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ARTICLE INFO	Fabio Lamantia, Sergio Ortobelli and Svetlozar Rachev (2006). VaR, CVaR and Time Rules with Elliptical and Asymmetric Stable Distributed Returns. *Investment Management and Financial Innovations*, *3*(4)
RELEASED ON	Friday, 15 December 2006
JOURNAL	"Investment Management and Financial Innovations"
FOUNDER	LLC “Consulting Publishing Company “Business Perspectives”

NUMBER OF REFERENCES	NUMBER OF FIGURES	NUMBER OF TABLES
0	0	0

VaR, CVaR AND TIME RULES WITH ELLIPTICAL AND ASYMMETRIC STABLE DISTRIBUTED RETURNS [1]

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Abstract

This paper proposes several parametric models to compute the portfolio VaR and CVaR in a given temporal horizon and for a given level of confidence. Firstly, we describe extension of the EWMA RiskMetrics model considering conditional elliptically distributed returns. Secondly, we examine several new models based on different stable Paretian distributional hypotheses of return portfolios. Finally, we discuss the applicability of temporal aggregation rules for each VaR and CVaR model proposed.

Key words: Elliptical distributions, domain of attraction, stable distribution, time aggregation rules.

JEL Classification: G21, C32, C53.

1. Introduction

This paper presents and discusses risk management models with the same computational complexity of the most used ones in literature. In particular, the paper serves a threefold objective: 1) studying and understanding elliptical EWMA VaR and CVaR models; 2) examining some distributional stable Paretian approaches applied to the evaluation of the risk of a given portfolio; 3) discussing the application and the limits of temporal aggregation rules of EWMA-type VaR and CVaR models.

The Value at Risk (VaR) and the Conditional Value at Risk (CVaR) are simple risk measures used by financial institutions to evaluate the market risk exposure of their trading portfolios. The main characteristic of VaR and CVaR is that of synthesizing, in a single value, the possible losses which could occur with a given probability in a given temporal horizon.

An important issue in calculating VaR and CVaR is the identification of the so called profit/loss distribution. In the model proposed by RiskMetrics (see Longerstaey and Zangari, 1996), the main assumption is that the profit/loss distribution, conditional upon the portfolio standard deviation, is Gaussian. The main consequence of this hypothesis is that the percentiles and the conditional expected loss, therefore VaR and CVaR, can be simply calculated by multiplying the portfolio standard deviation times a constant which is function of the given confidence level. On the other hand, the possibility of on-line "Gaussian" VaR and CVaR computation has represented the main "success" of these parametric models. As a matter of fact, in the last years there has been a growth in the number of those investors who prefer manage on-line their own portfolios. Moreover, to forecast the weekly, monthly, yearly losses under risk practitioners use scaling with opportune factor daily "Gaussian" VaR and CVaR estimates. However, although this temporal rule is very useful from a practical point of view, it is not generally valid except when we consider independent Gaussian distributed returns. In addition, many empirical studies show that the return conditional distributions diverge from the Gaussian one. In particular, it has been observed that the

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[1] We wish to express our most sincere gratitude to Gennady Samorodnitsky for his useful advices and suggestions that help us to overcome the limits of a preliminary version of the paper. We also are grateful to Hercules Vladimirou and to several anonymous readers for helpful comments.

[2] Sergio Ortobelli's research has been partially supported under Murst 40%-60% 2004, 2005, 2006 and CNR-MIUR-Legge 95/95.

[3] Svetlozar Rachev's research has been supported by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara and the Deutsche Forschungsgemeinschaft.

profit/loss distributions present asymmetries and fat tails. As shown in Longerstaey and Zangari (1996), the VaR calculated under the normal assumption underestimates the actual risk, given that the distribution of the observed financial series are leptokurtic with respect to those implied by a conditional normal distribution.

This paper presents several alternative models for the calculation of VaR and CVaR taking into consideration the skewness and the kurtosis (fat tail effect) that mark the empirical profit/loss distributions. In order to maintain the simplicity of the RiskMetrics model we first extend it to an exponential weighted moving average (EWMA) model with conditional elliptically distributed returns and finite variance. In particular, we discuss the opportunity of using temporal rules of aggregated EWMA models that have not been correctly justified by RiskMetrics researchers (see Longerstaey and Zangari, 1996). Thus, we show that time aggregation rules can be used only when we assume independently distributed returns to approximate the future VaR and CVAR estimates, and then we focus our attention on returns with a conditional multivariate elliptical distributions. Secondly, we propose many stable Paretian VaR and CVaR models. Several empirical and theoretical studies on the asymptotic behavior of financial returns (see, among others, Mandelbrot, 1963a-b; Fama, 1965) justify the assumption of stable paretian distributed returns. Therefore, many stable models have been proposed in recent literature to study financial applications of the stable distributions (see Samorodnitsky and Taqqu, 1994; Rachev and Mittnik, 2000, and references therein). In particular, Stoyanov et al. (2006) have proved some closed form solutions to compute conditional value at risk of a given elliptical and/or stable paretian distribution. However, all stable VaR models recently proposed in financial literature either describe simulating models (see Rachev et al., 2003) or propose models that do not take into account the dependence structure among asset returns and their autoregressive behavior (see, for example, Mittnik, et al., 2002). In contrast, in this paper we first present two parametric autoregressive stable models and we propose two relative time aggregation rules for the associated unconditional models where the return series are independent and identically distributed (i.i.d.). In the first stable model we consider conditional α-stable sub-Gaussian distributed returns. The joint stable sub-Gaussian family is an elliptical family that has been recently used in portfolio theory (see Rachev et al., 2004; Mittnik et al., 2002). As for the elliptical unconditional model with finite variance, we describe particular temporal rules of VaR and CVaR. In order to consider the asymmetry of financial series, we assume conditional jointly asymmetric α-stable distributed returns. The asymmetric stable model results from a new conditional version of the stable three fund separation model proposed by Rachev et al., (2004) to study the portfolio choice problem with asymmetric returns. In this case too, when we assume that the returns series are i.i.d., we obtain further time aggregation rules for VaR and CVaR.

The paper is organized as follows: in section 2 we formalize the RiskMetrics model and its elliptical extension. Section 3 introduces alternative approaches to the EWMA VaR and CVaR models with finite variance. Finally, we briefly summarize the paper.

2. RiskMetrics approach and elliptical EWMA models with finite variance

Value at Risk is the maximum loss among the best θ% cases that could occur in a given temporal horizon. If we denote with τ the investor's temporal horizon, with $W_{t+\tau} - W_t$ the profit/loss realized in the interval [t,t+τ] and with θ the level of confidence, then VaR is given by the loss such that,

$$VaR_{\theta,[t,t+\tau]}(W_{t+\tau} - W_t) = \inf\left\{q \mid \Pr\left(W_{t+\tau} - W_t \le q\right) > 1-\theta\right\}. \qquad (1)$$

Hence, the VaR is the percentile at the (1-θ)% of the profit/loss distribution in the interval [t,t+τ]. The temporal horizon τ and the level of confidence θ are chosen by the investor. The choice of τ depends on the frequency with which the investor wishes to control his/her investment. Alternatively to VaR, the recent literature on risk measures (see Szegö, 2004) has proposed the conditional value at risk (CVaR), also called *expected shortfall* or *expected tail loss*, to evaluate the exposure to market risks. The conditional value at risk is a coherent risk measure i.e. it is a positively homogeneous, translation invariant, subadditive and monotone risk measure. Even if there is no

doubt that VaR_{ϑ} provides useful information, VaR_{ϑ} is not a coherent risk measure (see Artzner et al., 1999; and Stoyanov et al., 2006) and it cannot offer exhaustive information about the expected future losses[1]. The conditional value at risk measures the expected value of profit/loss given that the Value at Risk has not been exceeded, that is

$$CVaR_{\theta,[t,t+\tau]}(W_{t+\tau}-W_t)=\frac{1}{1-\theta}\int_0^{1-\theta} VaR_{q,[t,t+\tau]}\left(W_{t+\tau}-W_t\right)dq \qquad (2)$$

and if we assume a continuous distribution for the profit/loss distribution, we obtain $CVaR_{\theta,[t,t+\tau]}(W_{t+\tau}-W_t)=E\left(W_{t+\tau}-W_t \mid W_{t+\tau}-W_t \le VaR_{\theta,t+\tau}\right)$. The RiskMetrics model assumes that the conditional distribution of the continuously compounded return $R_t(\tau)=\log\left(W_{t+\tau}/W_t\right)$ is a Gaussian law. In particular, RiskMetrics simplifies the VaR calculation for those portfolios with many assets. If we denote with $w=[w_1,...,w_n]'$ the vector of the positions taken in *n* assets forming the portfolio, then the return portfolio at time t+1 is given by

$$z_{(p),t+1}=\sum_{i=1}^{n} w_i z_{i,t+1},$$

where $z_{i,t+1}=\log\left(P_{i,t+1}/P_{i,t}\right)$ is the (continuously compounded) return of *i*-th asset during the period [t,t+1], and $P_{i,t}$ is the price of *i*-th asset at time *t*. RiskMetrics assumes that within a short period of time, the expected return is null and that the return vector

