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Risk parity versus other *μ-free strategies*: a comparison in a triple view

Abstract

This article proposes a comparison of risk parity strategy versus other asset allocation methodologies that don't require expected returns as input (naive risk parity, minimum-variance, equally weighting). Specifically, we empirically test if risk parity is consistently better than other *μ-free strategies* using two datasets that differ in terms of market conditions and in terms of the number of asset classes in the investment universe. The comparison is undertaken considering three evaluation dimensions: financial efficiency, diversification and asset allocation stability. Relative to the existing literature, we strongly expand the set of tools to be implemented in order to capture these aspects. The findings suggest that risk parity cannot be considered consistently superior relative to other *μ-free strategies* on the basis of the adopted triple view. The results are in line with, and more robust and more well-verified than those achieved by Maillard, Roncalli e Teiletche (2010) and disagree with Chaves et al. (2012).

Keywords: risk parity, risk contribution, marginal risk, estimation risk, optimization algorithm, reward-to-variability ratios.
JEL Classifications: G11, G12.

Introduction

In the asset allocation context, the mean-variance optimization developed by Markowitz (1952, 1959) over 50 years ago represented the cornerstone of Modern Portfolio Theory (MPT) and provided the appropriate methodology for allocating wealth to different risky investment alternatives. Basically, the Markowitz model derives portfolio weights such that expected return is maximized for a given standard deviation (variance) level or such that volatility is minimized for a given expected return level.

Despite its elegance, simplicity and rationality, Markowitz's approach suffers from serious drawbacks when practically implemented. They mainly raise from the estimation risk associated with the necessary inputs (expected returns, risks and correlations/covariances) or, to be more precise, from the fact that mean-variance optimization is commonly implemented without recognizing the parameters' uncertainty (Jorion, 1992; Kan and Zhou, 2007).

As indicated by Michaud (1989), ignoring the existence of measurement error in the optimization inputs leads to undesirable features for optimal portfolios. They can be summarized in their counter-intuitive nature, instability, un-uniqueness and poor out-of-sample performance. Several authors (Best and Grauer, 1991; Chopra and Ziemba, 1993; Jorion, 1986) have recognized serious influences on optimized portfolios especially from estimation errors in expected returns, while uncertainty in second moments is less critical.

Starting approximately from the second half of the eighties, different methodologies have been proposed to deal with the problem of estimation risk/errors. They can be distinguished between heuristic and Bayesian approaches. However all these methods keep constant the same and original framework while facing asset allocation problems: try to optimize the trade-off between the mean and the standard deviation (variance) of portfolio returns. The set of required inputs is also unchanged relative to Markowitz's framework.

An alternative way to address the problem of estimation risk is the use of asset allocation strategies that give less room to estimation error because they require fewer types of parameters to be estimated. Specifically, these strategies don't need expected returns as input in the portfolio construction process and therefore are distinct with respect to the classic mean-variance setting. We can simply label them as *μ-free strategies* given that μ traditionally denotes the expected returns. Different solutions corresponding to the above description have been suggested in the literature and have received increasing attention in the marketplace. In particular, emphasis has been put on the minimum-variance portfolio and on the equally weighted portfolio (DeMiguel, Garlappi and Uppal, 2009; Clarke, de Silva and Thorley, 2006, 2011).

With the global financial crisis started in 2008 we have seen a growing number of papers and a growing interest by institutional investors about a new *μ-free* asset allocation approach that is called risk parity strategy or equivalently risk parity portfolio. This approach suggests a portfolio composition such that each asset class contributes equally to portfolio risk. The novelty it brings in the portfolio construction process is to pay attention to risk allocation as recommended by the risk budgeting literature (Qian, 2005, 2006; Scherer, 2010).

Similarly to the studies performed by Maillard, Roncalli and Teiletche (2010), by Chaves et al. (2011, 2012) and by Anderson, Bianchi and

Goldberg (2012), our objective in this paper is to propose and evaluate a competition between risk parity strategy and other simplified asset allocation strategies. Unlike Anderson et al. (2012), we consider all and exclusively the *μ-free strategies* without admitting any intrusion (like a CAPM portfolio or mean-variance portfolios) because we are not interested in a comparison between portfolios that are ex-ante mean-variance efficient, but exposed to the mis-specification of the expected returns, and portfolios that show opposite features. Effectively, in this paper we deem it of interest to compare exclusively strategies that provide the same (partial) answer to the issue of estimation risk: remove the performance dimension in the portfolio construction process. Our set of rival strategies then includes risk parity, naive risk parity, minimum variance and equally weighting. Our main goal is to investigate if a strategy, the risk parity, that is more challenging and tricky on the computational side, consistently dominates the alternative strategies that have in common the absence of expected returns as input but distinguish themselves for the easy way they obtain solutions for portfolio weights. Differently from existing papers, we clearly highlight three different evaluation dimensions for our comparison (financial efficiency, diversification and asset allocation stability) and considerably expand the set of tools to be used to quantify these dimensions. Our empirical investigation is based on two datasets with different number of asset classes in order to understand if our results are extremely sensitive or not to the asset class inclusion decision. While implementing each *μ-free strategy*, we keep similar constraints (full investment, no short positions allowed) to make the competition fair.

