“Feasible portfolios under tracking error, β, α and utility constraints”

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FEASIBLE PORTFOLIOS UNDER TRACKING ERROR, β, α AND UTILITY CONSTRAINTS

Abstract

The investment nous of active managers is judged on their ability to outperform specified benchmarks while complying with strict constraints on, for example, tracking errors, β and Value at Risk. Tracking error constraints give rise to a tracking error frontier – an ellipse in risk/return space which encloses theoretically possible (but not necessarily efficient) portfolios. The β frontier is a parabola in risk/return space and defines the threshold of portfolios subject to a specified β requirement. An α-TE frontier is similarly shaped: portfolios on this frontier have a specified TE for a maximum TE Utility and associated risk aversion have also been explored for constrained portfolios. This paper contributes by establishing the impossibility of satisfying more than two constraints simultaneously and explores the behavior of these constraints on the maximum risk-adjusted return portfolio (defined arbitrarily here as the optimal portfolio).

Keywords tracking error frontier, beta, alpha, utility, optimal portfolios

JEL Classification C52, G11

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INTRODUCTION

Active portfolio managers aim to outperform their benchmarks while adhering to constraints imposed by principals. One of the most commonly-used of these constraints is the tracking error (TE). Researchers also refer to this quantity as the tracking error volatility, but the phrase has largely fallen out of use amongst practitioners and became simply tracking error), the annualized standard deviation of the difference between the fund and benchmark returns (whether ex post or ex ante). Optimizing portfolio performance over a benchmark, while constrained to a TE, is non-trivial. Problems begin with the definition of optimal. Optimal portfolios have been variously defined as those constrained by a TE which: (1) outperform the benchmark by the greatest amount with no regard to portfolio volatility (Roll, 1992), (2) have the same volatility as the benchmark and highest excess return over the benchmark (Jorion, 2003), (3) have the highest risk-adjusted return (Maxwell, Daly, Thomson, & van Vuuren, 2017), and others.

Contemporary active managers are not only constrained by TE, others include portfolio α (Alexander & Baptista, 2010) and β (Roll, 1992; Betrand, 2009, 2010) from the capital asset pricing model (CAPM), Value at Risk (VaR) from a risk point of view (Palomba & Riccetti, 2013; Rodposhti & Sharareh, 2015), or utility, which combines risk and return (Stowe, 2014). These multiple restrictions are often incompatible. Increasing portfolio β, for example, decreases risk-adjusted re-

turns. Assembling portfolios which obey strict VaR requirements, as well as TE constraints, is often impossible. Despite these mutually exclusive objectives, active fund managers must comply with mandates bearing these impositions, indeed, their performance (and subsequent remuneration) is based on strict compliance with these mandates.

This paper traces the development of various frontiers and boundaries in risk/return (and sometimes in mean/variance) space, which define the limits of the active fund manager's investable universe. These limits characterize efficient portfolios in the sense that they establish maximal returns for given levels of risk, or β, α, VaR or other parameters, or combinations of these. Inevitably, the regions bordered by these limits shrink as constraints are added. Depending on the severity of the constraints, the potential universe of permitted investments is sometimes undefined. Navigating this narrow arena of possibilities and optimizing the returns generated from it is a complex task. The contribution of this paper is to assemble these frontiers and then populate the return/risk space within them, using the same small but stylized asset universe to demonstrate the consequences of the limitations.

This paper proceeds as follows. Section 1 discusses the relevant literature governing some of the constraints imposed on active managers, and traces the mathematical development which describes the plausible (and ever-diminishing) investment universe given the array of mandated constraints. The relevant mathematics is introduced, defined and contextualized in section 2. The data used are also described in this section. Section 3 presents the results and discusses the consequences of constraints on active portfolio optimization. Last section concludes.

1. LITERATURE REVIEW

The Markowitz framework, in a fund management context, establishes the relationship between expected portfolio returns and the variance of those returns given a universe of investable assets. This relationship gives rise to the well-known efficient frontier; the parabolic delineation in mean return/variance space. Active portfolio managers are, however, constrained by restrictions specified by the fund sponsors: poorly-assembled benchmarks (which are seldom Markowitz efficient), maximum tracking error (defined as the standard deviation of differences between the benchmark and the active portfolio's returns), minimum outperformance of the benchmark, active fund β, VaR, etc.

Active fund managers are commonly rewarded for generating expected returns (by outperforming mandated benchmarks) while simultaneously minimizing specified tracking errors. Roll (1992) called this the TEV (tracking error volatility) criterion and established conclusively that in attempting to satisfy it, fund managers intentionally do not produce mean/variance efficient Makowitz portfolios under all but the rarest of circumstances. Portfolios selected by active fund managers would always be dominated by other portfolios with higher average returns and lower volatilities although not lower tracking errors.

Roll (1992) formalized the problem of tracking error-constrained portfolios and established an elegant solution for the "TE frontier" (Figure 1), i.e. portfolios having a maximum total expected return possible for a given TE. Markers are placed at intervals of 1% in Figure 1, so the TE -constrained portfolio indicated represents the maximum excess return possible for a fund relative to its benchmark with a TE constraint of 4%, the point above and to the right of it, $TE = 5\%$, and so on.

Jorion (2003) augmented Roll's (1992) solutions by establishing the shape of constant TE portfolios, i.e. the locus of active portfolios with the same tracking error, being equidistant from the benchmark. Jorion (2003) established that this locus is an ellipse in mean/variance space, but not in the efficient frontier (μ/σ) plane, where μ represents the portfolio expected return and σ the active portfolio volatility (Figure 2). The shape of the constant TE frontier in (μ/σ) space is a distorted ellipse in which the bi-axial symmetry associated with ellipse is lost. "Ellipse" will be used here when referring to the shape in either space.