$$z_{t+1}=\left[z_{1,t+1},...,z_{n,t+1}\right]'$$

follows a conditional joint Gaussian distribution. That is, every return conditioned on the forecasted volatility level is distributed like a standardized normal: $z_{i,t+1}/\sigma_{ii,t+1|t}\sim N(0,1)$ and any linear combination of the returns $z_{(p),t+1}=w'z_{t+1}$ is conditionally normal distributed, i.e. $z_{(p),t+1}=\sigma_{(p),t+1/t}X$, where $X\sim N(0,1)$, $\sigma^2_{(p),t+1/t}=w'Q_{t+1/t}w$ is the variance of portfolio $z_{(p),t+1}$ and $Q_{t+1/t}=\left[\sigma^2_{ij,t+1/t}\right]$ is the forecasted variance covariance matrix. As far as the estimates of variances of the single assets $\sigma^2_{ii,t+1/t}$ and the corresponding covariances $\sigma^2_{ij,t+1/t}$ are concerned, RiskMetrics uses the exponentially weighted moving average model (EWMA). Thus, one estimates the variance and covariance matrix $Q_{t+1/t}$ considering the following recursive formulas

$$\sigma^2_{ii,t+1/t}=E_t(z^2_{i,t+1})=\lambda\sigma^2_{ii,t/t-1}+(1-\lambda)z^2_{i,t} \qquad (3)$$

$$\sigma^2_{ij,t+1/t}=E_t(z_{i,t+1}z_{j,t+1})=\lambda\sigma^2_{ij,t/t-1}+(1-\lambda)z_{i,t}z_{j,t}, \qquad (4)$$

where λ is the optimal smoothing factor (see Longerstaey and Zangari, 1996). In particular the conditional variance-covariance process $f_t=\sigma^2_{ij,t+1/t}$ of z_{t+1} is a martingale because

[1] In literature we can find different definitions of VaR and CVaR that change slightly with respect to the use done of the risk measure. For example, in portfolio theory a positive risk measure is generally used, thus typically the above definitions change for the sign of VaR and CVaR functions.

$E_t\left(\sigma^2_{ij,t+1/t}\right) = \sigma^2_{ij,t+1/t}$ and using the expectation operator at time t for any i and j, we can write the forecasted parameters over $s \geq 1$ periods as

$$E_t(f_{t+s}) = E_t\left(\sigma^2_{ij,t+s+1/t+s}\right) = E_t\left(\lambda\sigma^2_{ij,t+s/t+s-1} + (1-\lambda)E_{t+s-1}\left(z_{i,t+s}z_{j,t+s}\right)\right) = \\ = E_t\left(\sigma^2_{ij,t+s/t+s-1}\right) = \sigma^2_{ij,t+1/t} = f_t .$$

The EWMA model is an IGARCH(1,1) (integrated generalized auto-regressive conditional heteroskedastic) model. GARCH-type models provide an alternative view to volatility estimation. There is a growing literature on such models and we refer to Duffie and Pan (1997) for their application to VaR.

The explicit modeling of the volatility series captures the time-varying persistent volatility observed in real financial markets. Under the normality assumption for the conditional returns, the Value at Risk of $z_{(p),t+1}$ at (1-θ)% conditional the information available at time t (denoted by $VaR_{\theta,t+1/t}$) is given by simply multiplying the volatility forecast in the period [t,t+1], times the tabulated value of the corresponding standard Gaussian percentile, $k_{1-\theta}$, therefore,

$$VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1-\theta}\sigma_{(p),t+1/t} . \quad (5)$$

We observe that a similar simplification is also valid for the Conditional Value at Risk of portfolio $z_{(p),t+1}$ at (1-θ)% conditional the information available at time t (denoted by $CVaR_{\theta,t+1/t}$); i.e. CVaR is given by multiplying the volatility forecast in the period [t,t+1], times the tabulated value of the corresponding standard Gaussian CVaR, $c_{1-\theta} = E(X / X \leq k_{1-\theta})$ where X~N(0,1) (see Stoyanov et al., 2006). Thus,

$$CVaR_{\theta,t+1/t}(z_{(p),t+1}) = c_{1-\theta}\sigma_{(p),t+1/t} . \quad (6)$$

Moreover, in all the following discussion we continue to call the VaR and CVaR conditional the information available at time t simply VaR and CVaR. In addition, we can also study temporal aggregation rules of EWMA models.

2.1. Time rules

Let us recall that Engle and Bollerslev (1986), Drost and Nijman (1993), Meddahi and Renault (2004), Hafner and Rombouts (2003) have studied and proved the temporal aggregation of weak GARCH-type processes and volatility models. They have adopted three definitions of GARCH-type models of increasing generality. In particular, a strong GARCH-type requires that rescaled innovations are independent, while in weak GARCH-type models only projections of the conditional variance are considered. Since strong aggregated GARCH-type processes are generally weak GARCH-type processes (see Drost and Nijman, 1993), then the distribution of the aggregated process changes even if the main structural characteristics of the model are maintained. Thus, most of the time rules verified for the conditional variance cannot be applied to compute percentiles of the aggregated process because we do not know a priori the distribution of the aggregated process. However, many practitioners apply time rules to compute the VaR and CVaR for short time aggregation even for these conditional models.

Let us assume that a sample path of 1-day return vectors $\{Z_t\}_{t\in N}$ follows the Gaussian EWMA model above. Under these assumptions the vector of returns $z_{t+1} = \left[z_{1,t+1},...,z_{n,t+1}\right]'$ follows a strong IGARCH(1,1), that is also a particular ISR-SARV(1) process (Integrated Square-Root Stochastic Autoregressive Volatility process i.e., $E_t\left(z_{t+1}\right) = 0$ and

$E_t\left(z_{t+1}z'_{t+1}\right)+U=E_t\left(z_{t+2}z'_{t+2}\right)$ where U is a constant semi-definite positive matrix[1]). Then $z_{t+1}=\Sigma_{t+1/t}\varepsilon_{t+1}$ is conditionally Gaussian distributed, where the conditional variance covariance matrix $Q_{t+1/t}=\Sigma_{t+1/t}\Sigma'_{t+1/t}$ of z_{t+1} at time t follows the above recursive formulas (3), (4) and the vectors $\{\varepsilon_t\}_{t\in N}$ are i.i.d. standard Gaussian distributed N(0,**I**) (**I** is the identity matrix). Thus, the conditional distribution of T-day vector of returns $Z_{t+T}=\left[Z_{1,t+T},...,Z_{n,t+T}\right]'=\sum_{s=1}^{T}z_{t+s}=\sum_{s=1}^{T}\Sigma_{t+s/t+s-1}\varepsilon_{t+s}$ (where $Z_{i,t+T}=\log\left(P_{i,t+T}\mid P_{i,t}\right)$) is a mixture of Gaussian vectors. Using the same arguments of Meddahi and Renault (2004), the corresponding sample path of T-day returns $\{Z_{t+kT}\}_{k\in N}$ is still an ISR-SARV(1) process. Alternatively to the above EWMA model, we can assume additive values $u_{ij,t}$ in the high frequency IGARCH(1,1) process

$$\sigma^2_{ij,t+1/t}=\lambda\sigma^2_{ij,t/t-1}+(1-\lambda)z_{i,t}z_{j,t}+u_{ij,t} \tag{7}$$

with constant definite positive matrix $U=[u_{ij,t}]$. In this case the valuation of the parameters corresponding to the aggregated ISR-SARV(1) process Z_{t+T} follows the rules explained in Meddahi and Renault (2004). Moreover, these parameters depend on the error in the approximation of decay factor λ in (3), (4) computed for the high frequency process because the effect of this misspecification grows when the time T increases. Thus, the error in the estimation can influence dramatically the computation of the variance process. In this context, Kondor (2005) has proved that the decay factor λ must be near to 1 and we need many more observations than what suggested by RiskMetrics in order to reduce the amount of error in the approximation. On the other hand, the conditional variance-covariance matrix of Z_{t+T} at time t for the EWMA model follows the rules of aggregated ISR-SARV(1) process. In view of Meddahi and Renault's analysis, the aggregated process maintains the main structural characteristic of Gaussian EWMA model, but it is not generally a strong Gaussian IGARCH process. Therefore the classical time rule $\tilde{\sigma}^2_{ij,t+T/t}=T\sigma^2_{ij,t+1/t}$ applied for variance covariance matrix $\tilde{Q}_{t+T/t}=\left[\tilde{\sigma}^2_{ij,t+T/t}\right]$ of Gaussian aggregated processes Z_{t+T} is not generally valid except for independent identically distributed returns. Even if it has not been justified by RiskMetrics researchers (see Longerstaey and Zangari, 1996) it is common practice to predict the VaR and CVaR at time t for different temporal horizons assuming the approximating time aggregation rules:

$$VaR_{\theta,t+T/t}(w'Z_{t+T})\approx k_{1-\theta}\sigma_{(p),t+T/t}\text{, and}$$

$$CVaR_{\theta,t+T/t}(w'Z_{t+T})\approx c_{1-\theta}\sigma_{(p),t+T/t},$$

where $\sigma_{(p),t+T/t}=\sqrt{w'\tilde{Q}_{t+T/t}\,w}$. However, this approximated result could lead to mistakes because the distributional structure of the aggregated process is not as strong as the original Gaussian IGARCH. Thus, when the temporal horizon T is not too big (say ten days, see Lamantia et al., 2006), the following time rules:

$$VaR_{\theta,t+T/t}\approx\sqrt{T}VaR_{\theta,t+1/t}, \tag{8}$$

$$CVaR_{\theta,t+T/t}\approx\sqrt{T}CVaR_{\theta,t+1/t}, \tag{9}$$

[1] The semi-definite positive matrix U is equal to zero when we consider EWMA models.

are generally used by practitioners to forecast the (1-θ)% VaR and CVaR in the period [*t*,*t*+*T*] of EWMA Gaussian models. However, as underlined, by Diebold et al. (1998a-b), these time rules can be applied to compute VaR and CVaR only when the returns are independent and identically distributed.