The rest of the paper is organized as follows. First we describe the risk parity strategy and then we provide a brief review of the other *μ-free strategies*. After that a description of the datasets we use is given together with details about the way we implement our empirical investigation. Later we present the various tools we apply to put to the test risk parity strategy with respect to other *μ-free strategies* and discuss the results. We end with some conclusions.

1. Description of risk parity strategy

The idea of risk parity strategy is to identify portfolio weights in a way that asset classes contribute equally to the overall portfolio risk. To implement the strategy, it is helpful to be familiar with two important definitions. The first one is the marginal risk contribution. It tells us the variation caused in the portfolio risk (measured here as the standard deviation) by an infinitesimal change in an asset's weight. Considering a portfolio of N risky asset classes with individual weight w_i and individual volatility σ_i and covariance σ_{ij} between asset class i and j, the marginal risk (M_Risk_i) can be expressed as follows:

$$M\_Risk_i = \frac{\partial \sigma_P}{\partial w_i} = \frac{w_i \sigma_i^2 + \sum_{j \neq i}^{N} w_j \sigma_{ij}}{\sigma_P}. \tag{1}$$

The second definition refers to total risk contribution, also called component risk. It is the load on total risk contributed by the position w_i and is simply computed as the product of the allocation to asset class i and its marginal risk. Therefore it is given by (2):

$$Risk\_contribution_i = w_i \cdot \frac{\partial \sigma_P}{\partial w_i} = w_i \cdot \frac{w_i \sigma_i^2 + \sum_{j \neq i}^{N} w_j \sigma_{ij}}{\sigma_P}. \tag{2}$$

Since portfolio risk is a homogeneous function of degree 1, conditions for Euler's theorem are satisfied. Consequently, the overall portfolio risk can be expressed as the sum of component risks:

$$\sigma_P = \sum_{i=1}^{N} Risk\_contribution_i. \tag{3}$$

We can also show the percentage total risk contribution of each position ($PCTR_i$) by (4):

$$PCTR_i = \frac{w_i \cdot \frac{\partial \sigma_P}{\partial w_i}}{\sigma_P}. \tag{4}$$

Pursuing a risk parity strategy requires equalizing total risk contributions or component risks. Formally, this goal can be translated as follows:

$$w_i \cdot \frac{\partial \sigma_P}{\partial w_i} = w_j \cdot \frac{\partial \sigma_P}{\partial w_j} \qquad \forall i, j. \tag{5}$$

This is the reason why portfolios resulting from risk parity strategies are also called ERC portfolios, equal risk contributions portfolios. By definition, they are portfolios that include all the N asset classes in the selected investment universe. The weight assigned to an asset class becomes higher the lower is its volatility and correlation with other asset classes.

In presence of positivity and full investment constraints, finding closed-form solutions for optimal ERC portfolio weights is not possible due to an issue of endogeneity: w_i is a function of the risk contributions which, by definition, depends on w_i.

To this end, a numerical optimization is necessary[1]. We follow the optimization algorithm proposed by Maillard, Roncalli and Teiletche (2010):

$$\begin{aligned} &\underset{w^*}{Min}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w_i\cdot\frac{\partial\sigma_P}{\partial w_i}-w_j\cdot\frac{\partial\sigma_P}{\partial w_j}\right)^2 \\ &\text{s.t.} \\ &\sum_{i=1}^{N}w_i=1 \\ &0\le w_i\le 1. \end{aligned} \qquad (6)$$

The ERC portfolio is found when the function in (6), based on the variance of the risk contributions, is equal to zero.

Just in a special or hypothetical situation it is possible to carry out explicit solutions: the case where asset classes have the same pair-wise correlations. The optimal ERC portfolio weights would be then proportional to the inverse of the standard deviation. Obviously, in the real world, asset classes have pair-wise correlations that differ (different volatilities as well) but mentioning this special situation represents a good device to remember the existence and properties of the so called naive risk parity or simplified risk parity. Under this approach, optimal weights are computed as follows:

$$w_i=\frac{1/\sigma_i}{\sum_{i=1}^{N}1/\sigma_i}. \qquad (7)$$

With naive risk parity a true homogeneity in asset classes' contribution to portfolio standard deviation cannot be achieved because covariances/correlations are completely ignored.

2. Description of other *μ-free strategies*

As we said in the introduction, we also consider two additional *μ-free strategies*: the minimum-variance strategy and the equally weighting strategy.