In Figure 2(a), the active manager's dilemma is evident: the portfolio subject to a tracking error constraint which also generates the maximum outperformance of the benchmark has higher risk than the benchmark. Because of the flat shape (referring to the generally shallow angle of the ellipse's long axis to the volatility axis) of the ellipse, Jorion (2003) suggested active managers invest in the portfolio indicated in Figure 2(b): portfolios with the same risk as the benchmark (on the constant TE frontier). The decrease in expected return (from the maximum expected return) is minimal again because of the ellipse's flat shape, the portfolio outperforms the benchmark and has the same risk as the benchmark. Jorion (2003) also found that this constraint improved managed portfolio performance, particularly those with lower TEs and less efficient benchmarks. For these portfolios, the information ratio (IR), given by:

$$IR\Big| = \frac{r_p - r_b}{TE}, \tag{1}$$

where r_p are the portfolio returns and r_b the benchmark returns, is not maximized.

Maxwell, Daly, Thomson, and van Vuuren (2017) further explored portfolio optimization under TE constraints and set forth arguments in favor of maximizing the risk-adjusted expected returns (i.e. the maximum Sharpe ratio) on the constant TE frontier. Depending on the risk-free rate or return, this portfolio can lie to the left or right of Jorion's (2003) suggestion. In the current (2017) low interest rate environment, Maxwell et al.'s (2017) active portfolios lie to the left of Jorion's so these portfolios have a higher expected return and lower risk than the benchmark, the highest risk-adjusted rate of expected return and they satisfy the TE constraint. These portfolios have lower expected returns than Jorion's (whose returns are, in turn, lower than the maximum expected return), but again the flat shape of the ellipse means that the portfolios' other credentials more than compensate for this decrease.

Roll (1992) found that all actively-managed portfolios (under the TE constraint) with positive expected performance have $\beta > 1$, while portfolios that have higher expected returns and lower total volatility have $\beta < 1$. Roll (1992) generated TE frontiers with a β constraint and proved that it is impossible to produce a portfolio that is simultaneously constrained by a TE, a given expected performance and a specified β.

Bertrand (2010) allowed the tracking error to vary, but fixed the investor's level of risk aversion, thereby generating what he called aversion frontiers. Betrand's (2010) $\beta = 1$ is aversion frontier coincided with Roll's (1992) $\beta = 1$ frontier and found that to take advantage of an expected rise in the market (i.e. have a $\beta > 1$), the constructed portfolio must be assembled in the context of its aversion frontiers, not constant tracking error frontiers. While these conclusions are compelling, the clear majority (if not all) actively managed funds have a mandated tracking error, not a mandated aversion level, so Bertrand's (2010) work is moot.

Stowe (2014) noted that the conventional practices of β constraints, studied in Roll (1992), and TE volatility constraints, studied in Jorion (2003), assure utility improvements for the investor. If these constraints are sensibly implemented, the fund manager will be forced to manage a portfolio which is more efficient than the benchmark. Stowe's (2014) principal contribution was to establish the conditions under which fund managers could increase portfolio utility and found that the β constraint always has the potential to increase utility, while the tracking error constraint (which may increase utility) always lies below the constrained β frontier.

Alexander and Baptista (2010) devised a solution for determining the α-TE frontier i.e. that frontier which exhibits the minimum tracking error for various levels of ex ante α. The authors showed that sensible choices of ex ante α lead to the selection of less risky portfolios than active fund managers may otherwise select.

2. DATA AND METHODOLOGY

2.1. Data

The data comprised simulated realistic weights, returns, volatilities and correlations for a small benchmark comprising three assets. Portfolio constituents were derived only from the benchmark universe (including short-selling of bench-

mark components). We followed the example of Stowe (2014) who chose a simple simulated portfolio comprising four assets, and the descriptive statistics for which were chosen somewhat arbitrarily, but mainly for ease of exposition. Like Stowe (2014), we believe these examples are representative of a realistic scenario. The relevant inputs are provided in the Appendix.

Note that the "assets" which constitute the portfolio could be asset classes (such as equity, bonds or cash) or specific industry sectors within an asset class (e.g. an industrial equity index, a banking and finance index, etc.) or individual assets such as single name stocks or bonds.

Karp and van Vuuren (2017) describe variations in excess portfolio returns in an emerging market environment using a Fama-French three-factor model. These excess returns helped identify and describe relevant, realistic excess portfolio returns for the generation of simulated data. The Fama and French (2015) five-factor model approach was also used to further refine these estimates.

2.2. Methodology

To establish the methodologies required for the various frontiers, some definitions are necessary. These are recreated below in line with the notation developed by Roll (1992) and perpetuated by Jorion (2003).

Fund managers, tasked with outperforming benchmarks, must take positions in assets which may or may not be components of the benchmark (depending on the fund's mandate). The following definitions will be used throughout this paper.

q_b: vector of benchmark weights for a sample of N assets; x: vector of deviations from the benchmark; $q_p = (q_b + x)$: vector of portfolio weights; E: vector of expected returns; and V: covariance matrix of asset returns.

Net short sales are allowed, so the total active weight $q_i + x_i$ may be negative for any individual asset i. The universe of assets can generally exceed the components of the benchmark, but for Roll's (1992) methodology, assets in the benchmark must be included.

Expected returns and variances are expressed in matrix notation as:

$\mu_b = q_b' E$: expected benchmark return;

$\sigma_b^2 = q_b' V q_b$: variance of benchmark return;

$\mu_\varepsilon = x' E$: expected excess return;

$G = r_p - r_b$: gain, the fund manager's target or expected performance relative to the benchmark;

$\sigma_\varepsilon^2 = x' V x$: TE variance (defined as TE^2) and

$\beta = \dfrac{q_p' V q_b}{\sigma_b^2}$: sponsor-specified level of market risk (relative to the benchmark).

The active portfolio's expected return and variance are given by:

$$\mu_p \Big| = (q_b + x)' E = \mu_b + \mu_\varepsilon, \tag{2}$$

$$\begin{aligned} \sigma_p^2 &= (q_b + x)' V (q_b + x) = \\ &= \sigma_b^2 + 2 q_b' V x + x' V x = \sigma_b^2 + 2 q_b' V x + \sigma_\varepsilon^2. \end{aligned} \tag{3}$$

The portfolio must be fully invested, so:

$$(q_b + x)' 1 = 1, \tag{4}$$

where 1 represents an N- dimensional vector of 1s.