2.2. Limits and advantages of EWMA models

The main advantage and success of the EWMA model applied to the computation of VaR and CVaR are due to its simplicity and applicability to large portfolios. In addition, simple time rules allow for forecasts of *VaR* and *CVaR* of the future wealth under the hypothesis that the vectors of returns are jointly independent and identically distributed. The previous temporal rules simplify the computation of the minimum loss and the average loss that a portfolio can suffer in the θ% worst cases in a temporal horizon greater than unity. In contrast, EWMA model is not a strictly stationary process and the variance process converges a.s. to zero. If we assume the high frequency strong IGARCH(1,1) process (on the marginal distributions, or on the portfolios) $\sigma^2_{ii,t+1/t} = \lambda\sigma^2_{ii,t/t-1} + (1-\lambda)z^2_{i,t} + u_{ii,t}$ with $u_{ii,t} > 0$, then the process is strictly stationary, but not second-order stationary since the second moment is infinite (see Nelson, 1990). Even if the second moment of the residuals is not finite, the conditional variance is well defined since the squared residual process is non-negative. In the case of strictly stationary IGARCH (1,1), portfolio VaR and CVaR for the temporal horizon *T* cannot be derived by the rules of aggregated conditional variance process explained in Meddahi and Renault (2004). However in practice, practitioners use these time rules even if these are generally applied only for limited temporal horizon *T*. Therefore, we generally agree with the main critics on the use of temporal aggregation rules for long temporal horizons (see, among others, Diebold et al., 1998a-b) for two important reasons. First of all, the distributional structure of the aggregated process generally changes. Secondly, the effects of the empirical error in the parameter estimation of high frequency process generally grow when the time *T* increases. As a matter of fact, even if the Gaussian IGARGH(1,1) model presents good performance at high frequency, say daily or intraday returns, the Gaussian IGARCH(1,1) is often rejected at low frequency (see Lamantia et al., 2006). A potential explanation of this aspect is the long memory in the volatility of these Gaussian GARCH-type models (see, for instance, Bollerslev and Mikkelsen, 1996; Comte and Renault, 1998). However, the temporal aggregation of long memory volatility models does not enter in the main objectives of this paper. As far as large portfolios or on-line VaR and CVaR calculation are concerned, the implementation of strictly stationary GARCH-type models should be evaluated on the basis of the tradeoff between costs and benefits. On the other hand, the computational simplicity of the VaR time rule and its validity when unconditional independent Gaussian returns are considered, are the main reasons why the Gaussian time rule is still largely used. In addition, considering that the composition of large portfolios of institutional operators frequently change, then the application of time rules is often limited over time. Thus, in contrast to their non-stationariness, the EWMA type models are still simple versatile tools to estimate the risk financial expositions. This is a further reason to investigate on the financial impact of non Gaussian EWMA models and their temporal aggregation that, generally, do not present an excessive computational complexity even for large portfolios.

2.3. Elliptical non Gaussian EWMA models with finite variance

Recall that we say that an *n*-dimensional vector *z* is elliptically distributed with parameters μ, Q and *f* if for some $\mu \in R^n$ and some $n \times n$ nonnegative definite symmetric matrix Q the characteristic function $\Phi_{z-\mu}(t)$ of $z-\mu$ is a function of the quadratic form $t'Qt$, i.e. $\Phi_{z-\mu}(t) = f\left(t'Qt\right)$ and we write

$$z \sim Ell_n\left(\mu, Q, f\right).$$

Observe that a vector $z \sim Ell_n(\mu, Q, f)$ with $\text{rank}(Q) = s$ if and only if there exists a random variable $B \geq 0$ independent of *U*, an *s*-dimensional random vector uniformly distributed on the unit hypersphere $\mathrm{S_s} = \{u \in R^s / u'u = 1\}$ and an $n \times s$ matrix Σ with $\Sigma\Sigma' = Q$ such that

$$z \overset{d}{=} \mu + B\Sigma U .$$

Given an elliptical vector *z,* its representation $Ell_n(\mu, Q, f)$ is not unique. It uniquely determines μ but Q and *f* are only determined up to a positive constant. In particular, if the vector *z* is elliptical with finite variance, we can always choose the vector μ equal to the mean and the dispersion matrix *Q* equal to the variance covariance matrix. Therefore, in the following discussion when we assume elliptical vectors with finite variance $z \sim Ell_n(\mu, Q, f)$ we implicitly suppose that $\mu = E(z)$ is the mean vector, *Q* is the variance covariance matrix of *z*, and the characteristic function of the centered vector is given by $\Phi_{z-E(z)}(t) = f(t'Qt)$. The function f is also called *characteristic generator*. When *n*=1 a class of elliptical distributions coincides with a class of one-dimensional symmetric distributions uniquely determined by its characteristic function *f*. For further details on elliptical distributions we refer to Cambanis et al. (1981), Owen and Rabinovitch (1983) and Fang et al. (1987).

The RiskMetrics model is a particular EWMA model with conditional elliptically distributed returns and finite variance. In this class of EWMA models the vector $z_{t+1} = \left[z_{1,t+1}, \ldots, z_{n,t+1}\right]'$ is conditionally elliptically distributed with finite variance, null mean and conditional characteristic function

$$\Phi_{z_{t+1}}(m) = E_t(e^{im'z_{t+1}}) = f(m'Q_{t+1/t}m),$$

where $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right]$ is the variance covariance matrix that, for simplicity, we assume is invertible. Therefore, under these assumptions:

$$z_{t+1} = \Sigma_{t+1/t} B_{t+1} U_{t+1},$$

where B_{t+1} is the positive random variable independent of vector U_{t+1} which characterizes the elliptical family. In addition, B_{t+1} U_{t+1} *t*=0,…,*T* are i.i.d. *n*-dimensional vectors (because Q is invertible) where U_{t+1} are uniformly distributed on the unit hypersphere $\mathrm{S_n}$, while the entries of variance and covariance matrix $Q_{t+1/t} = \Sigma_{t+1/t}\Sigma'_{t+1/t}$ follow the above formulas (3) (4) i.e.

$$Q_{t+1/t} = \lambda Q_{t/t-1} + (1-\lambda) z_t z'_t .$$

Moreover, under these assumptions, the aggregated process is a particular ISR-SARV(1) process (see Meddahi and Renault, 2004) and the conditional distribution of $Z_{t+T} = \sum_{s=1}^{T} z_{t+s}$ is a mixture of elliptical $Ell_n(0, Q_{t+1/t}, f)$ vectors. The Value at Risk of portfolio $z_{(p),t+1} = w' z_{t+1}$, at (1-θ)% is given by

$$VaR_{\theta,t+1/t}(z_{(p),t+1}) = \tilde{k}_{1-\theta}\sigma_{(p),t+1/t} \tag{10}$$

where $\sigma^2_{(p),t+1/t} = w'Q_{t+1/t}w$ is the portfolio variance forecasted in the period [t,t+1] and $\tilde{k}_{1-\theta}$ is the tabulated value of the corresponding elliptical $Ell_1(0,1,f)$ percentile uniquely deter-

mined by the characteristic generator f. Similarly, the Conditional Value at Risk of portfolio $z_{(p),t+1}$ at (1-θ)% level is given by

$$CVaR_{\theta,t+1/t}(z_{(p),t+1}) = \tilde{c}_{1-\theta}\sigma_{(p),t+1/t}, \tag{11}$$

where $\tilde{c}_{1-\theta} = CVaR_{\theta}(X)$ is the tabulated CVaR value of the corresponding elliptical $Ell_1(0,1,f)$ determined by its characteristic generator f (see Stoyanov et al., 2006). In addition, among the elliptical models with finite variance, the RiskMetrics Gaussian unconditional model is the unique for which the temporal rules (8) and (9) can be used when the portfolios of returns are independent identically distributed. As a matter of fact, the Gaussian law is the unique elliptical distribution with finite variance such that the sum of elliptical i.i.d. random variables belongs to the same family of elliptical random variables, that is, the unconditional distributions of vectors

$$z_{t+1} = \left[z_{1,t+1},...,z_{n,t+1}\right]' \text{ and } Z_{t+T} = \sum_{s=1}^{T} z_{t+s} = \left[Z_{1,t+T},...,Z_{n,t+T}\right]'$$

