"Measuring and managing the longevity risk for an indexed life annuity"

AUTHORS	Albina Orlando Massimiliano Politano

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Albina Orlando (Italy), Massimiliano Politano (Italy)

Measuring and managing the longevity risk for an indexed life annuity

Abstract

This paper investigates the problem of longevity risk sharing between an annuity provider and the annuitants. In this field, the idea of reducing the annuity periodic payments in similar way to what happens in the context of securitization is gaining. In the following the authors refer to a contract in which the installments of life annuities are scaled by a demographic index. The main finding is that, scaling the periodic installments, would result in a significant reduction in the level of benefits. The conditions that allow to limit the reduction of benefits without worsening the insurer's position is investigated. The conclusion is that it is possible to achieve an equilibrium not only reducing the amount of the periodic installments but also moving forward the retirement age.

Keywords: longevity risk, forecasting mortality, stochastic mortality intensity, deferral annuity, indexed life annuities.
JEL Classification: G22, G23, C15.

Introduction

During the 20th century, human life expectancy have considerably increased for the populations of many developed countries. Although the past trends suggest that further changes in the level of mortality are to be expected, the future improvements of life expectancy are uncertain and difficult to be predicted.

This uncertainty about the future development of mortality gives rise to longevity risk. The real challenge for public pension systems and for private insurance companies consists precisely in the design of products able to absorb any adverse events concerning the future mortality. In other words, the challenge is how to deal with the longevity risk. When we treat benefits depending on the survival of a certain number of individuals, the calculation of the present values, used both for pricing and for reserving, requires an appropriate projection of mortality in order to avoid an underestimation of future costs. Therefore, actuaries have to employ projected life tables incorporating a forecast of future trends of mortality. The insurer bears the risk that the projections of mortality turn out to be incorrect and the annuitants live longer than expected. Different approaches for the construction of the projected tables have been developed until now (for a full report on this subject, see Pitacco, 2004), but no one turned out to be suitable for the problem solution. Actually, this problem is deeply felt by private insurance companies. Although the annuity market is not well developed in western countries, the reduction of the intervention field of public systems, due to the main goal of the cost containment and the gradual shift from defined benefit schemes to defined contribution systems, suggests a growing interest of individuals for annuities.

The main problem for insurers is to make the annuities market attractive to the insured. Indeed the risk borne out by insurers for insurance annuities, which is undoubtedly too high, is reflected in high premiums charged for these products that discourage individuals who are intending to purchase annuities.

For this reason, many insurance companies and pension funds providers focus in the issue of sharing the longevity risk. An ordinary way to solve this problem is through reinsurance, but this method often involves high costs. The securitization provides a viable alternative (see Denuit, Devolder and Goderniaux, 2007), but unfortunately the longevity bonds are not a very attractive business for investors. Denuit et al. in 2011 have proposed a very interesting idea based on the reduction of annuity periodic payments in a similar way to what happens in the context of securitization. In this work, we try to develop this concept relying on past mortality experience of the Italian population measured in the period of 1954-2008. A computational tractable approach based on a CIR type stochastic process for modeling the future uncertainty about the force of mortality is used. We find that the process of reducing the payments for the insured would result in a significant reduction in the level of benefits hardly acceptable by the annuitant. On the other hand, without a proper reduction of benefits, the pension provider would face and hardly sustainable level of risk. At this point, in our opinion, it is possible to reconcile the two positions not only reducing the amount of periodic payments but also moving forward the retirement age, depending on the level of risk borne out by the insurer. In this way, the unknown factor is deferral time of the annuity, which becomes the variable to be controlled in order to achieve the equilibrium.

The literature on the attractiveness of deferred annuities has addressed the issue of choosing

between the purchase of an immediate annuity and a deferred one (see Milevsky and Young, 2007; Blake Cairns and Dowd, 2006). Because of the longevity, the choice of a deferred annuity is often preferred by the annuitants (see Milevsky, 2005). We contribute to this literature by linking the time of deferral and the impact of longevity risk. In the following we refer to the case of a longevity indexed life annuity with the aim of finding an equilibrium between the reduction of benefits (or the increase of premiums) and the annuity deferral.

The paper is organized as follows. Section 1 describes the annuities indexing process. In section 2 the general issue of modeling the uncertainty in future mortality is fronted and a CIR type model for describing the future evolution of hazard rates is described. In section 3 the effects that a certain hypothesis about the future mortality can have on the longevity index values are deepened. In section 4 the authors look for the conditions that allow to reduce the loss of benefits to the insured by decreasing the period of payment of the annuity. The final section concludes and discusses the results.

1. Longevity index

Let us consider an individual aged x in the calendar year t. His remaining life is indicated by the notation $T_x(t)$. Therefore, the individual will die at age $x + T_x(t)$ in the calendar year $t + T_x(t)$. Then $q_x(t) = P(\mathrm{T}_x(t) \leq 1)$ is the probability that an individual aged x in calendar year t dies before reaching the age $x + 1$ and $p_x(t) = 1 - q_x(t) = P(T_x(t) > 1)$ is the probability that the same individual reaches the age $x + 1$.

Let $p_{x+k}^{mod}(t + k)$ $(k = 0,\ldots,\omega - x)$ be the predicted one year survival probability referred to an individual aged x in the calendar year t deducted by some survival model, where ω denotes the ultimate age. Therefore $p_{x+k}^{mod}(t + k)$ $(k = 0,\ldots,\omega - x)$ is the assumption that is made on the future mortality.

As time passes, the observed values of the one year survival probabilities $p_{x+k}^{obs}(t + k)$ $(k = 0,\ldots,\omega - x)$ become available, so that it is possible to compare the values predicted on the basis of a given model with the actual ones, by means of the following ratio:

$$i_{t+k} = \prod_{j=0}^{k-1} \frac{p_{x+j}^{mod}(t+j)}{p_{x+j}^{obs}(t+j)}, \qquad (1)$$

which can be assessed each future calendar year k.

The basic idea is that the annual payment due at time k to an individual buying a longevity indexed annuity at age x in calendar year t, is adjusted by the factor (1). Hence, if the contract specifies an annual

payment of 1, the annuitant receives a stream of payments $i_{t+1}, i_{t+2},\ldots$ as long as he or she survives. In practice, we consider a basic life annuity contract paying one monetary unit of currency at the end of each year as long as the annuitant survives. The single premium is given by

$$a_x(t) = E\left[\sum_{k=1}^{T_x(t)} 1^{(x,k)} v(t,k)\right] = \sum_{k=1}^{\omega-x} v(t,k)_k\, p_x(t), \qquad (2)$$

where $1^{(x,k)}$ is an indicator which equals one if the individual with age x at time t is alive in the future year k $(k = 1,\ldots,\omega - x)$, $v(t,k)$ is the deterministic discount factor, $_kp_x(t)$ is the ordinary survival probability will be defined rigorously in the next section.