Using Merton's (1972) terminology, the following parameters are also defined:

$a = E' V^{-1} E$, $b = E' V^{-1} 1$, $c = 1' V^{-1} 1$, and

$$d = \frac{a - b^2}{c}, \tag{5}$$

$$\Delta_1 = \mu_B - \frac{b}{c}, \tag{6}$$

where $\dfrac{b}{c} = \mu_{MV}$ and

$$\Delta_2 = \sigma_B^2 - \frac{1}{c}, \tag{7}$$

where $\dfrac{1}{c} = \sigma_{MV}^2$.

Roll (1992) showed that the three parameters a, b and c are related to the means and variances of two important portfolios on the efficient fron-

Table 1. Properties of portfolios 0 and 1 in terms of a, b and c

Portfolio	Mean	Variance	Weights
P_0	$E_0 = b/c$	$\sigma_0^2 = 1/c$	$q_0 = V^{-1}1/c$
P_1	$E_1 = a/b$	$\sigma_1^2 = a/b^2$	$q_1 = V^{-1}E/b$

tier (denoted P_0 and P_1). The first, portfolio P_0, is the global minimum variance portfolio and the second, portfolio P_1, is located where a line drawn from the origin passes through the global minimum variance portfolio and intersects the efficient frontier. Both are shown in Figure 1. These portfolios have the properties indicated in Table 1.

2.2.1. *Tracking error frontier*

The tracking error frontier is generated by maximizing $x'E$ subject to $x'1 = 1$ and $x'Vx = \sigma_\varepsilon^2$.

The solution for the vector of deviations from the benchmark x is:

$$x = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}}V^{-1}\cdot\left(E - \frac{b}{c}1\right). \tag{8}$$

2.2.2. Constant TE frontier

To generate the constant tracking error frontier:

Maximize $x'E$ subject to $x'1 = 0$, $x'Vx = \sigma_\varepsilon^2$ and $(q_b + x)'V(q_b + x) = \sigma_p^2$.

The solution for the vector of deviations from the benchmark x is:

$$x = -\frac{1}{\lambda_2 + \lambda_3}V^{-1}\left(E + \lambda_1 + \lambda_3 V q_b\right), \tag{9}$$

where

$$\lambda_1 = -\frac{\lambda_3 + b}{c}, \tag{10}$$

$$\lambda_2 = \pm(-2)\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2\Delta_2 - y^2}} - \lambda_3, \tag{11}$$

Source: Authors.

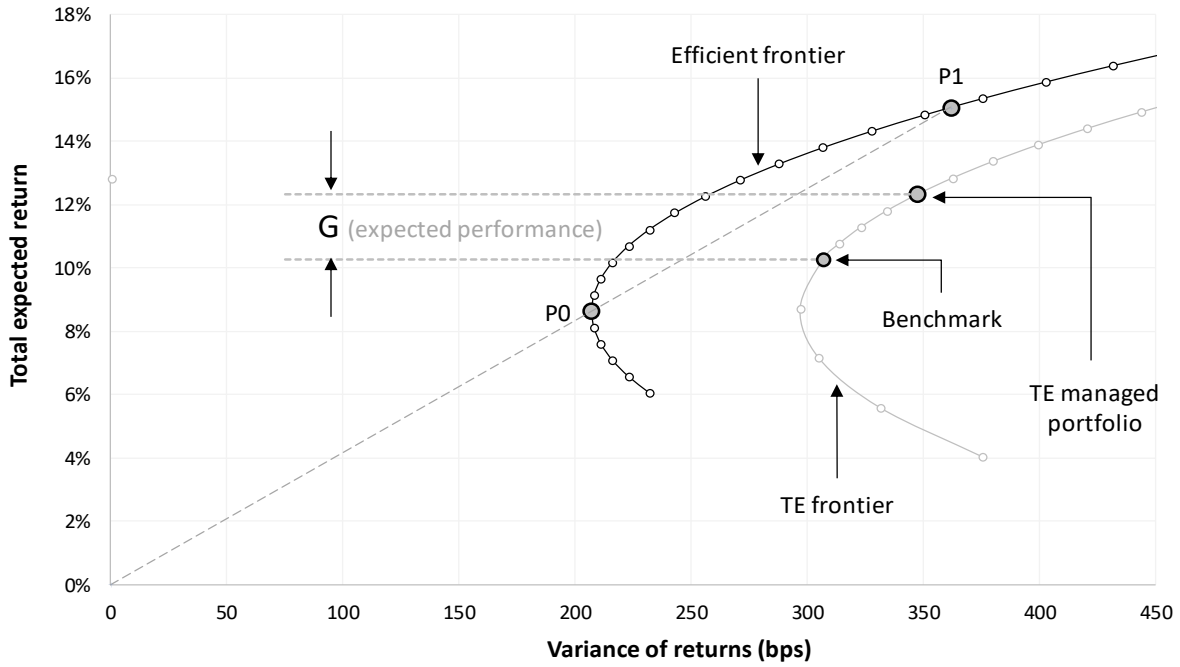

Figure 1. Positions of portfolios P_0 and P_1 on the efficient frontier and the gain $G = r_p - r_b$, the fund manager's outperformance target