belong to the same elliptical family with finite variance $Ell_n(0,Q,f)$ only if $z_{t+s} = \Sigma_{t+s/t+s-1}\varepsilon_{t+s}$ (s=1,...,T) are independent Gaussian distributed (that is do not follow the above EWMA model). However, we could prove a further time aggregation rule of VaR considering z_{t+s} (s=1,...,T) i.i.d. distributed and such that $z_{t+1} \sim Ell_n\left(0,Q_{t+1/t},f\right)$ and $z_{t+T} \sim Ell_n\left(0,\tilde{Q}_{t+T/t},\tilde{f}\right)$ admit different elliptically distributed returns with characteristic generators $f,\tilde{f}$. As a matter of fact, we recall that the sum of elliptical i.i.d. random variables is elliptically distributed but it does not necessarily belong to the same elliptical family (see Embrechts et al., 2003). For example, the sum of T i.i.d. univariate elliptical $X_s \sim Ell_n\left(0,1,f\right)$ (s=1,...,T) with characteristic function *f,* gives another symmetric random variable which is differently elliptically distributed with characteristic function $\tilde{f}$ and variance equal to the sum of variances, i.e. $\sum_{s=1}^{T} X_s \sim \sqrt{T}Ell_t\left(0,1,\tilde{f}\right)$. Similarly, if we sum T i.i.d. elliptical random vectors $z_{t+s} \sim Ell_n\left(0,Q_{t+1/t},f\right)$ (s=1,…,*T*), then we obtain $Z_{t+T} = \sum_{s=1}^{T} z_{t+s} \sim Ell_n\left(0,\tilde{Q}_{t+T/t},\tilde{f}\right)$ where the variance covariance matrix of Z_{t+T} is given by $\tilde{Q}_{t+T/t} = \left[\tilde{\sigma}^2_{ij,t+T/t}\right] = TQ_{t+1/t}$. Thus, when the vector of returns are i.i.d. elliptically distributed, we can apply the variance temporal rule to estimate at time t the (1-θ)% VaR and CVaR in the periods [t,t+1] and [t,t+T], that is $VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1,1-\theta}\sigma_{(p),t+1/t}$ and the temporal aggregation rule

$$VaR_{\theta,t+T/t} \approx k_{2,1-\theta}\sqrt{T}\sigma_{(p),t+1/t} = M_{\theta}\sqrt{T}VaR_{\theta,t+1/t} \tag{12}$$

holds where $M_{\theta} = \dfrac{k_{2,1-\theta}}{k_{1,1-\theta}}$, and $k_{1,1-\theta}$, $k_{2,1-\theta}$ are respectively the corresponding 1-θ elliptical $Ell_1(0,1,f); Ell_1(0,1,\tilde{f})$ percentiles. Similarly, we have that $CVaR_{\theta,t+1/t}(z_{(p),t+1}) = c_{1,1-\theta}\sigma_{(p),t+1/t}$ and

$$CVaR_{\theta,t+T/t} \approx c_{2,1-\theta}\sqrt{T}\sigma_{(p),t+1/t} = \tilde{M}_{\theta}\sqrt{T}CVaR_{\theta,t+1/t}, \tag{13}$$

Where $\widetilde{M}_\theta = \frac{c_{2,1-\theta}}{c_{1,1-\theta}}$, and $c_{1,1-\theta}, c_{2,1-\theta}$ are respectively the corresponding 1-θ elliptical $Ell_1(0,1,f); Ell_1(0,1,\tilde{f})$ conditional value at risk values. However, even if the above time rules are always theoretically justified when we assume log return processes with stationary and independent increments, they are not theoretically justified when we consider time aggregation of EWMA models. Moreover, we need to evaluate the 1-θ percentile $k_{2,1-\theta}$ and the associated conditional value at risk $c_{2,1-\theta}$ of the standardized sum of *m* i.i.d. elliptical distributions $Ell_1(0,1,f)$ (s=1,...,T), that is elliptical $Ell_1(0,1,\tilde{f})$ distributed. Thus, we can find the 1-θ percentile $k_{2,1-\theta}$ and the associated conditional value at risk $c_{2,1-\theta}$ considering the distribution $Ell_1(0,1,\tilde{f})$ derived from the convolution of i.i.d. random variables $Ell_1(0,1,f)$ distributed. All the above models and the following ones have been studied thorough an ex-post empirical comparison in a separate paper. Thus, a detailed discussion on the estimation of the parameters of each model has been done in Lamantia et al. (2006).

3. Alternative VaR and CVaR models with stable distributions

In this section we present some alternative models to compute VaR and CVaR. In particular, we focus our attention on two different stable models for the profit/loss distribution. A random variable *X* is stable distributed if it has a domain of attraction. That is, there exists a sequence of i.i.d. random variables $\{Y_i\}_{i\in N}$, a sequence of positive real values $\{d_i\}_{i\in N}$ and a sequence of real values $\{a_i\}_{i\in N}$ such that, as n→+∞

$$\frac{1}{d_n}\sum_{i=1}^{n} Y_i + a_n \xrightarrow{d} X ,$$

where "$\xrightarrow{d}$" shows the convergence in the distribution. Thus, the α-stable random variables describe a general class of distributions including the leptokurtic and asymmetric ones. The α-stable distribution is identified by four parameters: the index of stability α∈(0,2], the skewness parameter $\beta\in[-1,1]$; $\delta\in\Re$ and $\gamma\in\Re^+$ which are, respectively, the location and the dispersion parameter. If *X* is a random variable whose distribution is α-stable, we use the following notation to underline the parameter dependence:

$$X \simeq S_\alpha(\gamma,\beta,\delta).$$

When α=2, the α-stable distribution has a Gaussian density. A positive skewness parameter (β>0) identifies distributions whose tails are more extended towards the right, while a negative skewness parameter (β<0) characterizes distributions whose tails are extended towards the negative values of the distribution. A detailed analysis of stable distribution properties can be found in Samorodnitsky and Taqqu (1994).

The Functional Central Limit Theorem for normalized sums of i.i.d. random variables theoretically justifies the stable Paretian approach proposed by Mandelbrot (1963a-b) and Fama (1965) to model the behavior of asset returns. However, there is a considerable debate in literature concerning the applicability of α-stable distributions as they appear in Lévy's central limit theorems. A serious drawback of Lévy's approach is that in practice one can never know whether the underlying distribution is heavy tailed, or just has a long but truncated tail. Limit theorems for stable laws are not robust with respect to truncation of the tail or with respect to any change from light to heavy tail, or conversely. Based on finite samples, one can never justify the specification of a particular tail behavior. Hence, one cannot justify the applicability of classical limit theorems

in probability theory. Therefore, instead of relying on limit theorems, we can use the so-called pre-limit theorems which provide an approximation for distribution functions in case the number of observation T is "large" but not too "large". We refer to Klebanov et al. (2001) Klebanov et al. (2000), and paragraph 2.5 of Rachev and Mittnik (2000) for a theoretical description of pre-limit theorems considering that a formal analysis of these results is beyond the scope of this paper. In particular the "pre-limiting" approach helps to overcome the drawback of Lévy-type central limit theorems. As a matter of fact, we can assume that returns are bounded "far away", for example, say daily returns cannot be outside the interval [-0.5,0.5]. Thus, considering the empirical observation on asset returns, we can assume that the asset returns z_i are truncated α_i-stable distributed with support, [-0.5,0.5]. Thus pre-limit theorems show that, for any reasonable number of observations *T*, the truncated stable laws will be well approximated by a stable law. Next, we propose two parametric stable models. The main advantages of the following parametric models are:

1. a better empirical approximation than the RiskMetrics model;
2. the same computational complexity as the RiskMetrics model after parameters estimation;
3. VaR and CVaR time rules:

$$VaR_{\theta,t+T/t} \approx f_1(T) VaR_{\theta,t+1/t},$$

$$CVaR_{\theta,t+T/t} \approx f_2(T) CVaR_{\theta,t+1/t},$$

obtained when the returns are independent identically stable distributed.

3.1. A parametric model with symmetric α-stable distributed returns

In this subsection, we propose an exponentially weighted moving average model with α-stable distributions (2>α>1) that generalizes the classical EWMA model with Gaussian distribution (see Longerstaey and Zangari, 1996). In particular, we assume that the conditional distribution of the vector of returns $z_{t+1} = \left[z_{1,t+1}, \ldots, z_{n,t+1} \right]'$ is α-stable sub-Gaussian (in the period [t,t+1]) with characteristic function

$$\Phi_{z_{t+1}}(m) = E_t(e^{im' z_{t+1}}) = \exp\left(-\left(m' Q_{t+1/t} m \right)^{\alpha/2} + im' \mu_{t+1} \right),$$

where $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t} \right]$ is the conditional dispersion matrix (that we assume is invertible) and $\mu_{t+1} = E\left(z_{t+1} \right)$ even if we assume that within a short period of time the expected return is null. However, before describing the EWMA model with α-stable sub-Gaussian distributions, called the stable EWMA model, we describe some properties of the stable sub-Gaussian vectors.