At this point, if the predictions contained in the model are chosen such that the increase in longevity is greater than predicted, then the payments due to the insured are reduced accordingly. Substantially, the random longevity indexed life annuity is given by the following equation:

$$a_x^{L.I.}(t) = E\left[\sum_{k=1}^{T_x(t)} 1^{(x,k)} i_{t+k} v(t,k)\right] = \sum_{k=1}^{\omega-x} v(t,k) i_{t+k\,k}\, p_x(t) \quad (3)$$

The annuitant bears the non diversifiable risk that the predicted mortality trend departs from that of the reference population.

Our work focuses on evaluating ex post the effects that a certain hypothesis about the future mortality can have on the index values.

The aim of our work is twofold: on the one hand, the authors analyze the values of the index in order to quantify the effects that an incorrect choice by the insurer can have on the benefits paid to the insured, on the other the conditions that allow to reduce the loss of benefits to the insured by decreasing the period of payment of the annuity are deepened.

2. The mortality model

Let us consider an individual aged x in the calendar year t. As seen, $p_x(t) = P(T_x(t) > 1)$ is the probability that an individual reaches the age $x + 1$. Analogously $_kp_x(t)$ is the probability that an individual aged x in year t reaches age $x + k$ in the year $t + k$. If we consider the hazard rate for an individual aged $x + t$ in the year t μ_{x+t} we have

$$_k p_x(t) = E\left[e^{-\int_0^k \mu_{x+s}\,ds}\right]. \qquad (4)$$

We describe the evolution in time of mortality by a widely used stochastic mortality model, supposing

that the force of mortality at time t for an individual aged $x + t$ is given by

$$d\mu_{x+t} = \kappa(\gamma - \mu_{x+t})dt + \sigma\sqrt{\mu_{x+t}}dB_t,\qquad(5)$$

where κ and σ are positive constants, γ is the long-term mean and B_t is a Standard Brownian Motion. This model, referred as the CIR mortality model has the property that the mortality rates are continuous and remain positive. Moreover, for $2\kappa\gamma \geq \sigma^2$ the mortality rates does not reach zero, and the drift factor $\kappa(\gamma - \mu_{x+t})$ ensures the mean reversion of μ_{x+t} towards the long term mean γ.

For convenience, we now introduce the centered version of the model. Let us consider the shifted $\mu_{x+t}^* = \mu_{x+t} - \gamma$. The process is then centred around γ and the long-term mean converges almost everywhere to zero:

$$d\mu_{x+t}^* = \kappa\mu_{x+t}^*dt + \sigma\sqrt{\mu_{x+t}^* + \gamma}dB_t\qquad(6)$$

with initial condition given by the known value of μ_{x+t}. Its solution is given by

$$\mu_{x+t}^* = e^{-\kappa t}\mu_{x+0}^* + \sigma e^{-\kappa t}\int_0^t e^{\kappa u}\sqrt{\mu_{x+u}^* + \gamma}dB_u.\qquad(7)$$

The expected value, the covariance and the stationary variance functions immediately follow:

$$E[\mu_{x+t}^*] = e^{-\kappa t}\mu_{x+0}^*$$

$$\mathrm{cov}(\mu_{x+t}^*, \mu_{x+s}^*) = \sigma^2\frac{e^{-\kappa t} - e^{-\kappa(s+t)}}{\kappa}\mu_{x+0}^* +$$

$$+ \sigma^2\frac{e^{-\kappa(t-s)} - e^{-\kappa(s+t)}}{2\kappa}\gamma,\qquad s \leq t$$

$$\lim_{t->\infty} Var[\mu_{x+t}^*] = \frac{\gamma\sigma^2}{2\kappa}.$$

2.1. Parameter estimation procedure. Estimating the parameters of the stochastic mortality model requires the discrete representation of the model.

To this aim, we refer to the covariance equivalence principle (see Deelstra Parker, 1995 which requires that the expected values and the stationary variances of the continuous and discrete processes to be equal.

The discrete model representation is given by the following equation:

$$\mu_{x+t}^* = \phi\mu_{x+t-1}^* + \sigma_a\sqrt{\frac{2\phi}{1+\phi}\mu_{x+t-1}^* + \gamma}a_t.\qquad(8)$$

The expected value, the covariance and stationary variance functions of the previous equation are:

$$E[\mu_{x+t}^*] = \phi^t\mu_{x+0}^*$$

$$\mathrm{cov}(\mu_{x+t}^*, \mu_{x+s}^*) = 2\phi^t\sigma_a^2\mu_{x+0}^*\frac{1-\phi^s}{1-\phi^2} +$$

$$+ \phi^{t-s}\sigma_a^2\gamma\frac{1-\phi^{2s}}{1-\phi^2},\qquad s \leq t$$

$$\lim_{t->\infty}Var[\mu_{x+t}^*] = \frac{\sigma_a^2\gamma}{1-\phi^2}$$

The estimation procedure starts by finding the value of ϕ that minimizes the residual sum of squares function:

$$RSS = \sum_{t=1}^N \frac{(\mu_{x+t}^* - \phi\mu_{x+t-1}^*)^2}{\frac{2\phi}{1+\phi}\mu_{x+t}^* + \gamma}.$$

The least squares estimate of σ_a^2 is given by $RSS/N - 1$.

Finally the continuous model parameters are obtained by means of the parametric relationships between continuous and discrete models, derived by applying the covariance equivalence principle:

$$\phi = e^{-\kappa},$$

$$\sigma_a^2 = \sigma^2\frac{1-e^{-2\kappa}}{2\kappa}.\qquad(9)$$

At this point, by the Pitman and Yor formula, we can compute

$$_kp_x(t) = E\left[e^{-\int_0^k \mu_{x+s}ds}\right] = \frac{\exp\left\{-\frac{x}{\sigma^2}w\frac{1+\frac{\kappa}{w}\coth(wk/2)}{\coth(wk/2)+\frac{\kappa}{w}}\right\}}{(\cosh(wk/2)+(\kappa/w)\sinh\coth(wk/2))^{\frac{2\kappa\gamma}{\sigma^2}}},\qquad(10)$$

where $x = \mu_0 e$ and $w = \sqrt{\kappa^2 + 2\sigma^2}$.