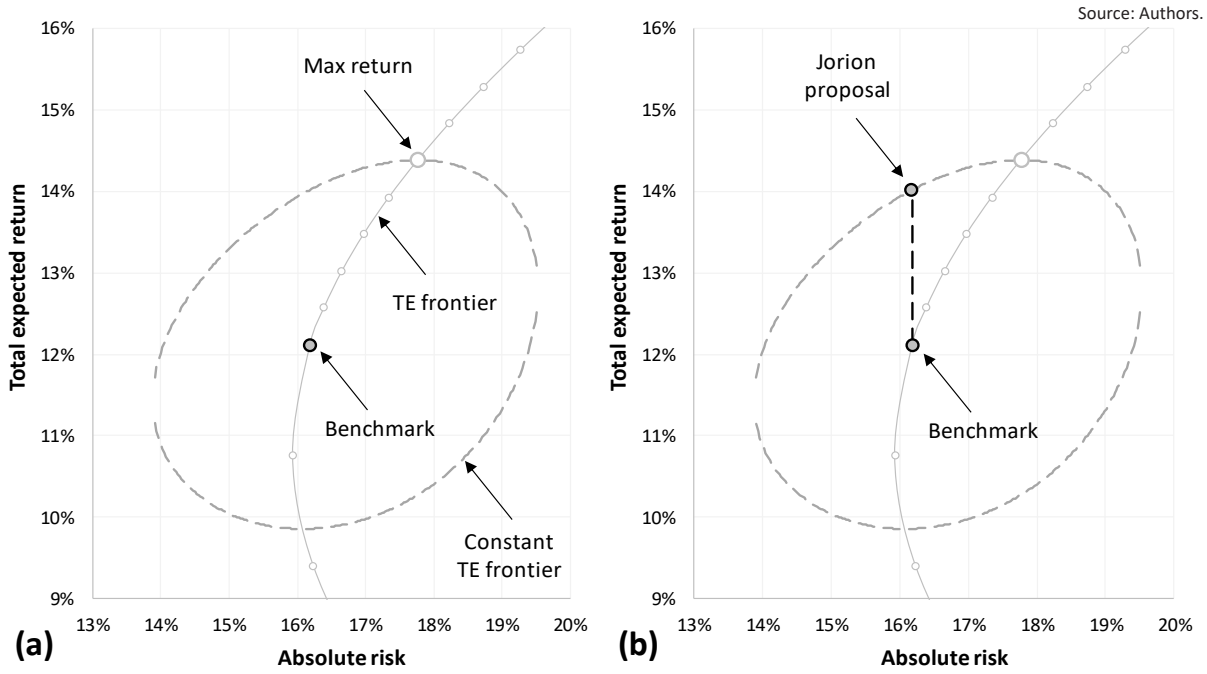

Figure 2. TE frontier, TE -constrained portfolio and constant TE frontier (with $TE = 5\%$). (a) shows the naïve portfolio: excess return is maximised for a given TE constraint. (b) shows Jorion's (2003) suggestion: observe constraints from (a), but restrict portfolio risk to that of the benchmark

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2\Delta_2 - y^2}}. \quad (12)$$

Jorion (2003) defined

$$z = \mu_p - \mu_b, \quad (13)$$

and

$$y = \sigma_p^2 - \sigma_b^2 - \sigma_\varepsilon^2, \quad (14)$$

and established that the relationship between y and z is:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4\sigma_\varepsilon^2\left(d\Delta_2 - \Delta_1^2\right) = 0, \quad (15)$$

which is a quadratic equation in both y and z.

Solving for z gives:

$$z\Big| = \frac{\Delta_1 y \pm \sqrt{\left(\Delta_1^2 - d\Delta_2\right)\cdot\left(y^2 - 4\Delta_2\sigma_\varepsilon^2\right)}}{2\Delta_2}, \quad (16)$$

which describes an ellipse – a constant TE frontier – in return/variance space (and a distorted ellipse in risk/return space – Figure 2), once the definitions of z (13) and y (14) have been reinstalled.

In Figure 2, each point on the ellipse represents a portfolio with $TE = 5\%$. The point on the ellipse corresponding to the largest outperformance of the portfolio over the benchmark is common to both the TE frontier and the constant TE frontier. Managers attempting to maximize excess return need to move up and to the right of the benchmark in the μ/σ plane, so the portfolio will always exhibit higher risk than that of the benchmark. This led Jorion (2003) to propose a constraint on total risk. Jorion (2003) suggested that the portfolio risk could be constrained to equal that of the benchmark (i.e. that $\sigma_p = \sigma_b$), which implies that $2q'Vx = -\sigma_\varepsilon^2$.

2.2.3. Constant β frontier

Assume a fund manager is mandated to assemble a portfolio P which minimizes the tracking error, generates an expected outperformance (or gain) G and maintains a specified β against the benchmark portfolio b. This optimization problem can be expressed as:

Minimize $x'Vx$ subject to $x'1=0$, $x'E=G$ and $q_p'Vq_b=\beta\cdot\sigma_b^2$.

The final constraint may be rearranged and written as:

$$\left(q_b-x\right)Vq_b\Big|=\beta\cdot\sigma_b^2, \tag{17}$$

$$x'Vq_b\Big|=\sigma_b^2\left(\beta-1\right). \tag{18}$$

Using Lagrange multipliers, the solution for the weights x, of the relevant portfolio which satisfies the above constraints is $x=\gamma_1q_1+\gamma_0q_0+\gamma_bq_b$ where:

$$\gamma_1=\frac{G\left(\sigma_b^2-\sigma_0^2\right)+\sigma_b^2\left(\beta-1\right)\left(E_0-\mu_b\right)}{\left(E_1-E_0\right)\left(\sigma_b^2-\sigma_{b^*}^2\right)}, \tag{19}$$

$$\gamma_0=\frac{G\left(\frac{\mu_b}{b-\sigma_b^2}\right)+\sigma_b^2\left(\beta-1\right)\left(\mu_b-E_1\right)}{\left(E_1-E_0\right)\left(\sigma_b^2-\sigma_{b^*}^2\right)}, \tag{20}$$

$$\gamma_b=\frac{G\left(\frac{\sigma_0^2-\mu_b}{b}\right)+\sigma_b^2\left(\beta-1\right)\left(E_1-E_0\right)}{\left(E_1-E_0\right)\left(\sigma_b^2-\sigma_{b^*}^2\right)}. \tag{21}$$

Roll (1992) also established that $G=\gamma_1E_1+\gamma_0E_0+\gamma_b\mu_b$ and $x'Vq_b=\sigma_b^2\left(\beta-1\right)$ as required.