Under the first strategy, portfolio weights are chosen through an optimization algorithm that requires only estimates of second moments. It can be written as follows:

$$\begin{aligned} &\underset{w^*}{Min}\sum_{i=1}^{N}\sum_{j=1}^{N}w_i w_j\sigma_{ij} \\ &\text{s.t.} \\ &\sum_{i=1}^{N}w_i=1 \\ &0\le w_i\le 1. \end{aligned} \qquad (8)$$

The resulting portfolio, usually defined as global minimum-variance portfolio (GMVP), is ex-ante the portfolio with the lowest possible standard deviation from which the efficient frontier starts. It is worth highlighting that minimum-variance strategy concentrates on reducing the overall portfolio risk as strongest as possible, but this doesn't mean it is diversified from the standpoint of component risks. In fact, the GMVP is characterized by homogenous marginal risks but according to (2) this doesn't imply that total risk contribution of all asset classes is identical.

The equally weighting strategy is the most straightforward to implement because it doesn't require an optimization model given that no objective function exists. It also doesn't need parameters estimates or any kind of additional information. In this case, the asset allocation problem is solved by simply assigning the same weight to each asset class, equal to $1/N$.

3. The empirical study: datasets and implementation

Our empirical study is based on two datasets. The first contains monthly returns for the 11 countries sub-indices that, until the end of 2012, were comprised in the index MSCI EMU calculated by Morgan Stanley Capital International (MSCI) for the Euro-zone geographical area. The sample period extends from January 1996 through December 2012, with a total of 204 monthly returns available. The second dataset includes monthly returns for the 5 countries sub-indices in the MSCI Emerging Markets Europe index calculated from February 1999 to January 2013, with a total of 168 observations. All returns are euro-denominated and calculated at total return level. The datasets characteristics are summarized in Table 1[2].

Table 1. Description of datasets used for the empirical investigation

Dataset	Description	Period	Frequency	Number of observations
DATASET 1	MSCI AUSTRIA, MSCI BELGIUM, MSCI FINLAND, MSCI FRANCE, MSCI GERMANY, MSCI GREECE, MSCI IRELAND, MSCI ITALY, MSCI NETHERLANDS, MSCI PORTUGAL, MSCI SPAIN	From 1/1996 to 12/2012	Monthly	204
DATASET 2	MSCI CZECH REPUBLIC, MSCI HUNGARY, MSCI POLAND, MSCI RUSSIA, MSCI TURKEY	From 2/1999 to 1/2013	Monthly	168

[1] Chaves et al. (2012) propose two algorithms to compute portfolio weights according to the risk parity strategy that do not involve optimization routines but they do not allow to explicitly insert constraints.
[2] The starting point for our time series coincides with euro-denominated sub-indices availability in Thomson-datastream.

We deeply explore the statistical properties of returns in the chosen datasets. Initially, for each dataset we estimate the first four moments (Table 2). The market conditions, in terms of return-risk combination, are rather different. Just to give an evidence, the highest mean return, on a monthly basis, for an asset class in the first dataset is 1.223%; it is 2.275% in the second dataset. The maximum monthly standard deviation is respectively 9.814% and 15.156%. The range of monthly volatilities is narrower for the first dataset; it goes from 5.538% to 9.814% while for the second dataset it goes from 7.122% to 15.156%.

Table 2. Descriptive univariate statistics on monthly returns

	Mean	Standard deviation	Skewness	Kurtosis
DATASET 1				
MSCI AUSTRIA	0.578%	6.740%	-1.047	6.288
MSCI BELGIUM	0.633%	5.721%	-1.590	8.267
MSCI FINLAND	1.223%	9.569%	0.137	4.469
MSCI FRANCE	0.745%	5.538%	-0.428	3.272
MSCI GERMANY	0.810%	6.529%	-0.545	4.673
MSCI GREECE	0.248%	9.814%	0.263	4.986
MSCI IRELAND	0.113%	6.321%	-0.783	3.844
MSCI ITALY	0.551%	6.332%	0.157	3.623
MSCI NETHERLANDS	0.698%	5.710%	-0.744	4.141
MSCI PORTUGAL	0.611%	5.822%	-0.331	4.076
MSCI SPAIN	0.990%	6.489%	-0.317	3.954
DATASET 2				
MSCI CZECH REPUBLIC	1.554%	7.122%	0.142	3.746
MSCI HUNGARY	0.855%	9.165%	-0.433	4.300
MSCI POLAND	0.941%	9.331%	0.013	3.444
MSCI RUSSIA	2.275%	11.707%	0.984	6.945
MSCI TURKEY	2.171%	15.156%	0.652	5.877

In Tables 3 and 4 we give the multivariate summary statistics on returns by reporting correlation matrices. Given that all equity indices in the tables are constituents of wider geographical area indices, not so surprisingly all pair-wise correlations are positive. Nevertheless, we observe a quite large variation.