Applying the described estimation procedure, the significant parameters of the mortality-CIR model are obtained and therefore the survival probabilities for each specific calendar year.

Our set of data relates to the Italian male population with annual age-specific death counts ranging from ages 64 to 89 over the period from 1954 to 2008 (data source: Human Mortality Database, www.mortality.org).

We refer to the class of the forward mortality models. These models study changes in the mortality rate curve for a specific age cohorts and capture dynamics of each age cohort over time for all ages greater than x in a specific year t (for example age x in the year t, $x + 1$ in the year $t + 1$ and so on). In this case, the mortality curves are modeled diagonally (for example see Dahl, 2004; Cairns et al., 2006, Bauer et al., 2008). In practice,

on the basis of data available for the previous 25 years, we can estimate the model parameters for the year t and, as a result, it is possible to get the forecasted survival probabilities.

For example, with the data of the period of 1954-1978 it is possible to obtain the column of the survival probabilities for the year 1979. This procedure is repeated thirty times in order to obtain the annual survival probabilities over the period from 1979 to 2008 and ranging from ages 64 to 89.

These probabilities can be compared with the corresponding survival rates obtained from the tables of the Human Mortality Database.

Regarding the choice of fixing the extreme age to 89, recent studies (Khalaf-Allah et al., 2006) have shown that the most damaging effects in terms of annuities present values for the provider are in the age range 73-80. Clearly this happens because the number of survival is still large at these ages. As a consequence, even modest improvements in the level of survival probabilities with respect to those used for pricing and reserving, result in large additional costs for the annuity provider. The results of the estimation procedure are summarized in the following table (Table 1). The parameters κ and σ^2 are obtained, for each year, by means of the relations (9), after the estimation of the discrete parameters in (8). We choose to calculate the long term mean γ as the simple mean of each historical series used to estimate the parameters.

Variable κ takes the same value for each calendar year. The reason can be found in the high autoregressive parameter of the discrete model $\phi = 0.999$, which is the same each year explaining the high correlation of each data of each series with the preceding one.

Table 1. CIR-estimated mortality parameters

Year	κ	σ^2	γ
1979	0.0010005	0.02154137	0.09879589
1980	0.0010005	0.02218555	0.09870146
1981	0.0010005	0.02196340	0.09855553
1982	0.0010005	0.02125207	0.09849015
1983	0.0010005	0.02006831	0.09848095
1984	0.0010005	0.02260864	0.09951799
1985	0.0010005	0.02051935	0.09773577
1986	0.0010005	0.02120267	0.09732236
1987	0.0010005	0.01981722	0.09758413
1988	0.0010005	0.01874663	0.09654774
1989	0.0010005	0.01883434	0.09567750
1990	0.0010005	0.01846146	0.09354197
1991	0.0010005	0.01880755	0.09122329
1992	0.0010005	0.01876966	0.09087511

Year	κ	σ^2	γ
1993	0.0010005	0.01765508	0.09013384
1994	0.0010005	0.01824302	0.08918243
1995	0.0010005	0.01765857	0.08768303
1996	0.0010005	0.01786665	0.08616861
1997	0.0010005	0.01748889	0.08443631
1998	0.0010005	0.01756343	0.08338616
1999	0.0010005	0.0182456	0.08166418
2000	0.0010005	0.01765565	0.07982632
2001	0.0010005	0.0170472	0.07782871
2002	0.0010005	0.01693314	0.07627881
2003	0.0010005	0.01732366	0.07500763
2004	0.0010005	0.01858599	0.07417957
2005	0.0010005	0.01731115	0.07317319
2006	0.0010005	0.01812203	0.07046889
2007	0.0010005	0.01662066	0.06851038
2008	0.0010005	0.01709801	0.06566561

Source: Human Mortality Database: Italian male population.

Figures 1a, 2a and 3a (see Appendix) show the comparison between the estimated annual survival probabilities obtained by means of the CIR model and the corresponding probabilities of the Italian male population. The results are shown year by year over the period 1979-2008.

## 3. The 'adjusted' longevity index

In the following we model the future uncertainty about mortality by means of the CIR type stochastic process described in section 2.

In practice, the longevity index (1) is computed as:

$$i_{t+k}^{CIR} = \prod_{j=0}^{k-1} \frac{p_{x+j}^{CIR}(t+j)}{p_{x+j}^{obs}(t+j)}, \qquad (11)$$

where $p_{x+j}^{CIR}(t+j)$ is the forecasted annual survival probability of a male aged 64 in 1983. The forecasted probabilities are obtained by means of the CIR type stochastic process on the basis of the estimated parameters; $p_{x+j}^{obs}(t+j)$ are the actual values of the annual survival probabilities deducted from the Italian male mortality tables over the period of 1983-2008.

In formula (11), $p_{x+j}^{CIR}(t+j)$ are calculated by means of (10), using the estimated parameters for the year 1983, based on the mortality experience over the period of 1958-1982. Figure 1 shows the comparison between the survival curve estimated by the model and the table available for the year 1983. The choice of the year 1983 can be explained as follows: an individual aged 64 in 1983 gets 89 in 2008. Knowing the real data until 2008, the estimated CIR probabilities can be compared with the real data.
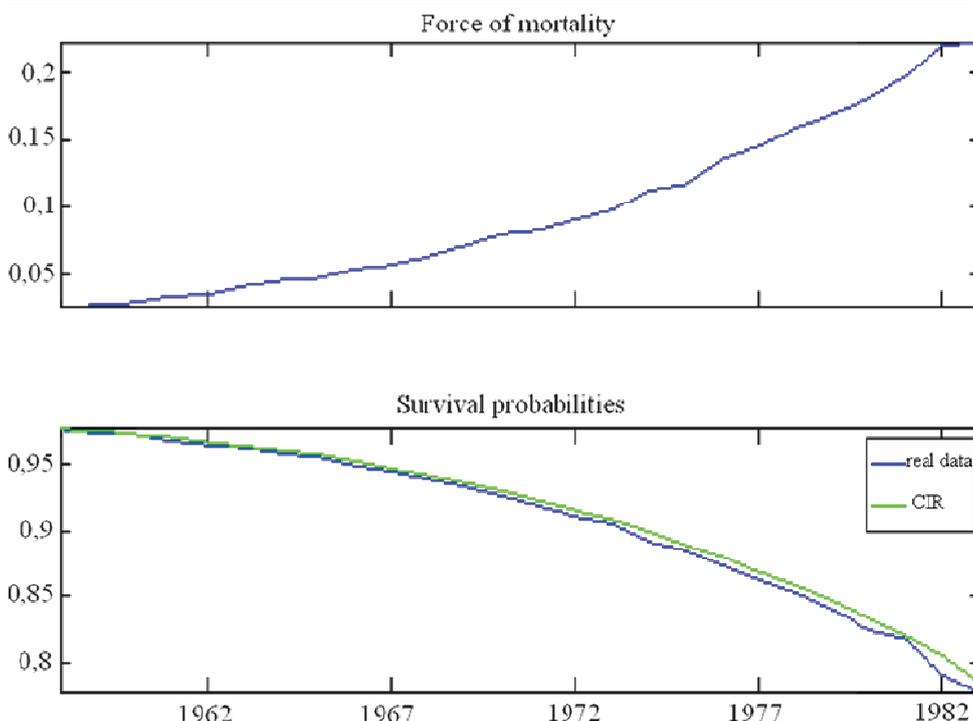
Notes: $\alpha = 0.0010005$; $\sigma^2 = 0.02008308$; $\gamma = 0.098480954$.