3.1.1. Unconditional α-stable sub-Gaussian distribution

A stable sub-Gaussian law is the elliptical extension of a Gaussian law when the variance is infinite. It is just one very particular stable law among the stable ones. In particular, an unconditional *α*-stable sub-Gaussian *n*-dimensional vector *z* with *α*<2 and characteristic function $\Phi_z(m) = \exp\left(-\left(m' V m \right)^{\alpha/2} + im' \mu \right)$ is an elliptically distributed vector with $n \times n$ dispersion matrix $V = \left[v^2_{ij} \right]$ and infinite variance. While, if α=2, the vector is Gaussian distributed. As any elliptical vector, even the vector *z* does not admit a unique representation. In order to fix one for any $\alpha \in (1,2)$, we can write

$$z = \mu + \Sigma \sqrt{B} G,$$

where $B \sim S_{\alpha/2}\left(2\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{2/\alpha},1,0\right)$ is $\alpha/2$-stable random variable (called *stable sub-ordinator*) independent of Gaussian vector $G=[G_1,...G_n]' \sim N(0,1)$ with identity covariance matrix. In addition, $\sqrt{B}\ G$ is also an α-stable sub-Gaussian vector where the components $\varepsilon_i = \sqrt{B}\ G_i$ are $S_\alpha(1,0,0)$ distributed, while the dispersion matrix V (that for simplicity we consider invertible) is obtained by the $n \times n$ matrix Σ i.e. $V=\left[v_{ij}^2\right]=\Sigma\Sigma'$. The term v_{ij}^2, that we call codispersion between asset *i* and asset *j,* is defined by

$$v_{ij}^2 = [\tilde{z}_i, \tilde{z}_j]_\alpha \left\|\tilde{z}_j\right\|_\alpha^{2-\alpha}, \tag{14}$$

where $\tilde{z}_i = z_i - \mu_i$ is the centered variable, $[\tilde{z}_i, \tilde{z}_j]_\alpha = \int_{S_2} s_i \left|s_j\right|^{\alpha-1} \operatorname{sgn}(s_j)\gamma_{ij}(ds)$ is the covariation between two jointly symmetric α stable random variables z_i, z_j and γ_{ij}(ds) is the spectral measure with support on the unit circle S_2. In particular, $\left\|\tilde{z}_i\right\|_\alpha = \left(\int_{S_2} \left|s_i\right|^\alpha \gamma_{ii}(ds)\right)^{\frac{1}{\alpha}} = ([\tilde{z}_i, \tilde{z}_i]_\alpha)^{\frac{1}{\alpha}}$. Using properties 1.2.17 and 2.7.16 in Samorodnitsky and Taqqu (1994) we can write for every $q \in (1,\alpha)$

$$\frac{[\tilde{z}_i, \tilde{z}_j]_\alpha}{\left\|\tilde{z}_j\right\|_\alpha^\alpha} = \frac{E\left(\tilde{z}_i \tilde{z}_j^{\langle q-1\rangle}\right)}{E\left(\left|\tilde{z}_j\right|^q\right)},$$

where $\tilde{z}_j^{\langle q-1\rangle} = \operatorname{sgn}(\tilde{z}_j)\left|\tilde{z}_j\right|^{q-1}$ and for every $p \in (0,\alpha)$ the scale parameter can be written

$$v_{jj}^p = \left\|\tilde{z}_j\right\|_\alpha^p = \frac{p\int_0^\infty u^{-p-1}\sin^2 u du}{2^{p-1}\Gamma(1-p/\alpha)} E\left(\left|\tilde{z}_j\right|^p\right).$$

Besides, $\int_0^\infty u^{-p-1}\sin^2 u du = \frac{\Gamma\left(1-\frac{p}{2}\right)\sqrt{\pi}}{2p\Gamma\left(\frac{p+1}{2}\right)}$ for every $p \in (0,2)$. Then, it follows that for every $q \in (1,\alpha)$:

$$v_{ij}^2 = v_{jj}^2 \frac{E\left(\tilde{z}_i \tilde{z}_j^{<q-1>}\right)}{E\left(\left|\tilde{z}_j\right|^q\right)} = v_{jj}^{2-q} A(q) E\left(\tilde{z}_i \tilde{z}_j^{<q-1>}\right) \tag{15}$$

where $A(q)=\dfrac{\Gamma\left(1-\frac{q}{2}\right)\sqrt{\pi}}{2^{q}\Gamma\left(1-\frac{q}{\alpha}\right)\Gamma\left(\frac{q+1}{2}\right)}$. In addition, from relations (14) and (15) we obtain for every q∈ (1,α)

$$[\tilde{z}_i,\tilde{z}_j]_\alpha = v_{jj}^{\alpha-q}A(q)E\left(\tilde{z}_i\tilde{z}_j^{\langle q-1\rangle}\right). \tag{16}$$

Moreover the codispersion between two components of a stable sub-Gaussian vector can be obtained using the symmetry of the dispersion matrix $V=\left[v_{ij}^2\right]$. As a matter of fact, consider the α-stable sub-Gaussian vector z with dispersion matrix respectively $V=\left[v_{ij}^2\right]$, then the sum z_i+z_j is still α-stable distributed with dispersion $v_{z_i+z_j,z_i+z_j}^2 =: v_{i\oplus j,i\oplus j}^2 = v_{ii}^2+v_{jj}^2+2v_{ij}^2$, thus we get that

$$v_{ij}^2=\frac{v_{i\oplus j,i\oplus j}^2-v_{ii}^2+v_{jj}^2}{2}. \tag{17}$$

This result suggests the following estimator $V=\left[\hat{v}_{ij}^2\right]$ for the entries of the unknown covariation matrix for some q∈(1,α):

$$\hat{v}_{ij}^2=\frac{\hat{v}_{i\oplus j,i\oplus j}^2-\hat{v}_{jj}^2-\hat{v}_{ii}^2}{2},$$

where $\hat{v}_{jj}^2$ and $\hat{v}_{i\oplus j,i\oplus j}^2$ are estimated as follows for some p∈(0,α)

$$\hat{v}_{jj}^2=\left(A(p)\frac{1}{N}\sum_{k=1}^{N}\left|\tilde{z}_j^{(k)}\right|^p\right)^{2/p},\ \hat{v}_{i\oplus j,i\oplus j}^2=\left(A(p)\frac{1}{N}\sum_{k=1}^{N}\left|\tilde{z}_i^{(k)}+\tilde{z}_j^{(k)}\right|^p\right)^{2/p}.$$

Moreover, regarding the sum of independent α-stable sub-Gaussian vectors, the following lemma holds.

Lemma 1: Let $Z^{(k)}=\left[Z_1^{(k)},Z_2^{(k)}\right]'$ k=1,2,...,T be T independent α-stable sub-Gaussian vectors (α>1) with null mean and dispersion matrixes respectively $V^{(k)}=\left[v_{(k),ij}^2\right]$, i,j=1,2 k=1,2,...,T. Then $Z=\sum_{k=1}^{T}Z^{(k)}=\left[\sum_{k=1}^{T}Z_1^{(k)},\sum_{k=1}^{T}Z_2^{(k)}\right]'$ is still symmetric with null mean and

$$\left[\sum_{k=1}^{T}Z_1^{(k)},\sum_{k=1}^{T}Z_2^{(k)}\right]_\alpha=\sum_{k=1}^{T}\left[Z_1^{(k)},Z_2^{(k)}\right]_\alpha=\sum_{k=1}^{T}\lim_{q\nearrow\alpha}A(q)E\left(Z_1^{(k)}\left(Z_2^{(k)}\right)^{<q-1>}\right)=\left(\sum_{k=1}^{T}\frac{v_{(k),12}^2}{v_{(k),22}^{2-\alpha}}\right).$$

.

Proof: It is well known that the sum of independent α-stable vectors is α-stable. Observe that the collections $\left\{Z_1^{(k)}\right\}_{k=1,\ldots,T}$, $\left\{Z_2^{(k)}\right\}_{k=1,\ldots,T}$ are collections of independent random variables and, for any $m \neq n$, $Z_2^{(m)}$ and $Z_1^{(n)}$ are also independent. Then, as a consequence of properties 2.7.7, 2.7.11 and 2.7.15 in Samorodnitsky and Taqqu (1994) we obtain $\left[\sum_{k=1}^{T} Z_1^{(k)}, \sum_{k=1}^{T} Z_2^{(k)}\right]_\alpha = \sum_{k=1}^{T}\left[Z_1^{(k)}, Z_2^{(k)}\right]_\alpha$. Thus, from (16) we get that, for every $q \in (1,\alpha)$,

$$\sum_{k=1}^{T}\left[Z_1^{(k)}, Z_2^{(k)}\right]_\alpha = \sum_{k=1}^{T}\left\|Z_2^{(k)}\right\|_\alpha^{\alpha-q} A(q)E\left(Z_1^{(k)}\left(Z_2^{(k)}\right)^{\langle q-1\rangle}\right) =$$

$$= \sum_{k=1}^{T}\lim_{q\nearrow\alpha} A(q)E\left(Z_1^{(k)}\left(Z_2^{(k)}\right)^{\langle q-1\rangle}\right),$$

where the last equality is verified because the first equality must be true even for $q \nearrow \alpha$. As a consequence of formula (14), $\left[Z_1^{(k)}, Z_2^{(k)}\right]_\alpha = \dfrac{v_{(k),12}^2}{v_{(k),22}^{2-\alpha}}$, thus the thesis follows. □