Table 3. Correlation matrix for time series of monthly returns in dataset 1

	MSCI AUSTRIA	MSCI BELGIUM	MSCI FINLAND	MSCI FRANCE	MSCI GERMANY	MSCI GREECE	MSCI IRELAND	MSCI ITALY	MSCI NETHERLANDS	MSCI PORTUGAL	MSCI SPAIN
MSCI AUSTRIA	1	0.700	0.394	0.657	0.624	0.568	0.601	0.624	0.678	0.552	0.580
MSCI BELGIUM	0.700	1	0.450	0.758	0.713	0.525	0.663	0.686	0.805	0.630	0.633
MSCI FINLAND	0.394	0.450	1	0.698	0.648	0.417	0.488	0.596	0.613	0.530	0.541
MSCI FRANCE	0.657	0.758	0.698	1	0.909	0.593	0.638	0.860	0.888	0.726	0.811
MSCI GERMANY	0.624	0.713	0.648	0.909	1	0.567	0.634	0.771	0.854	0.662	0.735
MSCI GREECE	0.568	0.525	0.417	0.593	0.567	1	0.406	0.617	0.533	0.542	0.623
MSCI IRELAND	0.601	0.663	0.488	0.638	0.634	0.406	1	0.573	0.694	0.506	0.568
MSCI ITALY	0.624	0.686	0.596	0.860	0.771	0.617	0.573	1	0.771	0.700	0.796
MSCI NETHER-LANDS	0.678	0.805	0.613	0.888	0.854	0.533	0.694	0.771	1	0.657	0.733
MSCI PORTUGAL	0.552	0.630	0.530	0.726	0.662	0.542	0.506	0.700	0.657	1	0.741
MSCI SPAIN	0.580	0.633	0.541	0.811	0.735	0.623	0.568	0.796	0.733	0.741	1

Table 4. Correlation matrix for time series of monthly returns in dataset 2

	MSCI CZECH REPUBLIC	MSCI HUNGARY	MSCI POLAND	MSCI RUSSIA	MSCI TURKEY
MSCI CZECH REPUBLIC	**1**	0.665	0.675	0.402	0.374
MSCI HUNGARY	0.665	**1**	0.777	0.483	0.489
MSCI POLAND	0.675	0.777	**1**	0.503	0.482
MSCI RUSSIA	0.402	0.483	0.503	**1**	0.577
MSCI TURKEY	0.374	0.489	0.482	0.577	**1**

As a second step, we perform various tests of normality for the univariate returns distributions in the datasets. We apply the most widely used test due to Jarque and Bera (JB test), but we also use the Omnibus test that takes into account the finite sample bias and a test based on densities, the Lilliefors test (LI test). We summarize the results from testing for normality in Table 5. The results for acceptance of the null hypothesis of normality at 5% confidence level are shown in bold. The third step is to test for the null hypothesis that our datasets come from a multivariate normal distribution. We then perform the Mardia test and give the results in Table 5.

We reject in general the normality assumption for our univariate return series and strongly reject the multinormality hypothesis. However the departure from normality looks more severe for the dataset containing the sub-indices in the MSCI EMU. Additionally, those asset classes seem to be prevalently featured by negative skewness.

Table 5. Tests for normality on monthly returns

	JB test		Omnibus test		LI test	
	Stat.	p-value	Stat.	p-value	Stat.	p-value
DATASET 1						
MSCI AUSTRIA	129.214	0.100%	202284.061	0.000%	0.093	0.100%
MSCI BELGIUM	321.797	0.100%	510076.835	0.000%	0.112	0.100%
MSCI FINLAND	18.991	0.292%	57489.169	0.000%	0.065	3.402%
MSCI FRANCE	6.868	3.381%	50215.134	0.000%	0.065	3.402%
MSCI GERMANY	33.900	0.100%	81697.915	0.000%	0.077	0.490%
MSCI GREECE	35.874	0.100%	67695.911	0.000%	0.067	2.719%
MSCI IRELAND	26.918	0.106%	85276.182	0.000%	0.069	2.073%
MSCI ITALY	4.130	**9.580%**	47250.569	0.000%	0.049	**27.551%**
MSCI NETHERLANDS	29.903	0.100%	89407.722	0.000%	0.089	0.100%
MSCI PORTUGAL	13.571	0.714%	58667.053	0.000%	0.047	**34.071%**
MSCI SPAIN	11.148	1.152%	56323.361	0.000%	0.072	1.249%
DATASET 2						
MSCI CZECH REPUBLIC	4.464	**7.948%**	32069.806	0.000%	0.060	**15.035%**
MSCI HUNGARY	17.091	0.422%	44252.366	0.000%	0.053	**28.942%**
MSCI POLAND	1.383	**44.192%**	28753.401	0.000%	0.061	**13.331%**
MSCI RUSSIA	136.056	0.100%	134338.329	0.000%	0.079	1.184%
MSCI TURKEY	69.837	0.100%	75359.716	0.000%	0.088	0.315%

MULTIVARIATE STATISTICS			
	Stat.	Critical value	p-value
MARDIA test (dataset 1)	800.1227	3.2751	0.000%
MARDIA test (dataset 2)	494.3939	50.9984	0.000%

We believe that observing departure from normality is important in order to select the reward-to-variability ratios to use in the comparative evaluation of the selected μ*-free strategies* in terms of financial efficiency. In dealing with non-normal distributions, the exclusive application of the Sharpe ratio as in the previous literature would be inadequate. As argued by Farinelli and Tibiletti (2008) and Farinelli et al. (2008), ratios based on one-sided type measures can better serve the purpose.