2.2.4. The α-*TE* frontier

A portfolio is deemed to be on the α-*TE* frontier if there is no portfolio with the same α and a smaller *TE*. The methodology to generate this frontier involves first calculating three useful parameters:

$$k_1=\frac{b-\left(\frac{\mu_b-r_f}{\sigma_b^2}\right)}{c}, \tag{22}$$

$$k_2=a+\left(\frac{r_f^2-\mu_b}{\sigma_b^2}\right)-\frac{\left[b-\left(\frac{\mu_b-r_f}{\sigma_b^2}\right)\right]^2}{c}, \tag{23}$$

$$k_3=\frac{d}{c}+\left(\frac{b}{c}-\mu_b\right)\cdot\left(\frac{\mu_b-r_f}{\sigma_b^2}\right), \tag{24}$$

where r_f is the risk-free rate and the other symbols have been defined previously.

Define

$$\gamma_{0,\alpha}=\frac{-\alpha ck_1}{k_2}, \tag{25}$$

$$\gamma_{1,\alpha}=\frac{\alpha b}{k_2}, \tag{26}$$

then the vector of portfolio weights on the α-*TE* frontier, q_α, are generated from:

$$q_\alpha=\gamma_{0,\alpha}q_0+\gamma_{1,\alpha}q_1+\left(1-\gamma_{0,\alpha}-\gamma_{1,\alpha}\right)q_b, \tag{27}$$

where q_0 and q_1 retain the definitions established in Table 1.

2.2.5. *Fund utility*

Stowe (2014, 2017) used the popular "quadratic-style" utility function to find the portfolio, constrained by a *TE*, which maximized the investor's utility which increases with expected return $q_p'E$, and decreases with risk $\sqrt{q_p'Vq_p}$. The relationship is:

$$U=\mu_p-\theta\sigma_p^2, \tag{28}$$

where U is the utility function, μ_p the fund's expected return, σ_p^2 the variance of the portfolio's return and θ the coefficient of risk aversion (Stowe, 2014, 2017). The maximization setup is:

$$\max_w\ q_p'E-\theta q_p'Vq_p, \tag{29}$$

for which Stowe (2014, 2017) provides elegant derivations for the optimal portfolio. We use his results, but do not find the optimal utility portfolio on the constant *TE* frontier. Instead, we investigate the utility around the portfolio which lies on the constant *TE* frontier and has maximal risk-adjusted return, i.e. the maximum Sharpe ratio portfolio constrained to the constant *TE* frontier. This augments and extends our work on this portfolio under tracking error constraints (Maxwell et al., 2017).

Differentiating (28) gives:

$$\frac{dU}{d\sigma_p}=2\sigma_p\theta, \tag{30}$$

and where this slope takes the value of the maximum Sharpe ratio, the investor's utility for the maximum Sharpe ratio portfolio is determined. Maxwell et al. (2017) showed that solving for μ_p

in (31) gives the coordinates of the maximum Sharpe ratio portfolio in mean/standard deviation space (from whence it is trivial to determine the maximum Sharpe ratio, knowing r_f:

$$\left.\frac{\left(r_f-\mu_b\right)}{\sigma_p^2}\right| + \frac{\dfrac{\left(\Delta_1^2-d\Delta_2\right)\cdot\left(\sigma_p^2-\sigma_b^2-\sigma_\varepsilon^2\right)}{\sqrt{\left(\Delta_1^2-d\Delta_2\right)\left[\left(\sigma_p^2-\sigma_b^2-\sigma_\varepsilon^2\right)^2-4\Delta_2\sigma_\varepsilon^2\right]}}+\Delta_1}{\Delta_2} \quad (31)$$

$$\left.-\frac{\sqrt{\left(\Delta_1^2-d\Delta_2\right)\left[\left(\sigma_p^2-\sigma_b^2-\sigma_\varepsilon^2\right)^2-4\Delta_2\sigma_\varepsilon^2\right]}+\Delta_1\cdot\left(\sigma_p^2-\sigma_b^2-\sigma_\varepsilon^2\right)}{2\Delta_2\sigma_p^2}\right| = 0,$$

with $\Delta_1 = \mu_B - \frac{b}{c}$ $\left(\frac{b}{c} = \mu_{MV}\right)$

and $\Delta_2 = \sigma_B^2 - \frac{1}{c}$ $\left(\frac{1}{c} = \sigma_{MV}^2\right)$

as defined by Jorion (2003).

Equating the maximum Sharpe ratio and (30) gives:

$$\left.\frac{dU}{d\sigma_p} = 2\sigma_p^2\theta_S\right| = \text{Maximum Sharpe ratio}, \quad (32)$$

so:

$$\left.\theta_S\right| = \frac{\text{Maximum Sharpe ratio}}{2\sigma_p}, \quad (33)$$

where θ_S is the risk aversion coefficient tangent to the constant TE frontier at the maximum Sharpe ratio portfolio.

3. RESULTS AND DISCUSSION

Armed with the mathematics and methodologies detailed in the previous section, the various frontiers were constructed and a small portfolio comprising three stylized assets used to investigate the range of possible portfolios, given various mandated constraints.

3.1. The constant tracking error frontier

Roll (1992) developed the tracking error frontier (Figure 1). Jorion (2003) first described the constant TE frontier (Figure 2), the boundary whose upper edge defines the maximum return possible, at various risk levels, for a given fixed TE. This region, an ellipse in mean/variance space, led to the proposition that a portfolio with the same risk as the benchmark on this frontier would perform better than the naïve 'maximum return' portfolio (whose risk is greater than that of the benchmark).

3.2. The β frontier

Adding a β constraint generates a β frontier on which all portfolios have the same CAPM β. Portfolios not geared to the market (in this case, it is assumed that the "market" is the benchmark) have $\beta = 1$: this frontier necessarily passes through the benchmark (Figure 3a). For $\beta < 1$, the frontier moves to the left, so portfolios with lower risk yet higher returns than the benchmark, in bear markets, are possible. For $\beta > 1$, the frontier moves to the right in risk/return space, so only portfolios with higher volatility than the benchmark are feasible. This result was also found by Roll (1992).

For all values of β, only one sensible intersection with the constant TE frontier exists (i.e. on the upper half of the ellipse). This point may not be the maximum return possible (given the TE constraint) nor the maximum risk-adjusted return – the crux is that this intersection point may not be "optimal" in any sense. Active fund managers with portfolios subject to this combination of constraints (TE and β) and wishing to maximize excess returns are confined to a single point in risk/return space: a highly restrictive arrangement.