Even if $Z^{(k)} = \left[Z_1^{(k)}, Z_2^{(k)}\right]'$ k=1,2,...,T are T independent α-stable sub-Gaussian vectors, $Z = \sum_{k=1}^{T} Z^{(k)}$ is symmetric α-stable distributed, but it is not generally α-stable sub-Gaussian distributed (a simple counterexample is given by example 2.13 in Samorodnisky and Taqqu, 1994). Moreover if $Z = \sum_{k=1}^{T} Z^{(k)}$ is α-stable sub-Gaussian, then the entries of dispersion matrix $V = \left[v_{ij}^2\right]$ are given by $v_{jj}^\alpha = \sum_{k=1}^{T} v_{(k),jj}^\alpha$ and $v_{ij}^2 = \left(\sum_{k=1}^{T} v_{(k),jj}^\alpha\right)^{\frac{2-\alpha}{\alpha}}\left(\sum_{k=1}^{T}\dfrac{v_{(k),ij}^2}{v_{(k),jj}^{2-\alpha}}\right)$ where i,j=1,2. As a matter of fact, it is well known that the dispersion of sum of independent α-stable random variables with dispersion $v_{(k),jj}$ k=1,…,T satisfies the relation $v_{jj}^\alpha = \sum_{k=1}^{T} v_{(k),jj}^\alpha$. On the other hand, by the previous Lemma we deduce that the codispersion is given by:

$$v_{12}^2 = \left\|\sum_{k=1}^{T} Z_2^{(k)}\right\|_\alpha^{2-\alpha}\left(\sum_{k=1}^{T}\left[Z_1^{(k)}, Z_2^{(k)}\right]_\alpha\right).$$

Thus, it follows $v_{ij}^2 = \left(\sum_{k=1}^{T} v_{(k),jj}^\alpha\right)^{\frac{2-\alpha}{\alpha}}\left(\sum_{k=1}^{T}\dfrac{v_{(k),ij}^2}{v_{(k),jj}^{2-\alpha}}\right)$. A very particular example is the sum of T i.i.d. α-stable sub-Gaussian vectors with null mean and dispersion matrix $W = \left[w_{ij}^2\right]$, that is an α-stable sub-Gaussian vector with null mean and dispersion matrix $\widetilde{W} = T^{2/\alpha}W$.

As a consequence of the symmetry of α-stable sub-Gaussian random variables, for any p<α, the dispersion v_{jj}^{p} of the centered random variable $\tilde{z}_j$ can be seen as the variance of the random variable $\left|\tilde{z}_j\right|^{\frac{p}{2}} \operatorname{sgn}\left(\tilde{z}_j\right)\sqrt{A(p)}$ i.e. $v_{jj}^{p} = \text{variance}\left(\left|\tilde{z}_j\right|^{\frac{p}{2}} \operatorname{sgn}\left(\tilde{z}_j\right)\sqrt{A(p)}\right)$ for any p<α. Then, if the sum $X = \sum_{k=1}^{T} \tilde{z}_k$ of T independent centered α-stable symmetric random variables z_i, j=1,2,...,T, satisfies the following relation for any p<α,

$$v^{\alpha} = \lim_{p \nearrow \alpha} v^{p} = \lim_{p \nearrow \alpha} \text{variance}\left(\left|\sum_{k=1}^{T} \tilde{z}_k\right|^{\frac{p}{2}} \operatorname{sgn}(X)\sqrt{A(p)}\right) =$$

$$= \sum_{k=1}^{T} \lim_{p \nearrow \alpha} \text{variance}\left(\left|\tilde{z}_k\right|^{\frac{p}{2}} \operatorname{sgn}(\tilde{z}_k)\sqrt{A(p)}\right) = \lim_{p \nearrow \alpha} \text{variance}\left(\sum_{k=1}^{T}\left(\left|\tilde{z}_k\right|^{\frac{p}{2}} \operatorname{sgn}(\tilde{z}_k)\sqrt{A(p)}\right)\right).$$

The third equality above derives from the previous discussion, while the last equality is a consequence of the independence of random variables $\left|\tilde{z}_j\right|^{\frac{p}{2}} \operatorname{sgn}\left(\tilde{z}_j\right)\sqrt{A(p)}$, j=1,2,...,T.

3.1.2. The Stable EWMA model

Suppose that the conditional distribution of the returns vector $z_{t+1} = \left[z_{1,t+1},\ldots,z_{n,t+1}\right]'$ is α-stable sub-Gaussian (α∈(1,2)) with characteristic function

$$\Phi_{z_{t+1}}(m) = E_t(e^{im'z_{t+1}}) = \exp\left(-\left(m'Q_{t+1/t}m\right)^{\alpha/2} + im'\mu_{t+1}\right).$$

Under these assumptions we assume

$$z_{t+1} = \mu_{t+1} + \Sigma_{t+1/t}\sqrt{B_{t+1}}G_{t+1},$$

where $B_{t+1} \sim S_{\alpha/2}\left(2\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{2/\alpha}, 1, 0\right)$ is a stable subordinator independent of Gaussian vector $G_{t+1} = [G_{1,t+1},\ldots,G_{n,t+1}]' \sim N(0,1)$ with identity covariance matrix. In addition, $\sqrt{B_{t+1}}\ G_{t+1}$ t=0,…,T are i.i.d. α-stable sub-Gaussian vectors, where the components $\varepsilon_{i,t+1} = \sqrt{B_{t+1}}G_{i,t+1}$ are $S_{\alpha}(1,0,0)$ distributed, while the entries of dispersion matrix $Q_{t+1/t} = \left[\sigma_{ij,t+1/t}^{2}\right] = \Sigma_{t+1/t}\Sigma'_{t+1/t}$ are generated as follows:

$$\tilde{z}_{i,t+1} = z_{i,t+1} - \mu_{i,t+1} = \sigma_{ii,t+1/t}\varepsilon_{i,t+1}$$

$$\sigma_{ii,t+1/t}^{p} = E_t\left(\left|\tilde{z}_{i,t+1}\right|^{p}\right)A(p) = \lambda\sigma_{ii,t/t-1}^{p} + (1-\lambda)A(p)\left|\tilde{z}_{i,t}\right|^{p} \qquad (18)$$

$$\sigma_{i\oplus j,i\oplus j,t+1/t}^{p} = E_t\left(\left|\tilde{z}_{i,t+1} + \tilde{z}_{j,t+1}\right|^{p}\right)A(p) = \lambda\sigma_{i\oplus j,i\oplus j,t/t-1}^{2} + (1-\lambda)A(p)\left|\tilde{z}_{i,t} + \tilde{z}_{j,t}\right|^{p}$$

$$\sigma^2_{ij,t+1/t} = \frac{\sigma^2_{i\oplus j,i\oplus j,t+1/t} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2}, \tag{19}$$

where $A(q) = \dfrac{\Gamma\left(1-\frac{q}{2}\right)\sqrt{\pi}}{2^q\,\Gamma\left(1-\frac{q}{\alpha}\right)\Gamma\left(\frac{q+1}{2}\right)}$, p∈(0,α), λ is the decay factor that regulates the weighting on past covariation parameters. These assumptions are consistent with the structure of dispersion matrix of an α-stable sub-Gaussian vector. In fact, we require that $\sigma^2_{ij,t+1/t} = \dfrac{\left(E_t\left(\left|\tilde{z}_{i,t+1}+\tilde{z}_{j,t+1}\right|^p\right)A(p)\right)^{2/p} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2}$ and $\sigma^p_{ii,t+1/t} = E_t\left(\left|\tilde{z}_{i,t+1}\right|^p\right)A(p)$. The scale parameter of *i*-th return, $\sigma_{ii,t+1/t}$, is given by

$$\sigma_{ii,t+1/t} = \left(E_t\left(\left|\tilde{z}_{i,t+1}\right|^p\right)A(p)\right)^{1/p} \approx \left((1-\lambda)A(p)\sum_{k=0}^{K}\lambda^{K-k}\left|\tilde{z}_{i,t-K+k}\right|^p\right)^{1/p},$$

where for any given tolerance level $tl = (1-\lambda)\sum_{k=K+1}^{\infty}\lambda^k$ we can determine the number of useful observations $K = \dfrac{\log(tl)}{\log(\lambda)}$ as per the RiskMetrics model. Thus, considering a tolerance level *tl*=0.001 and a decay factor λ=0.97 we obtain that $K \simeq 228$. Similarly the scale parameter of the sum $\tilde{z}_{i,t+1} + \tilde{z}_{j,t+1}$ is given by

$$\sigma_{i\oplus j,i\oplus j,t+1/t} = \left(E_t\left(\left|\tilde{z}_{i,t+1}+\tilde{z}_{j,t+1}\right|^p\right)A(p)\right)^{1/p} \approx \left((1-\lambda)A(p)\sum_{k=0}^{K}\lambda^{K-k}\left|\tilde{z}_{i,t-K+k}+\tilde{z}_{j,t-K+k}\right|^p\right)^{1/p}.$$

Instead $\sigma^2_{ij,t+1/t}$ is the stable covariation parameter between the *i*-th and the *j*-th returns and it holds from formula (19). Observe that the decay factor λ determines the relative weights that are applied to return observations. Thus, the most recent observations are more weighted than the old ones. Thus, the above model is a particular Stable GARCH(1,1) model (see, among others, Rachev and Mittnik, 2000) and it is also an EWMA model and an ISR-SARV(1) process applied to the random variables $X_{j,t} = \left|\tilde{z}_{j,t}\right|^{\frac{p}{2}}\operatorname{sgn}\left(\tilde{z}_{j,t}\right)\sqrt{A(p)}$, because the dispersion $\sigma^p_{jj,t+1/t}$ is the conditional variance of $X_{j,t+1}$ for every j=1,2,...,*n*.