Our empirical investigation consists of an out-of-sample analysis that relies on a rolling-window approach for the four μ*-free strategies* across the two datasets. Specifically, given a T-month-long dataset of asset class returns (with T = 204 and 168, respectively, for dataset 1 and 2) we use a rolling estimation window of 60 months. With these observations, we estimate the parameters required to implement a particular strategy that is variances/ standard deviations and covariances for risk parity and minimum variance, standard deviations/variances for naive risk parity[1]; obviously nothing is needed for the equally weighting strategy. The estimated parameters are then used to obtain optimal portfolio weights according to algorithms and algebraic procedure illustrated in equations (6), (7) and (8). These portfolio allocations are held for the next 6 months. Afterwards, we move the estimation window 6 months forward, re-estimate the parameters and determine the adjusted portfolio allocations[2]. Hence, in our empirical analysis *μ-free strategies* are rebalanced semi-annually. The procedure just described is repeated until the end of the sample period in each dataset. With this approach, 25 allocation experiments are reached when *μ-free strategies* refer to the first dataset. This number decreases to 19 for the second dataset. For all asset allocation experiments, we keep the same constraints: non-negative portfolio weights that sum to one. This is a necessary condition in order to compare the risk parity portfolios with other *μ-free strategies* in a fair way. The final outcome of our rolling-window based procedure is a series of 144 (for dataset 1) and 108 (for dataset 2) out-of-sample monthly returns generated by each of the *μ-free strategies* under investigation. On the basis of two different tests we have performed without reporting the results (Jarque Bera and Lilliefors), we can reject the hypothesis they follow a Gaussian distribution[3]. It is worth pointing out that these series do not suffer from a look-ahead bias. In other words, they come from portfolio allocations computed without using forward-looking information.

In Figures 1 and 2 (see Appendix) we can observe the time series of portfolio weights suggested for the different strategies. Whatever the dataset considered, the minimum-variance strategy, that will be the top performing, shows strong variations in portfolio weights. Obviously, this is not the case for equally weighting strategy. Risk parity shows portfolio compositions that seem a bit more changeable than naive risk parity.

4. Tools for evaluating risk parity strategy against other *μ-free strategies*

Our goal is to compare empirically the out-of-sample performance of risk parity allocation strategy relative to that of other *μ-free strategies* in order to understand if the former is consistently better. To address this issue, it is useful to keep in mind that investors look for several characteristics; they like efficient, diversified and stable portfolios. So it is not advisable to confine the comparative analysis to a single criteria, it is definitely better to consider different evaluation dimensions. They can be listed as follows: financial efficiency, diversification and asset allocation stability.

Even if three of the four asset allocation strategies we consider are entirely risk-based, we remember that this is motivated by the intention to mitigate estimation risk and this doesn't mean to completely disregard the reward achieved. So also proponents of *μ-free strategies* appreciate more to less efficiency. Therefore it is reasonable to acknowledge some space to risk-adjusted performance measures.

However, as a preliminary comparison we will explore separately the return and risk characteristics of the strategies. With reference to the first quantity, we merely propose measures of cumulative performance, geometric and arithmetic return. Concerning risk, we are aware that different ideas are plausible. Consequently, in addition to the traditional standard deviation of returns, we compute measures that focus on the variability of underperformance with respect to an exogenous threshold level. They are downside risk measures (see Nawrocki, 1999; and Sortino, Van der Meer and Platinga, 1999). Also drawdown measures are included in our analysis. Specifically, we consider the maximum drawdown and the average value of drawdowns we can count over the entire out-of-sample path for each strategy. In the end, we take into account forward looking measures of risk. In particular, we calculate both one-month VaR and one-month Expected Shortfall at 99% confidence level. Given that the out-of-sample returns from the selected *μ-free strategies* are not normally distributed we prefer to apply non parametric approaches and estimate VaR and Expected Shortfall using bootstrapped historical simulations with 10000 draws.

After that we will turn our attention to the category of reward-to-risk variability ratios. From this class, we select the most popular index that is the Sharpe Ratio and for its computation we use monthly returns from JPM Euro Cash 1Month index as a proxy for risk free rate.

Obviously Sharpe ratio is not the best fitting ratio in presence of non-Gaussian distributions so we also use ratios that modify the risk measure in order to integrate the downside risk or to consider as a penalty element the drawdowns encountered by each strategy. However, the violation of the normal distribution assumption is not the unique reason for

[1] An investigation of different estimation techniques for second moments is beyond the scope of our contribution. However, given that we detected serial correlation across squared returns, it would be interesting in further research to consider models of conditional variance and covariance.