Fig. 1. Force of mortality and survival probabilities for an individual aged 64 in 1983 computed by means of the CIR type stochastic mortality model and Italian male mortality table referred to the period of 1983-2008

Figure 2 shows the evolution in time of the longevity index computed by means of (11) and setting $x = 64$, $t = 1983$, $j = 0,1,2,....24$.

We observe that, if the annuitant absorbs all the systematic risk, annuity payments may become arbitrarily low in old ages. In fact, at the age 89, the benefits fall below 50 percent compared with those expected without of the longevity index.

The problem, of course, is that the model fails in projection and is not able to capture the decrease in time of the parameters σ and γ because of the well known phenomena of rectangularization and expansion of the Lexis point. In other words, using the probability of the model and comparing them with those available on the tables ex post you get ratios that are significantly less than one.
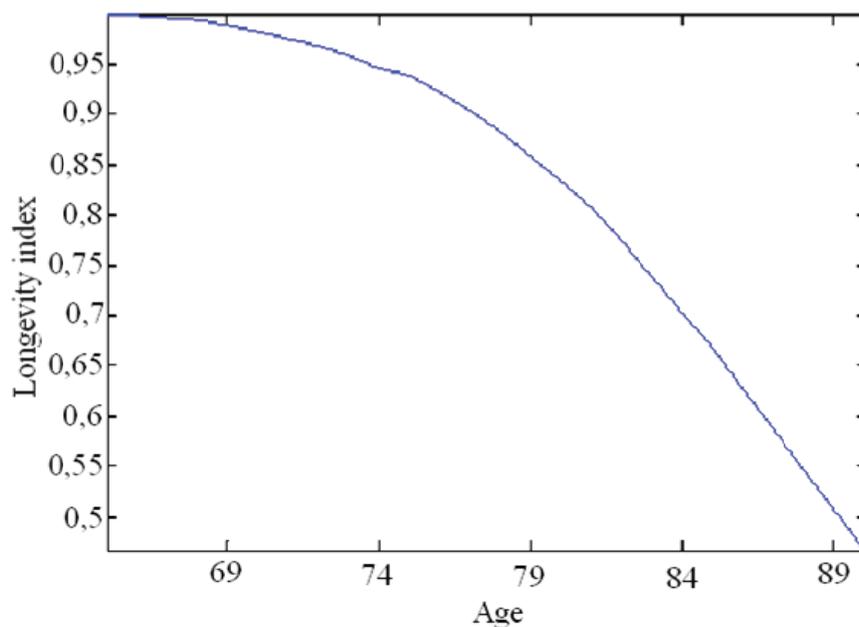
Fig. 2. Longevity index for an individual aged 64 in 1983 computed by means of the CIR type stochastic mortality model and Italian male mortality table referred to the period 1983-2008

Denuit et al. propose to reduce the systematic longevity risk passed to the annuitants limiting the impact of the index on the annuity payments. In particular they suppose that at most 20% of variation is allowed. In practice the longevity index (1) is replaced with the capped version

$$i_{t+k}(i_{min}, i_{max}) = \max\{\min(i_{t+k}, i_{max}), i_{min}\}, \quad (12)$$

with $i_{max} = 1{,}2$ and $i_{min} = 0{,}8$.

This means that the minimum value the index can reach is 0.8, and beyond this threshold the risk is borne by the insurer. Following the guidelines by Denuit at al. we compute a sort of 'adjusted' longevity index assuming the same hypothesis.

First of all we can observe that having, as shown in Figure 2, the longevity index a decreasing trend, surely the last value is the lowest one.

On the basis of the previous considerations the following procedure is implemented:

Referring to equation (5), a large number of paths for the force of mortality are simulated. Each path allows to compute a simulated set of probabilities, that is $p_{x+j}^{CIR}(t+j)$ with $x = 64$, $t = 1983$ and $j = 0,1,2,\ldots,24$. For each simulated set, the lowest value (the last value) of the longevity index is computed. Therefore 10000 values of (11) are obtained by setting $x = 64$, $t = 1983$, $k = 25$.
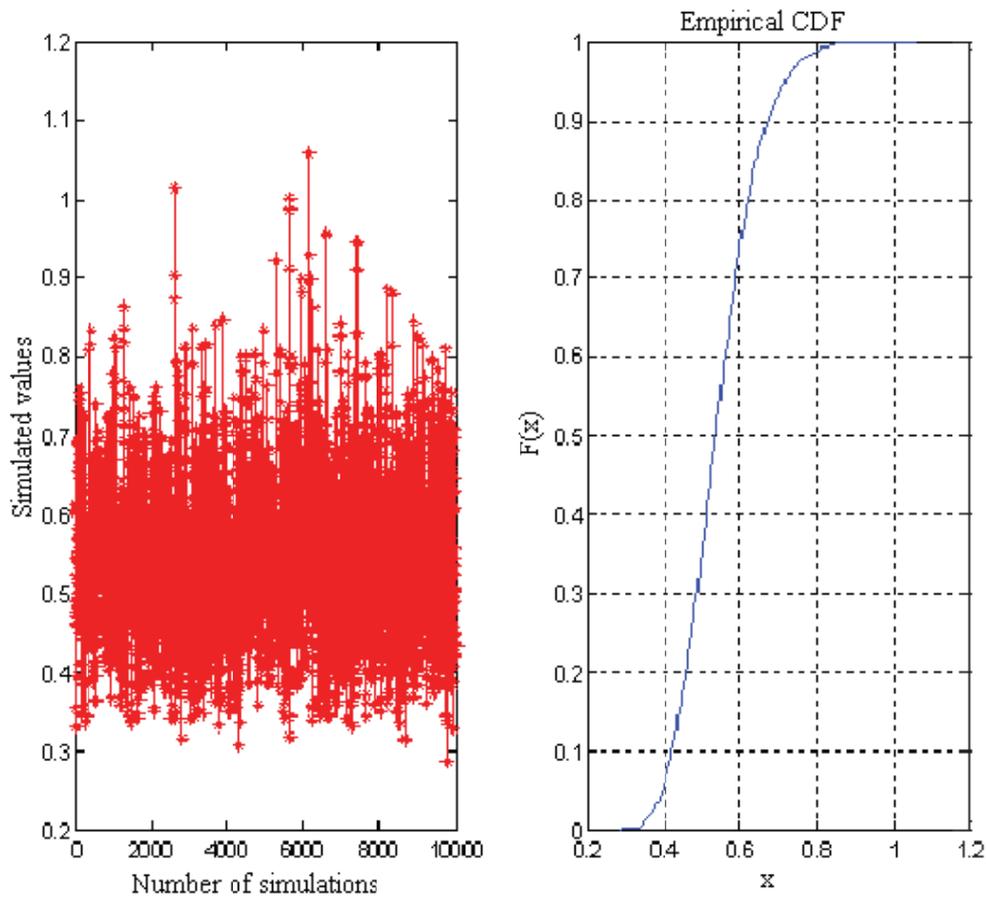

Fig. 3. Simulations of the last longevity index value computed assuming $\omega = 89$ and empirical CDF

Among the N simulated values, the one equal to 0.8 is chosen. Then the simulated vector of annual probabilities generating the index value 0.8, is deducted. In the following these probabilities are referred to as 'adjusted';

The 'adjusted' longevity index is then calculated as follows:

$$i_{t+k}^{ADJ} = \prod_{j=0}^{k-1} \frac{p_{x+j}^{ADJ}(t+j)}{p_{x+j}^{obs}(t+j)}, \quad (13)$$

where $p_{x+j}^{ADJ}(t+j)$ are the simulated annual survival probabilities generating the index value of 0.8.

As we can observe looking at Figure 4, the adjusted longevity index values are higher than the ones of the longevity index given by (11), and the minimum value reached is 0.8. In this way, the insured receives higher benefits. Obviously, the increase in the longevity profile leads to an increase of the survival probability and therefore to an increase of the premium paid by the insured.

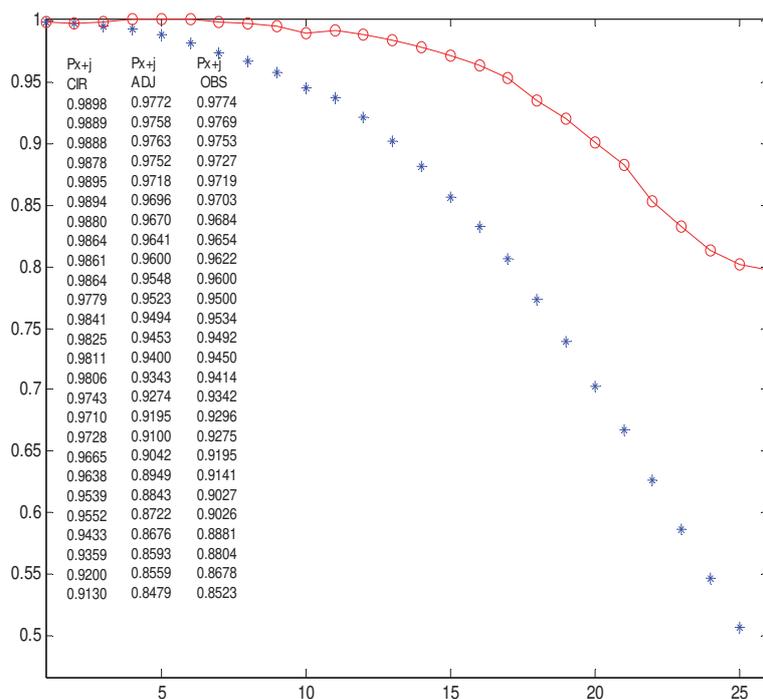| Px+j CIR	Px+j ADJ	Px+j OBS
0.9898	0.9772	0.9774
0.9889	0.9758	0.9769
0.9888	0.9763	0.9753
0.9878	0.9752	0.9727
0.9895	0.9718	0.9719
0.9894	0.9696	0.9703
0.9880	0.9670	0.9684
0.9864	0.9641	0.9654
0.9861	0.9600	0.9622
0.9864	0.9548	0.9600
0.9779	0.9523	0.9500
0.9841	0.9494	0.9534
0.9825	0.9453	0.9492
0.9811	0.9400	0.9450
0.9806	0.9343	0.9414
0.9743	0.9274	0.9342
0.9710	0.9195	0.9296
0.9728	0.9100	0.9275
0.9665	0.9042	0.9195
0.9638	0.8949	0.9141
0.9539	0.8843	0.9027
0.9552	0.8722	0.9026
0.9433	0.8676	0.8881
0.9359	0.8593	0.8804
0.9200	0.8559	0.8678
0.9130	0.8479	0.8523

Fig. 4. Comparison of longevity index (denoted by asterix) and 'adjusted' longevity index (denoted by circles)

The condition that the index doesn't fall below 80% limits the risk passed to the insured to a maximum of 20%. The insurer doesn't take any risk; indeed the insurer should bear a risk if the index should go under 80%.

Therefore, on the basis of an ex post analysis we observe that indexing the life annuity can lead to very low periodic installments. This condition doesn't make the contract attractive to the insured.

If the contract includes the capped version (12) the insured is encouraged to buy the annuity. On the other hand, the insurer is still exposed to the risk that the index goes below 80% if the predicted survival probabilities are estimated by means of the CIR stochastic model (or by means of any other model, knowing that at the moment doesn't exist a stochastic model immune to the projection risk).