Under these assumptions, any portfolio is defined by $z_{(p),t+1} = \sum_{i=1}^{n} w_i z_{i,t+1} = \sqrt{w' Q_{t+1/t} w}\,\bar{\varepsilon}_{t+1}$ where $\bar{\varepsilon}_{t+1} \simeq S_\alpha(1,0,0)$ and $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right]$. Then, the (1-θ)% VaR in the period [t,t+1] is obtained by multiplying the corresponding percentile, $k_{1-\theta,\alpha}$, of the standardized α-stable $S_\alpha(1,0,0)$, times the forecast volatility $\sigma_{(p),t+1/t} = \sqrt{w' Q_{t+1/t} w}$, that is

$$VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1-\theta,\alpha}\sigma_{(p),t+1/t}.$$

Similarly, the Conditional Value at Risk at (1-θ)% confidence level is given by

$$CVaR_{\theta,t+1/t}(z_{(p),t+1}) = c_{1-\theta,\alpha}\sigma_{(p),t+1/t}.$$

Where $c_{1-\theta,\alpha} = E\left(X / X \le k_{1-\theta,\alpha}\right)$ is the CVaR of the corresponding standard α-stable $X \sim S_\alpha(1,0,0)$. Even in this case, we can consider the aggregated process $Z_{t+T} = \sum_{s=1}^{T} \tilde{z}_{t+s}$. If the high frequency process z_t follows the above α-stable sub-Gaussian EWMA model, then the conditional distribution of Z_{t+T} is a mixture of α-stable sub-Gaussian vectors. In particular, the aggregated process is still an *ISR-SARV*(1) process if we consider the variance of the random variables $\left|\tilde{z}_{j,t}\right|^{\frac{p}{2}} \operatorname{sgn}\left(\tilde{z}_{j,t}\right)\sqrt{A(p)}$. According to Lemma 1, the aggregated process $Z_{t+T} = \sum_{s=1}^{T} \tilde{z}_{t+s}$ at time *t* is still α-stable sub-Gaussian distributed only if the vectors of returns $\tilde{z}_{t+s}$ are i.i.d. α-stable sub-Gaussian distributed. In this case, the dispersion matrix $\widetilde{Q}_{t+T/t}$ of Z_{t+T} follows the time rule

$$\widetilde{Q}_{t+T/t} = T^{2/\alpha} Q_{t+1/t}.$$

Therefore, if the aggregated process Z_{t+T} is the sum of i.i.d. α-stable sub-Gaussian distributed vector of returns, we can predict the (1-θ)% VaR and CVaR over the period [t,t+*T*] with the time rule:

$$VaR_{\theta,t+T/t} \approx (T)^{1/\alpha}\, VaR_{\theta,t+1/t},$$

$$CVaR_{\theta,t+T/t} \approx (T)^{1/\alpha}\, CVaR_{\theta,t+1/t}.$$

However, if we assume a Stable EWMA model for the returns, the previous aggregation time rules are only an approximation of future VaR and CVaR estimates and they can be applied only when T is not too big. Moreover, among the elliptical distributions, the α-stable sub-Gaussian with α∈(0,2] (with α=2 we obtain the Gaussian case) are the unique elliptical distributions such that the sum of i.i.d. elliptical random variables belongs to the same family of elliptical random variables.

3.2. An α-stable model with asymmetrically distributed returns

As an alternative to the previous model, we can take into account the asymmetry of stable distributions generalizing the model proposed by Rachev et al. (2004). Under these assumptions the vector of centered return is conditional α-stable distributed, i.e.

$$\tilde{z}_{t+1} = z_{t+1} - \mu_{t+1} = b_{t+1}Y_{t+1} + \Sigma_{t+1/t}\sqrt{B_{t+1}}G_{t+1}, \tag{20}$$

where $\mu_{t+1} = E(z_{t+1})$ the factor $Y_{t+1} \simeq S_\alpha\left(\sigma_{y_{t+1}}, \beta_{y_{t+1}}, 0\right)$ is an α-stable asymmetric (i.e. $\beta_{Y_{t+1}} \neq 0$) centered index return with dispersion and the skewness respectively equal to $\sigma_{Y_{t+1}}$ and $\beta_{Y_{t+1}}$. Besides the residual random vector $\tilde{z}_{t+1} - b_{t+1}Y_{t+1} = \Sigma_{t+1/t}\sqrt{B_{t+1}}G_{t+1}$ is independent of factor Y_{t+1} and it is conditional α-stable sub-Gaussian distributed, as the above Stable EWMA model with zero mean and dispersion matrix

$Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right] = \Sigma_{t+1/t}\Sigma'_{t+1/t}$. Therefore, the centered return vector $\tilde{z}_{t+1} = \left[\tilde{z}_{1,t+1},...,\tilde{z}_{n,t+1}\right]'$ is conditionally α-stable distributed with conditional characteristic function:

$$\Phi_{\tilde{z}_{t+1}}(m) = E_t(e^{im'\tilde{z}_{t+1}}) = \exp\left(-\left(\left|m'Q_{t+1/t}m\right|^{\alpha/2} + \left|m'b_{t+1}\sigma_{Y_{t+1}}\right|^{\alpha}\right)\times\right.$$

$$\left.\times\left(1 - i\frac{\left|m'b_{t+1}\sigma_{Y_{t+1}}\right|^{\alpha}\beta_{Y_{t+1}}\operatorname{sgn}(m'b_{t+1})}{\left(\left|m'Q_{t+1/t}m\right|^{\alpha/2} + \left|m'b_{t+1}\sigma_{Y_{t+1}}\right|^{\alpha}\right)}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right).$$

Observe that this characteristic function is obtained by the sum of two independent multivariate α-stable laws (see Samorodnitsky and Taqqu (1994) for further details). In particular, we assume that the centered returns $\tilde{z}_{i,t+1}$ are generated as follows

$$\tilde{z}_{i,t+1} = b_{i,t+1}Y_{t+1} + \sigma_{ii,t+1/t}\varepsilon_{i,t+1} = \left(\sigma^{\alpha}_{ii,t+1/t} + \left|b_{i,t+1}\sigma_{Y_{t+1}}\right|^{\alpha}\right)^{1/\alpha} X_{i,t+1},$$

$$\sigma^p_{ii,t+1/t} = E_t\left(\left|\tilde{z}_{i,t+1}\right|^p\right)A(p) = \lambda\sigma^p_{ii,t/t-1} + (1-\lambda)A(p)\left|\tilde{z}_{i,t} - b_{i,t}Y_t\right|^p \tag{21}$$

$$\begin{aligned}\sigma^p_{i\oplus j,i\oplus j,t+1/t} &= E_t\left(\left|\tilde{z}_{i,t+1} + \tilde{z}_{j,t+1} - (b_{i,t+1} + b_{j,t+1})Y_{t+1}\right|^p\right)A(p) = \\ &= \lambda\sigma^2_{i\oplus j,i\oplus j,t/t-1} + (1-\lambda)A(p)\left|\tilde{z}_{i,t} + \tilde{z}_{j,t} - (b_{i,t} + b_{j,t})Y_t\right|^p\end{aligned} \tag{22}$$

$$\sigma^2_{ij,t+1/t} = \frac{\sigma^2_{i\oplus j,i\oplus j,t+1/t} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2}, \tag{23}$$

where p∈(0,α). For any i and t, $X_{i,t} \sim S_\alpha\left(1, \frac{\left|b_{i.t}\sigma_{Y_t}\right|^{\alpha}\beta_{Y_t}\operatorname{sgn}(b_t)}{\left(\sigma^{\alpha}_{ii,t/t-1} + \left|b_{i.t}\sigma_{Y_t}\right|^{\alpha}\right)}, 0\right)$ and $\varepsilon_{i,t} = \sqrt{B_t}G_{i,t} \sim S_\alpha(1,0,0)$ because the conditional distribution of residual vector $\tilde{z}_{t+1} - b_{t+1}Y_{t+1}$ is α-stable sub-Gaussian and it is independent of factor Y_{t+1}. We require that the structure of dispersion matrix $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right]$ of residual vectors is defined according to formulas (14) and (17). The parameter λ is a "decay factor" that regulates the weighting of the past covariation parameters. The vector $b_t = \left[b_{1,t},...,b_{n,t}\right]'$ is estimated considering the OLS estimator. Thus, if we assume that the vector $b_t \equiv b = \left[b_1,...,b_n\right]'$ is constant over the time, then