[2] To give an example of the empirical procedure we use, consider the first asset allocation experiment with dataset 1: we use observations from January 1996 through December 2000 to estimate parameters on which are based the four portfolios (one for each strategy) that will be held (without rebalancing) from January 2001 to June 2001. At the following step, with observations from July 1996 through June 2001, we obtain portfolios to be held from July 2001 to December 2001.

[3] We had just an exception: the null hypothesis of normality was accepted for the minimum-variance strategy in the second dataset according to the Lilliefors test.

the selected risk-adjusted performance measures. Risk parity has attracted many institutional investors in conjunction with a growing fear for collapses, crashes. So, the stronger need for protection and the increasing demand for control of bad risk and not just for the instability around the mean are the rationale for the chosen ratios in the ex-post analysis.

According to the above reasoning we give space to the Sortino ratio, with threshold level or MAR (minimum acceptable return) represented by both the risk-free rate (as in the original version) and a null return and also to the Performance Ulcer Index proposed by Martin and McCann (1989). Denoting by $\overline{R}_k$ the arithmetic mean return for strategy *k*, by $\overline{R}_{rf}$ the risk free rate and by DD a drawdown, the Performance Ulcer Index for strategy *k* is defined as follows:

$$Performance\,Ulcer\,Index_k = \frac{\overline{R}_k - \overline{R}_{rf}}{\sqrt{\dfrac{\left(\sum_{t=1}^{T} DD_{k,t}^2\right)}{T}}}. \quad (9)$$

Even if these reward-to-variability ratios overcome the problem of "Gaussian dependency" and recognize the existence of good and bad risk, they don't allow to put a subjective and different emphasis on favorable events (overperforming a threshold) and unfavorable ones (underperforming a threshold) and they don't incorporate one-sided measure for reward evaluation. We can include these desirable properties by carrying out the calculation of the performance ratio proposed by Farinelli and Tibiletti (2008) that is given by the following general form:

$$\Phi_{threshold}^{p;q}\left(R_k\right) = \frac{E^{1/p}\left[\left\{\left(R_k - threshold\right)^{+}\right\}^{p}\right]}{E^{1/q}\left[\left\{\left(R_k - threshold\right)^{-}\right\}^{q}\right]}, \quad (10)$$

where p and q (> 0) are said, respectively, right and left orders. The higher the order the higher the weighting given to large deviations relative to small deviations above (in evaluating the reward) and under (in evaluating risk) the threshold level. We have chosen what can be considered two common parametrization of Farinelli-Tibiletti ratio: the Upside Potential ratio by Sortino, Van der Meer and Plantinga (1999) which corresponds to $\Phi_{threshold}^{1;2}$, and the Omega Index by Cascon, Keating and Shadwick (2002) which corresponds to $\Phi_{threshold}^{1;1}$. In both applications, MAR is our threshold level that we represent through the risk free rate. Keeping the threshold unchanged, we add the following third parametrization where $p = 0.5$ and $q = 2$:

$$\Phi_{threshold}^{0.5;2}\left(R_k\right) = \frac{\left[\frac{1}{T}\sum_{t=1}^{T}\sqrt{max\left(R_{t,k} - threshold;0\right)}\right]^2}{\sqrt{\frac{1}{T}\sum_{t=1}^{T} min\left(R_{t,k} - threshold;0\right)^2}}. \quad (11)$$

We believe this parametrization is particularly adept for very risk-averse investors and, as we said, this is likely a profile to be associated with investors in risk parity strategy. The growing interest for ERC portfolios in a period of financial turmoil suggests that risk parity portfolio investors are scared by huge losses and perceive them as catastrophic events; their desire for safety is much stronger that the desire for exceptional performance. Coherently, the parametrization in (11), relative to the Upside Potential ratio, lightens the gratification from over-performing a threshold.

The second evaluation dimension in our comparative analysis of *μ-free strategies* is represented by the level of diversification or, conversely, of concentration. This aspect is measured both in terms of weights and in terms of (percentage) total risk contributions for each strategy. It is captured by different tools. We start with the Shannon Entropy measure (*SE*)[1] that reaches its maximum value (ln *N)* in the case each asset weight or risk contribution is identical. The other extreme value is 0 and occurs when just an asset weight or risk contribution is equal to 1 and the rests are all zero. Therefore, when using the Shannon Entropy measure the higher the value, more diversified is the portfolio or the dissemination of risk among asset classes. This interpretation has to be reversed when different statistics are used. Actually we also report the Herfindahl index (*H*) and the Gini coefficient (*G*). The former is given by the sum of the squared asset allocation weights or by the sum of the squared total risk contributions but we adopt a normalized version[2] that ranges between 0% (perfect equality or diversification) and 100% (extreme inequality or concentration). For Gini coefficient computation we follow Chaves et al. (2012)[3]. Finally, with reference only to the asset class weights, we provide the diversification ratio proposed by Choueifaty and

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[1] It is defined as follows:

$SE(\pi) = -\sum_{i=1}^{N}\pi_i \ln \pi_i$, where π_i represents, alternatively, an asset class weight or a risk contribution.