Figure 3b shows the range of possible β values for various levels of tracking error. This range is effectively the values of β at which the β frontier tangentially intersects the constant TE frontier at either end of the ellipse. Either end represents, respectively, the active fund's minimum and maximum volatility, so it is doubtful active fund managers would be interested in these extreme portfolios in the first place. Thus, the range of possible βs is lower than the stylized values in Figure 3b.

3.3. The α-TE frontier

Figure 4 shows the α-TE frontier, that frontier which exhibits the minimum tracking error for various levels of ex ante CAPM α. These á s are

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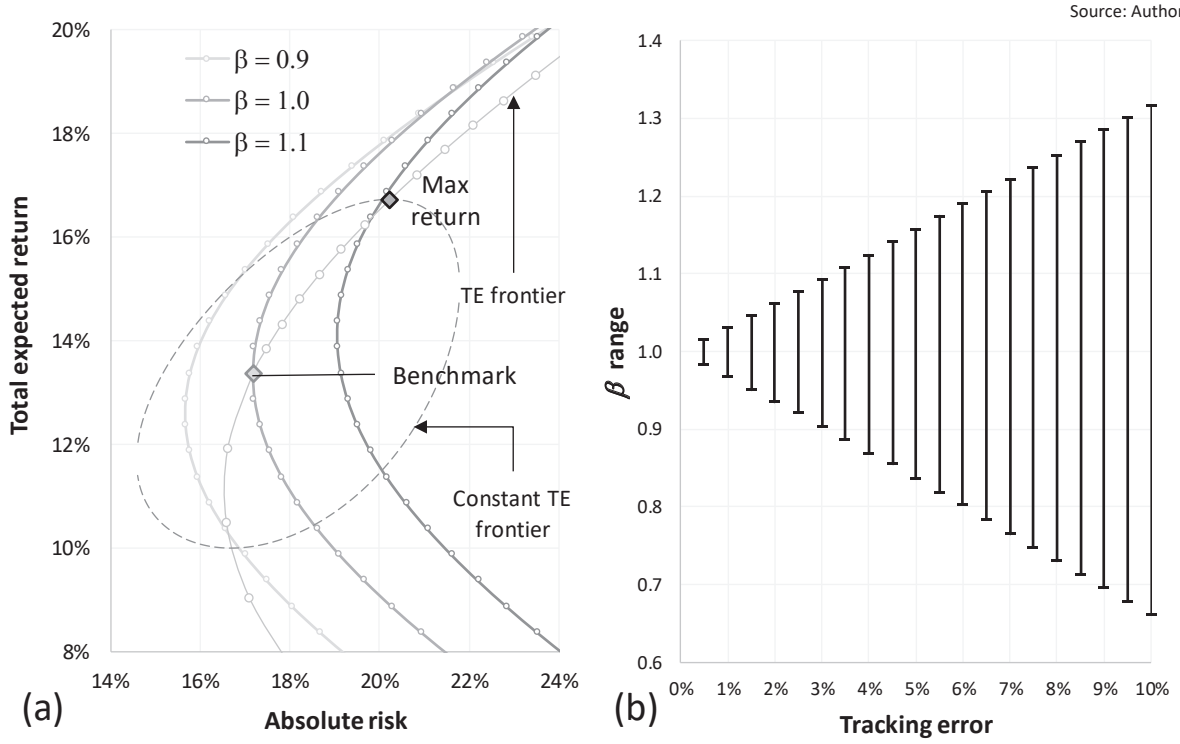

Figure 3. (a) Position of β frontier for $\beta = 0.9, 1.0$ and 1.1 and (b) maximum and minimum β values for changing tracking errors

indicated on the frontier. The elements of Figure 4 were generated using stylized asset parameters. It is clear that feasible, active portfolios, which simultaneously satisfy α and prescribed TE constraints, as well as maximize excess returns, are impossible (except in the case of the vanishingly small probability that the á constraint coincides with the intersection on the TE frontier). Combining α, β and TE constraints is impossible as shown in Figure 4. Note, however, that using the α -frontier leads to the selection of less risky portfolios than managers might otherwise select. In Figure 4, for example, the (naïve) maximum return portfolio is considerably riskier than the intersection of the α -frontier and the constrained TE frontier.

Note also that the $\alpha = 0\%$ portfolio is necessarily coincident with the benchmark.

3.4. Utility constraints

While variance minimization is an approach to analyzing portfolio selection, maximizing investor utility is another. The parameter θ, the coefficient of risk aversion, measures the sensitivity or the trade-off. Stowe (2014) found that the weights for the portfolio which maximized utility under tracking error constraints were given by:

$$w = \frac{1}{2\theta} V^{-1} \left(E - \frac{EV^{-1}1}{1'V^{-1}1} 1 \right) + q_b. \qquad (34)$$

We did not use Stowe's (2014) maximum utility portfolio. Instead, we determined the utility function which is tangent to the constant TE frontier at the maximum Sharpe ratio portfolio (Figure 5a) using (32) and backed out the appropriate risk aversion coefficient.

Figure 5b shows the θ surface as a function of r_f and TE. While θ decreases for increasing r_f (understandable because, all else equal, the slope of the maximum Sharpe ratio decreases with increasing r_f), a clear maximum θ exists at a certain TE, for all r_f. It is not obvious why this should be.

Figure 6a further explores θ as a function of TE. A maximum θ occurs, for this stylized example, at about $TE = 7\%$. Maxwell et al. (2017) established that the maximum Sharpe ratio also increases with TE before flattening off and de-

Source: Authors.