Finally, if the 'adjusted' longevity index (13) is used, the risk passed to the insured is still limited to a maximum of 20% but the insurer doesn't bear any risk because the adjusted longevity index doesn't go below 0.8. Of course in this last case the insured will pay an higher premium.

4. Longevity indexed deferral annuities

At this point, in order to avoid a monetary penalty for the insured, you can search for an equilibrium between the reduction in benefits (or increase in premiums) and the deferral of the annuity. Essentially, it is possible to find a balance between economic penalty and time penalty to the insured.

In the following we study two cases in which the equilibrium could be reached: on the one hand considering the probability estimated by the CIR model and those limiting the index to 0.8, on the other hand considering the adjusted probabilities and the observed ones. The technical rate is fixed to 3%.

In the first case the unique premium $a_x^{ADJ}(t)$ calculated using the survival probabilities generating the index value of 0.8 and the one calculated using the CIR survival probabilities $a_x^{CIR}(t)$ both referred to the calendar year t are given, respectively, by the following expressions:

$$a_x^{ADJ}(t) = \sum_{k=1}^{\omega-x} v(t,k) i_{t+k}^{ADJ}{}_k p_x^{ADJ}(t), \qquad (14)$$

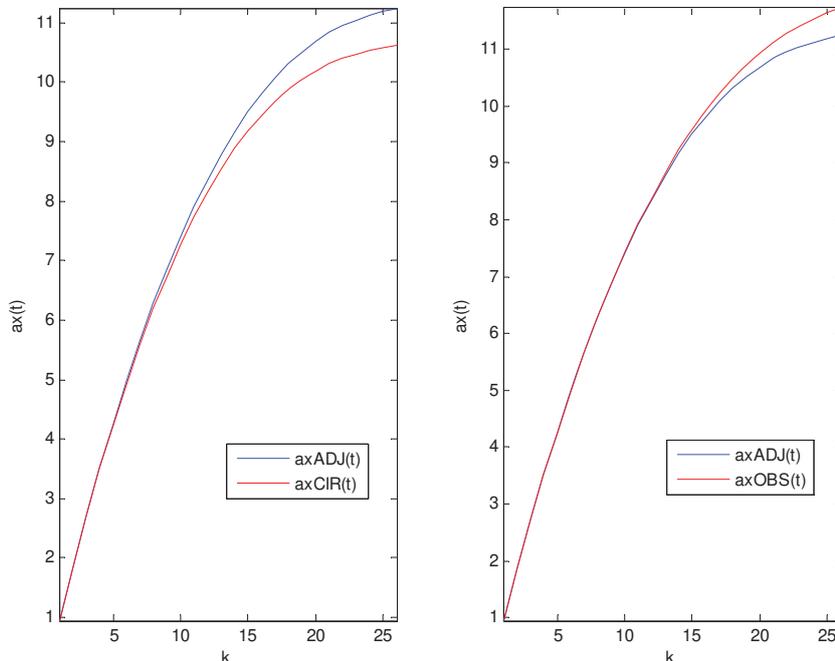$$a_x^{CIR}(t) = \sum_{k=1}^{\omega-x} v(t,k) i_{t+k}^{CIR}{}_k p_x^{CIR}(t), \qquad (15)$$

Obviously $a_x^{ADJ}(t) > a_x^{CIR}(t)$, since, other parameters being equal, $a_x^{ADJ}(t)$ is calculated using survival probabilities higher than $a_x^{CIR}(t)$. The difference $a_x^{ADJ}(t) - a_x^{CIR}(t)$ is equal to the present value of an annuity calculated on the basis of the adjusted probabilities, with duration equal to τ.

$$a_x^{ADJ}(t) - a_x^{CIR}(t) = \sum_{k=1}^{\tau} v(0,k)_k p_x^{ADJ}(t).$$

The previous can be rewritten as:

$$_{\tau/}a_x^{ADJ}(t) = a_x^{CIR}(t)$$

so that we can determine the value of τ realizing the identity. Here, the annuitant can choose to accept a deferral equal to tau in return for a loss of benefit equal at most to twenty percent. In our numerical example, the deferral period is approximately eight months.

In the second case the unique annuity premium $a_x^{OBS}(t)$ calculated using the observed survival probabilities referred to the calendar year t is given by:

$$a_x^{OBS}(t) = \sum_{k=1}^{\omega-x} v(t,k)_k \, p_x^{OBS}(t), \qquad (15)$$

Notes: $x = 64$; $t = 2008$; $k = 0,1,2,3.....,26$

Fig. 5 Comparison between: $a_x^{ADJ}(t)$ and $a_x^{CIR}(t)$ (subplot 1), $a_x^{ADJ}(t)$ and $a_x^{OBS}(t)$ (subplot 2)

Also in this case $a_x^{OBS}(t) > a_x^{ADJ}(t)$, where $a_x^{ADJ}(t)$ is given by (8). The difference $a_x^{OBS}(t) - a_x^{ADJ}(t)$, is equal to the present value of an annuity calculated on the basis of the observed probabilities, with duration equal to τ. At the end we obtain:

$$_{\tau/}a_x^{OBS}(t) = a_x^{ADJ}(t),$$

so that we can determine the value of τ realizing the identity. The annuitant can choose whether to accept a further deferral equal to tau nullifying the loss of benefit. In our numerical example, the deferral period is approximately six months.

It should be emphasized that deferring the annuity allows to reach the actuarial equilibrium instead of reducing benefits (or increase premiums).

It is stressed that the cases described in this section avoid any risk to the insurer. The reason can be found the premise of this analysis: the longevity index does not fall below 0.8.

Our contribution is to study the insured position in terms of the annuity attractiveness. He can choose between a loss of benefits (or an increase in the level of premium) or the annuity deferral that, on the basis of the previous considerations, is very short.

Concluding remarks

Although the annuity market is not well developed in western countries, the reduction of the intervention field of public systems and the gradual shift from defined benefit schemes to defined contribution systems, suggests both a growing interest of individuals for annuities and a considerable development of their market in coming years. The task of actuaries is to make this market more attractive than it is now. In fact, because of longevity, the risk borne out by insurers for insurance annuities, which is undoubtedly too high, is reflected in high premiums charged for these products that discourage individuals who are intending to purchase annuities. On the other hand, the idea of reducing the annuity periodic payments in similar way to what happens in the context of securitization could be reflected either in a significant reduction in the level of benefits for the annuitants or in a modest reduction of the risk for the insurer.