[2] The normalized version of the Herfindahl index is given by: $\dfrac{\left(H - 1/N\right)}{1 - 1/N}$.

[3] First, weights or (percentage) total risk contributions are sorted in ascendant order and then the following quantity is computed: $\frac{2}{N}\sum_{i=1}^{N} i\left(w_i - \overline{w}\right)$ or $\frac{2}{N}\sum_{i=1}^{N} i\left(PCTR_i - \overline{PCTR}\right)$.

Coignard (2008)[1]. Two things need to be précised about reporting these statistics regarding the level of diversification/concentration. The first one is that, since we perform 25 and 19 asset allocation experiments for each strategy in our two datasets, we have to report average values. The second one is that we compute ex ante measures of diversification/ concentration of risk given that reported statistics refer to each rebalance date. In other words, we do not account for the possible varying behavior of these measures during the 6 months until portfolios are rebalanced again[2].

The last evaluation dimension included in our empirical analysis is that of portfolio stability. In order to capture this feature, we indicate, for each strategy and for each dataset, the average value across the asset allocation experiments of the turnover. For example, when the investment universe is represented by countries in the Eurozone, the average turnover for strategy k is defined as follows:

$$\frac{1}{24}\sum_{t\_reb=1}^{24}\sum_{i=1}^{11}\left(\left|w_{k,i,t\_reb^{+}.}-w_{k,i,t\_reb^{-}.}\right|\right), \tag{12}$$

where $w_{k,i,t\_reb^{-}}$ is the portfolio weight in asset class i under strategy k before re-estimation of portfolio allocation and $w_{k,i,t\_reb^{+}}$ is the suggested portfolio weight after rebalancing[3]. This quantity can be interpreted as the average percentage of portfolio wealth that needs to be traded at each rebalancing date in order to implement each *μ-free strategy*. In general, high values for turnover mean higher transaction costs. Together with the average turnover, we also give evidence of maximum turnover.

5. Results

We start by displaying in Table 6 the performance of the four *μ-free strategies*. To help better understand them, we make a preliminary note: the 12-year period from January 2001 to December 2012 at the back of the out-of-sample returns series for strategies based on Eurozone developed countries is characterized by downturns (excluding Austria, Belgium and Germany). On the contrary, the 9-year period from February 2004 to January 2013 covered up by the out-of-sample performances of strategies involving the emerging European equity markets shows strong up-movements (especially Czech Republic and Turkey).

Whatever the market conditions, the minimum-variance strategy has the best cumulative return (and consequently also the best annual and monthly compounded return), followed by risk parity strategy. The order between the two is reversed just if we consider the arithmetic mean return.

Table 6. Return statistics for *μ-free strategies*

	RISK PARITY	NAIVE RISK PARITY	MINIMUM VARIANCE	EQUALLY WEIGHTED
	DATASET 1: MSCI EMU EQUITY INDICES			
Cumulative performance	-11.597%	-16.156%	26.425%	-21.443%
Annualized geometric return	-1.022%	-1.458%	1.973%	-1.991%
Geometric mean return (monthly)	-0.086%	-0.122%	0.163%	-0.167%
Arithmetic mean return (monthly)	0.060%	0.025%	0.299%	-0.015%
	DATASET 2: MSCI EMERGING MARKETS EUROPE EQUITY INDICES			
Cumulative performance	191.231%	181.585%	201.887%	175.290%
Annualized geometric return	12.611%	12.191%	13.062%	11.909%
Geometric mean return (monthly)	0.995%	0.963%	1.028%	0.942%
Arithmetic mean return (monthly)	1.274%	1.242%	1.272%	1.237%

When we consider the risk profile of the *μ-free strategies* (Table 7), minimum-variance strategy again dominates in terms of standard deviation. In particular we can confirm and extend the order of volatilities documented by Maillard, Roncalli and Teiletche (2010):

$$\sigma_{Minimum-variance} \le \sigma_{Risk\ parity} \le \\ \le \sigma_{Naive\ risk\ parity} \le \sigma_{Equally\ weighting} \tag{13}$$

The ordering remains the same when we consider one-side measure of variability like the downside risk. On the basis of VaR and Expected shortfall, as well, the risk parity strategy is dominated by the minimum-variance strategy. The risk parity strategy is behind the minimum-variance strategy also in terms of drawdown statistics: the minimum-variance strategy shows a lower number of drawdowns, a smaller maximum drawdown and, in the case of dataset 1, also an inferior value for average drawdown.

[1] The diversification ratio is defined as follows:

$$\frac{\sum_{i=1}^{N} w_i\sigma_i}{\sqrt{\sum_{i=1}^{N} w_i^2\sigma_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} w_i w_j\sigma_{ij}}}.$$

[2] The measurement of ex-post diversification/concentration in percentage total risk contributions would be particularly interesting and feasible to implement with more observations available and with a longer time interval between two rebalancing dates.

[3] In equation (12), we have 24 instead of 25 terms in the average because at the starting time each portfolio is already allocated according to each asset allocation strategy.

