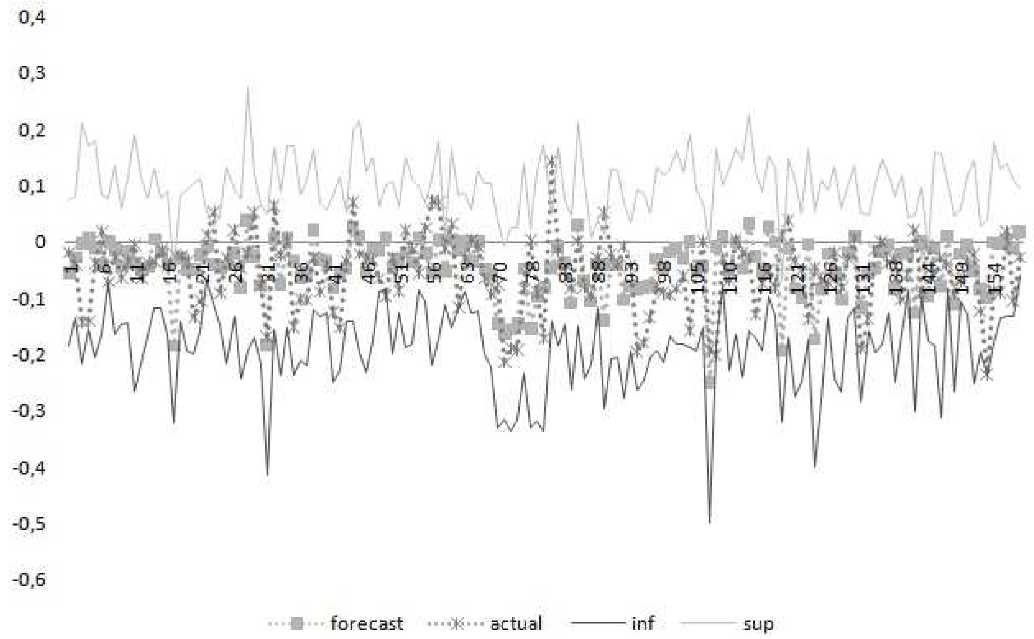

Fig. 5. Estimated confidence interval 95% and forecast, actual values

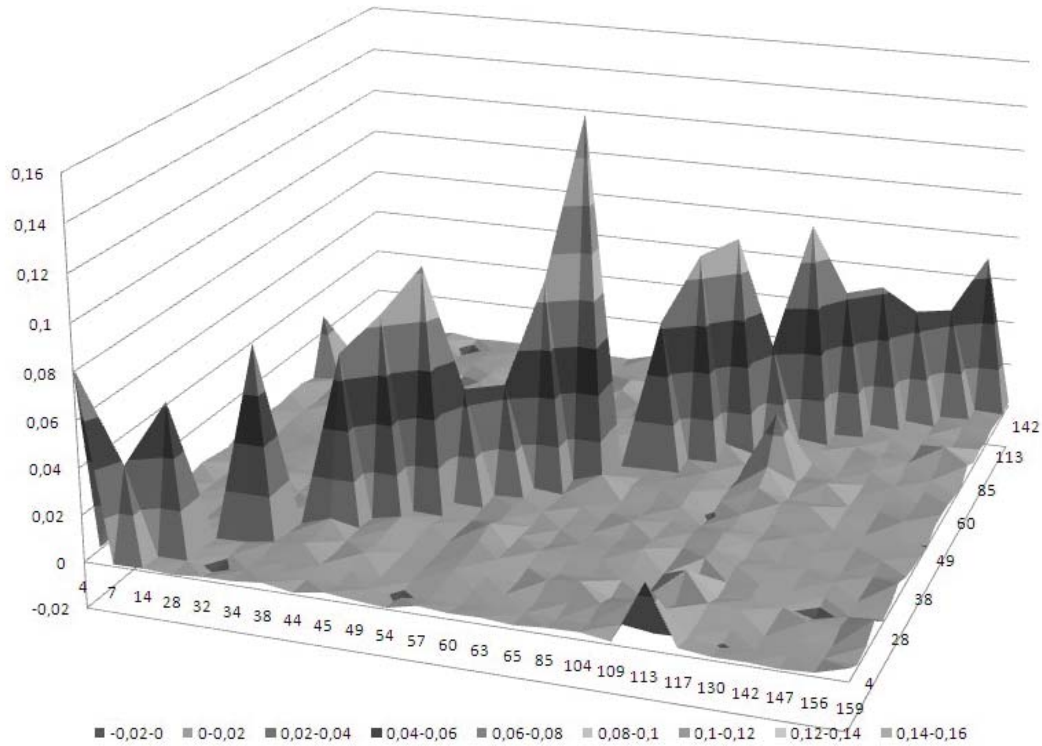

Fig. 6. Estimates of the elements of the volatility matrix (n = 25)