In this context, this paper looks for the conditions that allow to achieve an equilibrium between the reduction of the benefits and the annuity deferral. Based on past experience of the Italian population mortality measured in the period of 1954-2008, we find that a

modest deferment of the starting point of the annuity can balance the needs of the insurer and the insured. In any case, the choice of the decrease in performance over time and the deferral of annuity should encourage to buy this kind of contract.

Further research on this subject could be oriented in deepening the topic of stochastic interest rates. Moreover, we could also consider the choice of different mortality models in order to quantify the so called model risk.

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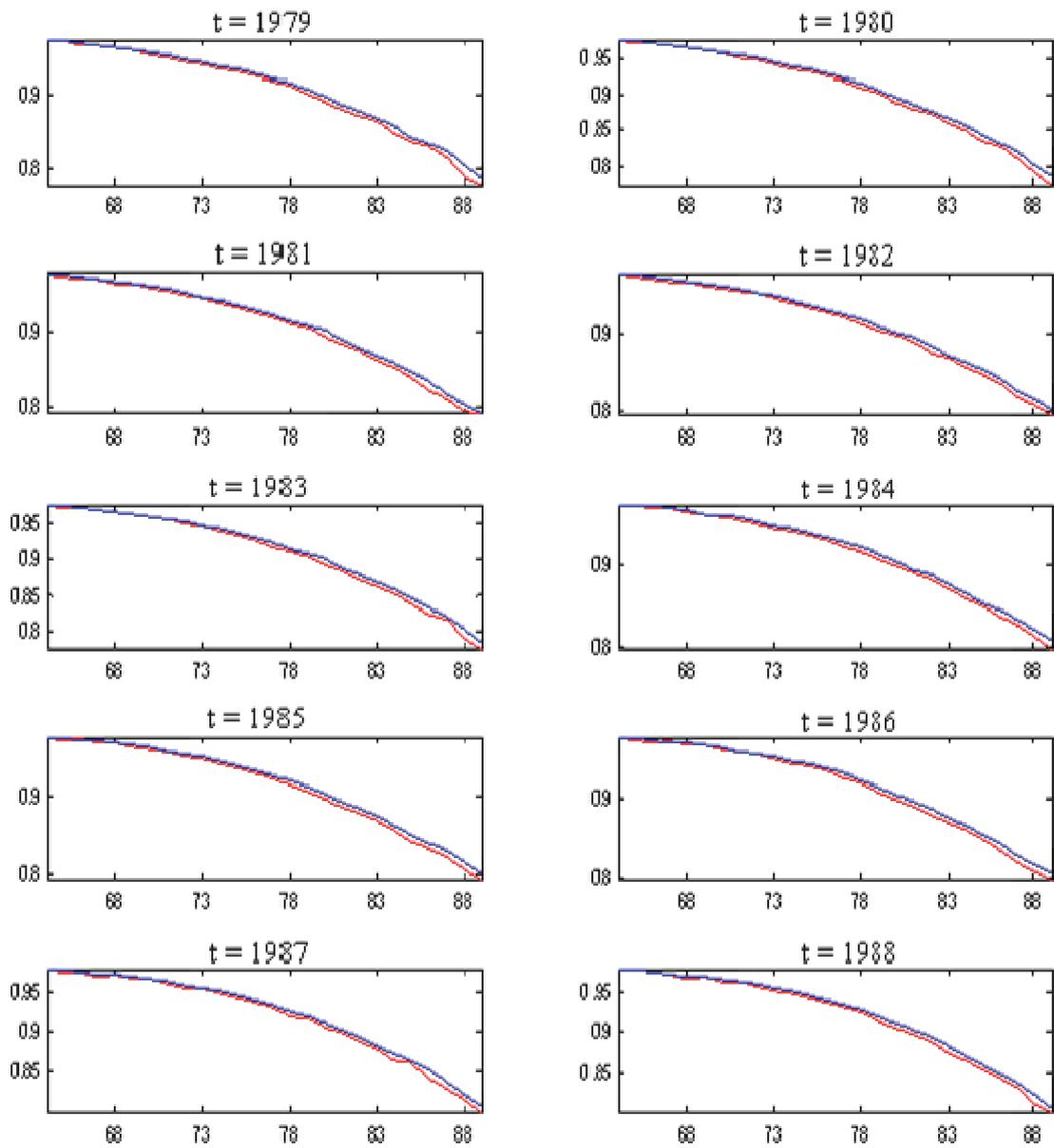
Appendix