$\hat{b}_i = \dfrac{\sum_{k=1}^{N} Y^{(k)} \tilde{z}_i^{(k)}}{\sum_{k=1}^{N} \left(Y^{(k)}\right)^2}$, i=1,...,*n*. Götzenberger et al. (1999), Paulauskas and Rachev (2003) discuss the asymptotic properties of OLS estimators applied to stable models. However, Kurz-Kim et al. (2005) have shown that other estimators could present better asymptotic properties. The forecast scale parameter of *i*-th residual, $\sigma_{ii,t+1/t}$ is defined by:

$$\sigma_{ii,t+1/t} = \left(E_t \left(\left| \tilde{z}_{i,t+1} - b_{i,t+1} Y_{t+1} \right|^p \right) A(p) \right)^{1/p} \approx \left((1-\lambda) A(p) \sum_{k=0}^{K} \lambda^{K-k} \left| \tilde{z}_{i,t-K+k} - b_{i,t-K+k} Y_{t-K+k} \right|^p \right)^{1/p}.$$

While the time t+1 stable covariation parameter between the i-th and the j-th residual is defined by $\sigma^2_{ij,t+1/t} = \dfrac{\sigma^2_{i\oplus j,i\oplus j,t+1/t} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2}$ where

$$\sigma^2_{i\oplus j,i\oplus j,t+1/t} \approx \left((1-\lambda) A(p) \sum_{k=0}^{K} \lambda^{K-k} \left| \tilde{z}_{i,t-K+k} + \tilde{z}_{j,t-K+k} - (b_{i,t-K+k} + b_{j,t-K+k}) Y_{t-K+k} \right|^p \right)^{1/p}.$$

Under these assumptions, the forecasted (1-θ)% Value at Risk and Conditional Value at Risk of portfolio $z_{(p),t+1} = \sum_{i=1}^{n} w_i \tilde{z}_{i,t+1} = w' \tilde{z}_{t+1}$ in the period [t,t+1], are given by the corresponding percentile and CVaR, of the α-stable distribution $S_\alpha(\sigma_{(p),t+1/t}, \beta_{(p),t+1/t}, 0)$ where

$$\sigma_{(p),t+1/t} = \left(\left| w' Q_{t+1/t} w \right|^{\alpha/2} + \left| w' b_{t+1} \sigma_{Y_{t+1}} \right|^\alpha \right)^{1/\alpha}$$

is the volatility forecast and

$$\beta_{(p),t+1/t} = \frac{\left| w' b_{t+1} \sigma_{Y_{t+1}} \right|^\alpha \beta_{Y_{t+1}} \operatorname{sgn}(w' b_{t+1})}{\left(\left| w' Q_{t+1/t} w \right|^{\alpha/2} + \left| w' b_{t+1} \sigma_{Y_{t+1}} \right|^\alpha \right)}$$

is the skewness forecast. Moreover, in order to contemplate the evolution of the index Y, we assume that the dispersion parameter σ_{Y_t} follows the recursive formula

$$\sigma^p_{Y_t} = \frac{A(p) E_t \left(\left| Y_{t+1} \right|^p \right)}{g\left(\alpha_Y, \beta_{Y_t}, p\right)} \approx \frac{A(p) \sum_{k=0}^{K} \left| Y_{t-K+k} \right|^p}{K g\left(\alpha_Y, \beta_{Y_t}, p\right)} \approx \sigma^p_{Y_{t-1}} + A(p) \frac{\left| Y_t \right|^p - \left| Y_{t-K-1} \right|^p}{K g\left(\alpha_Y, \beta_{Y_t}, p\right)}$$

where

$$g\left(\alpha_Y, \beta_{Y_t}, p\right) = \left(1 + \beta^2_{Y_t} \left(\tan^2 \left(\frac{\alpha_Y \pi}{2} \right) \right) \right)^{\frac{p}{2\alpha_Y}} \cos\left(\frac{p}{2\alpha_Y} \arctan\left(\beta_{Y_t} \left(\tan\left(\frac{\alpha_Y \pi}{2} \right) \right) \right) \right).$$

We refer again to Samorodnitsky and Taqqu (1994) for further details on properties of stable laws. Under these assumptions, the distribution of aggregated process $Z_{t+T} = \sum_{s=1}^{T} \tilde{z}_{t+s}$ conditionally

on the knowledge of dispersion and skewness is a mixture of α-stable vectors. Therefore no time rules can be used to compute VaR and CVaR values of vector Z_{t+T} when the returns follow the above conditional autoregressive stable Paretian model. However, if we assume that the vectors $\tilde{z}_{t+s} = b_{t+s}Y_{t+s} + \Sigma_{t+s/t+s-1}\sqrt{B_{t+s}}G_{t+s}$ of formula (20) are i.i.d. α-stable distributed, then $Z_{t+T} = \sum_{s=1}^{T}\tilde{z}_{t+s}$ is also α-stable distributed. In particular the dispersion and the skewness of a portfolio $Z_{(p),t+T} = w'Z_{t+T}$ are:

$$\bar{\sigma}^{\alpha}_{(p),t+T/t} = \sum_{s=1}^{T}\left(\sigma^{\alpha}_{(p),t+s/t+s-1} + \left|w'b_{t+s}\sigma_{Y_{t+s}}\right|^{\alpha}\right),$$

$$\bar{\beta}_{(p),t+T/t} = \frac{\sum_{s=1}^{T}\left|w'b_{t+s}\sigma_{Y_{t+s}}\right|^{\alpha}\beta_{Y_{t+s}}\operatorname{sgn}(w'b_{t+s})}{\sum_{s=1}^{T}\left(\left|w'Q_{t+s/t+s-1}w\right|^{\alpha/2} + \left|w'b_{t+s}\sigma_{Y_{t+s}}\right|^{\alpha}\right)}.$$

In addition, the aggregated vector of the i.i.d. α-stable sub Gaussian distributed residuals $\Sigma_{t+s/t+s-1}\varepsilon_{t+s}$, where $\varepsilon_{t+s} = \sqrt{B_{t+s}}G_{t+s}$ (with null mean and dispersion matrix $Q_{t+s/t+s-1} = Q_{t+1/t}$ for any s=1, …, T)

$$W_{t+T} = Z_{t+T} - \sum_{s=1}^{T} b_{t+s}Y_{t+s} = \left[Z_{1,t+T} - \sum_{s=1}^{T} b_{1,t+s}Y_{t+s},\ldots,Z_{n,t+T} - \sum_{s=1}^{T} b_{n,t+s}Y_{t+s}\right]'$$

is itself α-stable sub Gaussian distributed with null mean and dispersion matrix $\tilde{Q}_{t+T/t} = \left[\tilde{\sigma}^{2}_{ij,t+T/t}\right]$ that follows the time rule $\tilde{Q}_{t+T/t} = T^{2/\alpha}Q_{t+1/t}$.

Let us assume that the parameters α, $\beta_{Y_t}, \sigma_{Y_t}, b_t$ are constant over the time and suppose that the vector Z_{t+T} is the sum of i.i.d. α-stable distributed vectors $\tilde{z}_{t+s}$ of formula (20), then the forecasted (1-θ)% VaR and CVaR of portfolio $Z_{(p),t+T} = w'Z_{t+T}$ in the period [t,t+T] are given by the corresponding (1-θ) percentile and CVaR, of the α-stable distribution $S_{\alpha}(\tilde{\sigma}_{(p),t+T/t}, \tilde{\beta}_{(p),t+T/t}, 0)$ where

$$\tilde{\sigma}^{\alpha}_{(p),t+T/t} = T\left(\sigma^{\alpha}_{(p),t+1/t} + \left|w'b_t\sigma_{Y_t}\right|^{\alpha}\right),$$

$$\tilde{\beta}_{(p),t+T/t} = \frac{T\left(\left|w'b_t\sigma_{Y_t}\right|^{\alpha}\beta_{Y_t}\operatorname{sgn}(w'b_t)\right)}{T\left(\left|w'Q_{t+1/t}w\right|^{\alpha/2} + \left|w'b_t\sigma_{Y_t}\right|^{\alpha}\right)} = \beta_{(p),t+1/t}.$$

Therefore, the temporal rules

$$VaR_{\theta,t+T/t} \approx (T)^{1/\alpha}\, VaR_{\theta,t+1/t},$$

$$CVaR_{\theta,t+T/t} \approx (T)^{1/\alpha}\, CVaR_{\theta,t+1/t},$$

hold when the vectors $\tilde{z}_{t+s}$ are i.i.d. α-stable distributed and follow the model (20). Thus, as for the previous cases, these time rules are not verified when we assume the previous autoregressive model and they could be considered only as an approximated result when T is not too big. On the

other hand, further studies on temporal aggregation of stable processes are beyond the scope of this paper, and they will be object of future discussions.

4. Conclusions

This paper proposes alternative models for the VaR and CVaR calculation. In the first part we describe several elliptical EWMA models with finite variance and discuss the applicability of some time rules. Then we introduce and discuss symmetric and asymmetric stable Paretian models to compute the percentiles and the expected losses. The symmetric stable model is an elliptical EWMA model with infinite variance, while the asymmetric stable model is a three fund separation conditional model with symmetric stable residuals. In particular, we prove simple temporal aggregation rules for each parametric stable model when the returns are i.i.d. stable distributed. However these rules are not valid for the conditional models even if in the classic Gaussian case they are still used by practitioners. On the other hand, all the parametric VaR and CVaR models and the respective time rules introduced here can be theoretically improved and empirically tested. This research is the starting point for further discussion, studies, and comparisons on temporal aggregation rules and subject for future research.

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