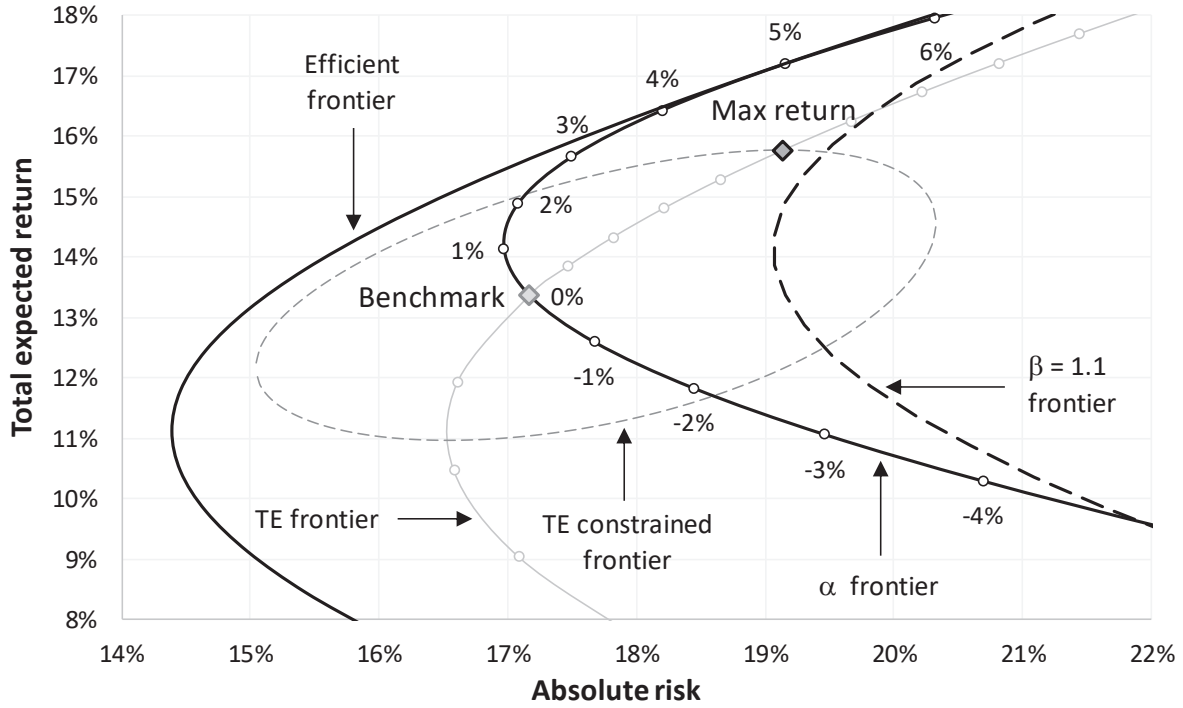

Figure 4. The α-TE frontier for various levels of α. Other frontiers are shown for comparison. Levels of α are indicated on the graph. $TE = 5\%,\ r_f = 2\%$

Source: Authors.

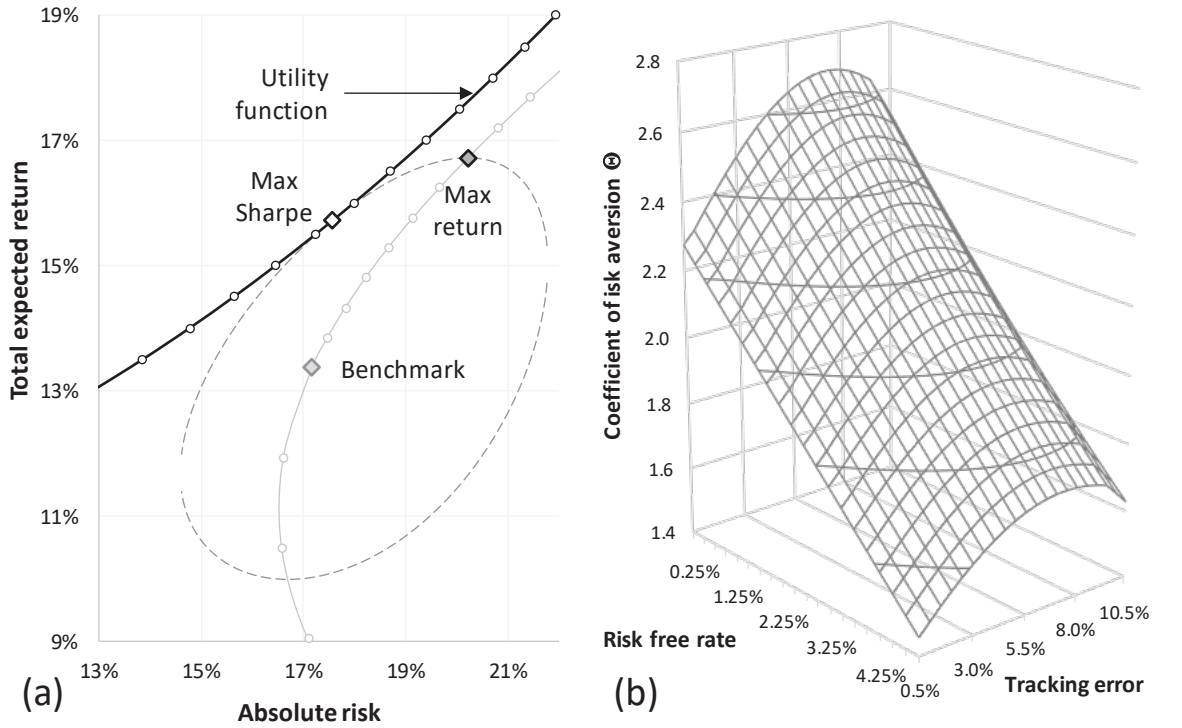

Figure 5. (a) Utility function tangential to the maximum Sharpe ratio portfolio on the constant TE frontier and (b) θ as a function of tracking error and risk-free rate

Source: Authors.