Fig. 1a. Annual survival probabilities $Px(t)$ **with** $x \in (64,89)$ **for each calendar year** t **ranging from 1979 to 1988. Comparison between CIR model (black line) and real data (grey line)**
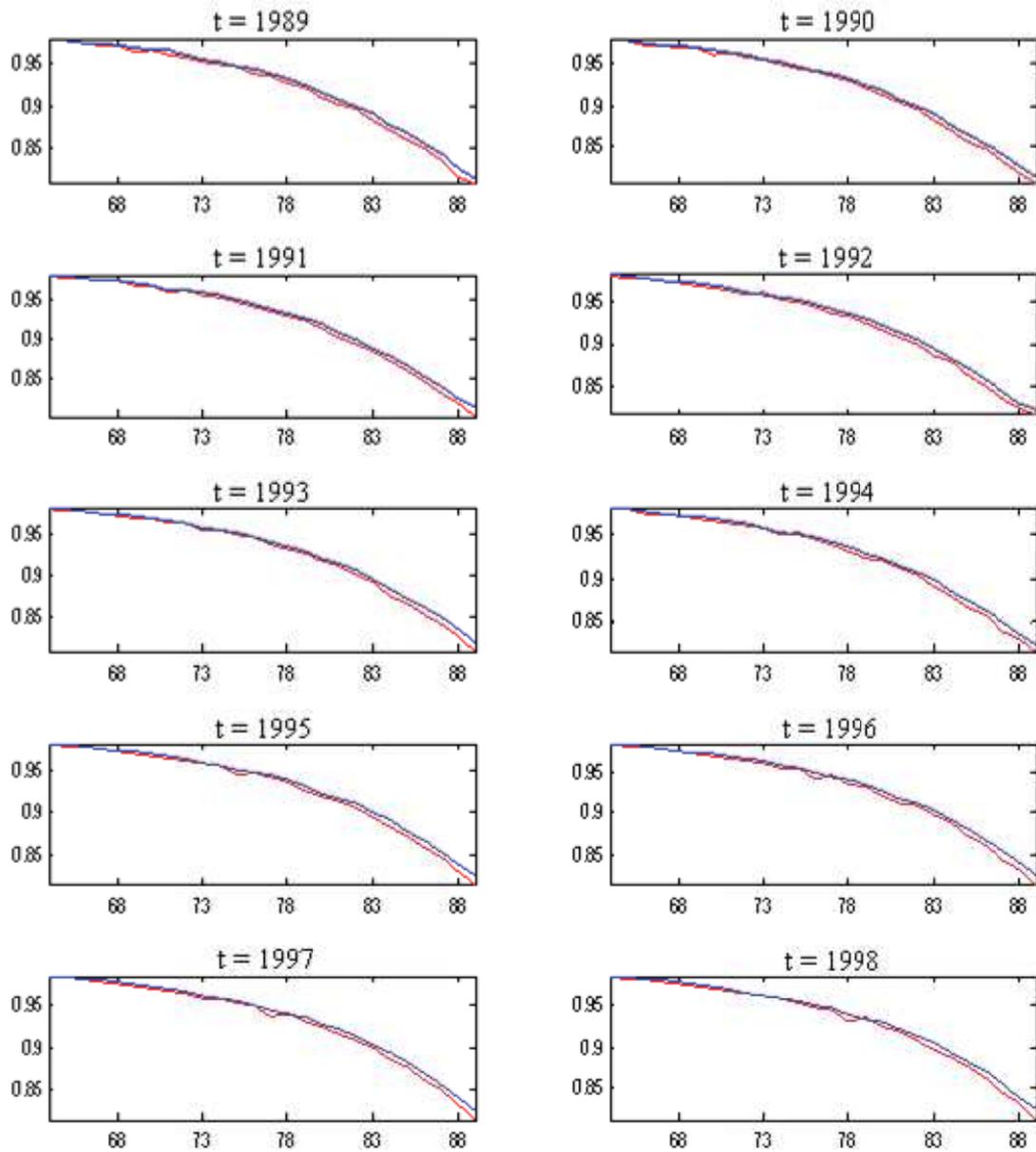
Fig. 2a. Annual survival probabilities *Px*(*t*) with *x* ∈ (64,89) for each calendar year *t* ranging from 1989 to 1998. Comparison between CIR model (black line) and real data (grey line)
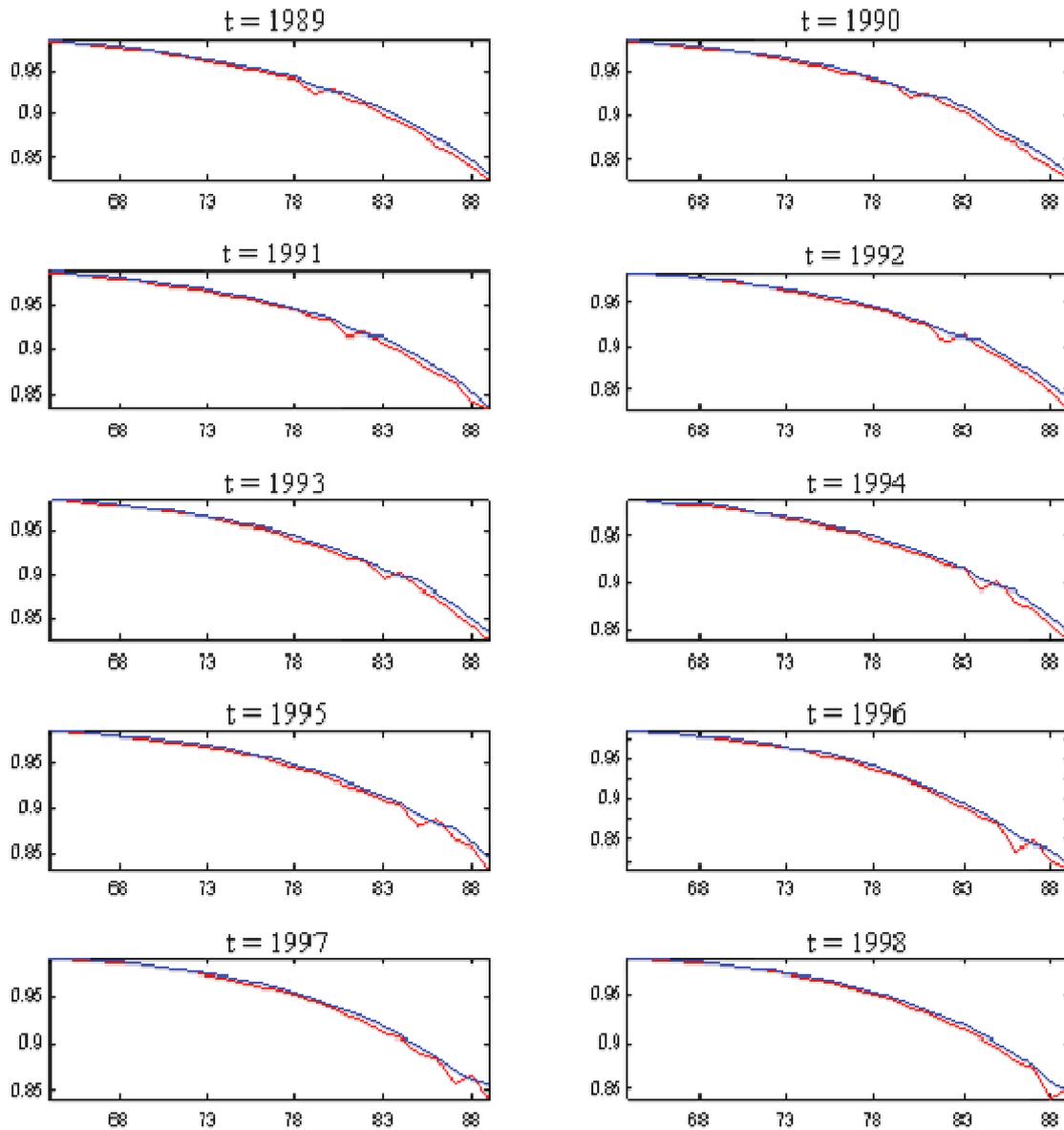
Fig. 3a. Annual survival probabilities $Px(t)$ **with** $x \in (64,89)$ **for each calendar year** t **ranging from 1999 to 2008. Comparison between CIR model (black line) and real data (grey line)**