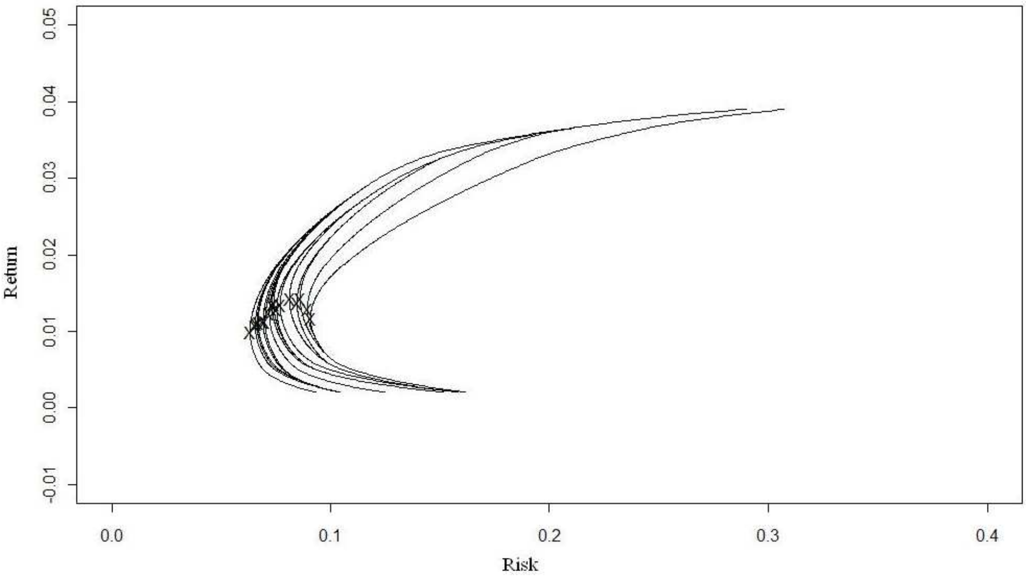

Fig. 7. Efficient simulated portfolio frontiers (upper curves' part)

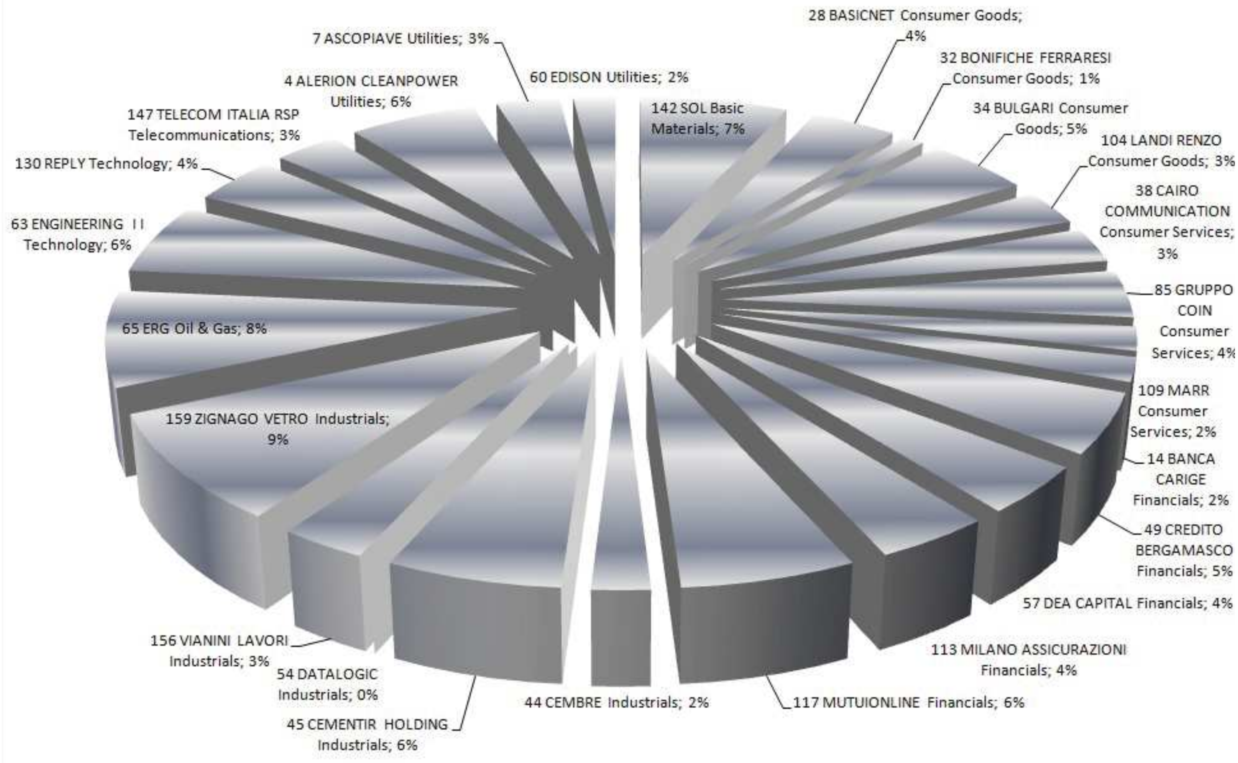

Fig. 8. Optimal portfolio allocation (*n* = 25)