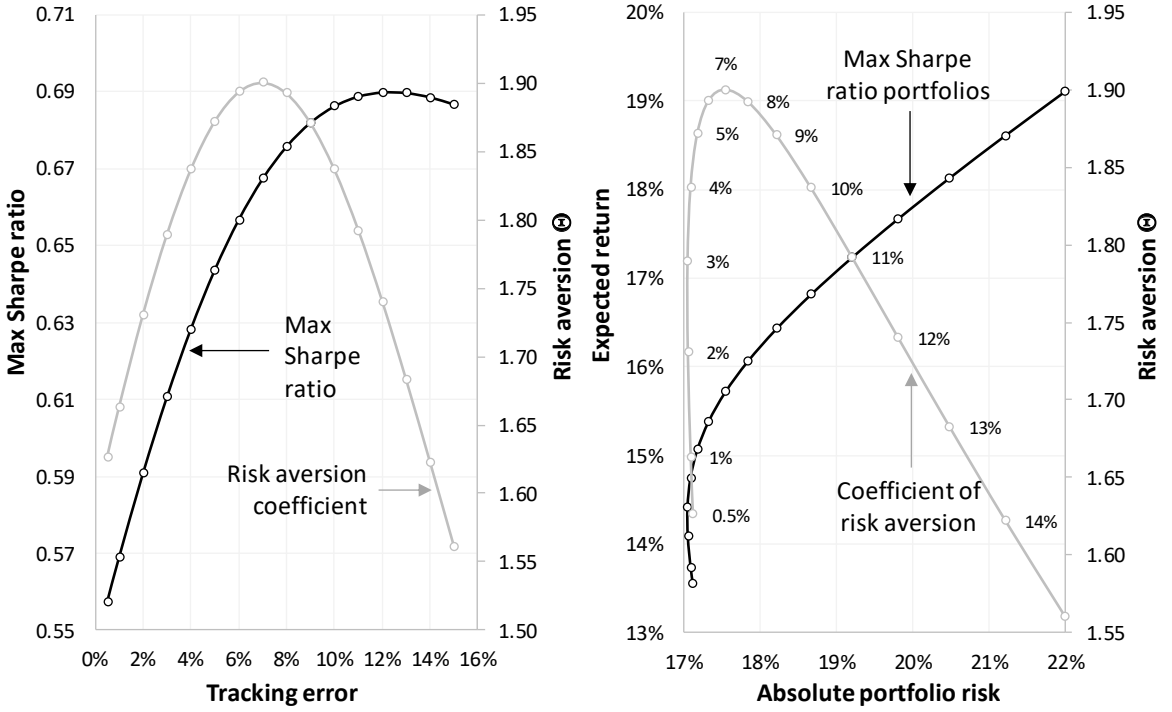

Figure 6. (a) Maximum Sharpe ratio as a function of tracking error and risk-free rate and (b) utility function (risk aversion) as a function of tracking error and portfolio risk

creasing slightly for large TEs $(>12\%)$. Thus, a portfolio exists for which the Sharpe ratio itself is maximized. The portfolios characterized by a maximum θ and a maximum Sharpe ratio are not the same, thus a maximum Sharpe ratio does not describe the reason for the maximum θ.

The locus of the maximum Sharpe ratio portfolio, as TE increases, moves up (increased return) and to the left (decreased risk) of the benchmark. At higher levels of TE, the portfolio continues to move up, but then moves to the right, i.e. absolute risk increases. At some value of TE, the maximum Sharpe ratio portfolio will be coincident upon Jorion's (2003) proposal (i.e. where $\sigma_b = \sigma_p$) and for higher TE values, the portfolio will continue to move up and right eventually coincident with the maximum return portfolio (coincident with the TE frontier and ellipse) in the clockwise direction (see Maxwell et al., 2017). Investor risk aversion increases dramatically with increasing TE (labels indicated on the graph) and increasing σ_p – the portfolio risk of the Sharpe ratio portfolio at that level of TE (Figure 6b).

CONCLUSION AND SUGGESTIONS

Portfolios subject to TE constraints are always sub-optimal to those that are not. Furthermore, as the TE constraint is in excess return space and relative to an investor or a somewhat arbitrarily defined benchmark (in general), there is potential for greater inefficiency should fund managers naïvely pursue maximum excess returns as a sole investment objective. As a TE is taken as a given in portfolio management, optimization under such constraint is of great interest to practitioners.

Before addressing the selection of an optimal portfolio under TE, α and β constraints and what that means in terms of maximizing excess and risk-adjusted returns, and the subsequent impact upon investor utility, it is important to note what is meant by an optimal portfolio. An optimal portfolio is one

which maximizes investor's utility, not necessarily that which minimizes risk, maximizes returns, nor that which maximizes risk-adjusted returns. Therefore, an optimal portfolio for one investor may not be the same as the optimal portfolio for another investor. Unfortunately, to assign a number to utility, one looks at risk aversion (or risk tolerance), but this does not encompass the full utility an investor derives from an investment (whether that be avoiding large losses or realizing large gains). There is a subjective component of utility, or investor satisfaction, which cannot be captured numerically. Furthermore, investors are not always aware what provides the most utility/satisfaction from an investment. Although the peak of the utility curve and the peak of the maximum Sharpe curve do not coincide, and the maximum risk-adjusted portfolio is not identified as the optimal portfolio (in utility terms), it could be argued that it should be. The understanding of optimality, risk and utility (and their interaction) requires re-adjustment. As an investor's utility cannot be fully captured, it cannot be concluded that the maximum Sharpe portfolio is not the optimal portfolio as it is the maximum risk-adjusted return portfolio and will provide the investor with far more utility than a more risk-averse portfolio. Therefore, as they are generally uninformed, investors should attempt to maximize risk-adjusted returns, rather than avoid risk.

Another important discussion is that between risk and volatility. A high volatility portfolio is not necessarily riskier than a lower volatility portfolio, as the volatility estimate employs both positive and negative returns. A high volatility portfolio could comprise of predominantly positive ("good") returns and a lower volatility portfolio could comprise of predominantly negative ("bad") returns. Portfolio risk, on the other hand, is the risk of losing capital. In the above example, the lower volatility portfolio contributes more risk than the higher volatility portfolio. Reducing portfolio risk would thus reduce the potential capital losses, and metrics such as VaR, conditional VaR and downside deviation should therefore be optimized. One could adjust the Sharpe ratio and attempt to maximize the Sortino ratio instead; this may provide substantially more utility to the risk-averse investor.

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APPENDIX

Correlation matrix:

1	+0.09	+0.16
	1	+0.12
		1

Volatility vector:	28%	25%	18%
Benchmark weights:	50%	22%	28%
Annualized *r*:	15%	19%	6%