Table 7. Risk statistics for μ-*free strategies*

	RISK PARITY	NAIVE RISK PARITY	MINIMUM VARIANCE	EQUALLY WEIGHTED
	DATASET 1: MSCI EMU EQUITY INDICES			
Monthly standard deviation	5.342%	5.374%	5.165%	5.460%
Monthly downside deviation (threshold = MAR = free risk)	4.242%	4.269%	3.982%	4.345%
Monthly downside deviation (threshold = 0%)	4.112%	4.136%	3.856%	4.214%
Var 1M (bootstrapped historical simulations – 10000, conf. level = 99%)	-15.313%	-15.745%	-15.311%	-15.168%
Expected shortfall 1M (bootstrapped historical simulations – 10000, conf. level = 99%)	-17.711%	-17.604%	-15.936%	-17.967%
Maximum drawdown	-62.415%	-60.870%	-61.810%	-62.266%
Number of drawdowns	123	128	115	127
Average drawdown	-32.090%	-31.536%	-29.657%	-32.581%
	DATASET 2: MSCI EMERGING MARKETS EUROPE EQUITY INDICES			
Monthly standard deviation	7.410%	7.411%	6.963%	7.616%
Monthly downside deviation (threshold = MAR = free risk)	5.114%	5.133%	4.568%	5.292%
Monthly downside deviation (threshold = 0%)	4.998%	5.017%	4.461%	5.175%
Var 1M (bootstrapped historical simulations – 10000)	-15.406%	-15.851%	-13.966%	-16.437%
Expected shortfall 1M (bootstrapped historical simulations – 10000)	-27.369%	-27.010%	-24.877%	-27.712%
Maximum drawdown	-66.604%	-66.309%	-60.757%	-67.461%
Number of drawdowns	81	84	79	83
Average drawdown	-23.831%	-23.259%	-24.306%	-24.506%

In Table 8 we report results about the first evaluation dimension we have chosen: financial efficiency. On the basis of the Sharpe ratio, we obtain results that contrast with those from Chaves et al. (2011, 2012). They documented, for a diversified investment universe of 10 asset classes, that in terms of Sharpe ratio the risk parity strategy (including its simplified version) was similar to equally weighted and outperformed the minimum-variance. They also observed that for a dataset including 28 commodity future sub-indices the Sharpe ratio of the minimum-variance was behind that of risk parity. Differently, with reference to both our datasets, the risk parity strategy does not overcome the minimum-variance strategy which shows the highest Sharpe ratio, but prevails both over naive risk parity and equally weighting. Our results are therefore more in line with those from Maillard, Roncalli and Teiletche (2010).

Table 8. Reward-to-variability ratios for μ-*free strategies* calculated on monthly out-of-sample returns

	RISK PARITY	NAIVE RISK PARITY	MINIMUM VARIANCE	EQUALLY WEIGHTED
	DATASET 1: MSCI EMU EQUITY INDICES			
Sharpe ratio	-0.0266	-0.0329	0.0187	-0.0398
Sortino ratio (threshold = MAR = free risk)	-0.0334	-0.0415	0.0242	-0.0500
Sortino ratio (threshold = MAR = 0%)	0.0147	0.0061	0.0775	-0.0036
Upside Potential ratio (threshold = MAR = free risk)	0.4759	0.4722	0.5271	0.4676
Farinelli-Tibiletti ratio (with parametrization p=0.5; q=2)	0.2313	0.2275	0.2662	0.2218
Omega Index (MAR = free risk)	0.9344	0.9193	1.0481	0.9034
Performance Ulcer index (or Martin ratio)	-0.0041	-0.0052	0.0031	-0.0061
	DATASET 2: MSCI EMERGING MARKETS EUROPE EQUITY INDICES			
Sharpe ratio	0.1485	0.1442	0.1577	0.1397
Sortino ratio (threshold = MAR = free risk)	0.2152	0.2082	0.2404	0.2010
Sortino ratio (threshold = MAR = 0%)	0.2549	0.2476	0.2851	0.2391
Upside potential ratio (threshold = MAR = free risk)	0.6808	0.6776	0.7075	0.6722
Farinelli-Tibiletti ratio (with parametrization p=0.5; q=2)	0.3685	0.3698	0.3455	0.3682
Omega Index (MAR = free risk)	1.4622	1.4437	1.5148	1.4267
Performance Ulcer index (or Martin ratio)	0.0448	0.0434	0.0449	0.0418

The hierarchy we have obtained for the Sharpe ratio within the two datasets is replicated when the downside risk is included in the risk-adjusted performance measure[1]. Once again the minimum-variance portfolio shows the highest Sortino ratio and the highest Upside Potential ratio followed by the risk parity strategy that slightly overcomes naive risk parity and equally weighting strategies. Also consideration for the Performance Ulcer index doesn't change this ordering. Through a closer examination of the ratios sharing the shape of the Farinelli-Tibiletti ratio, we note that even using

[1] In the case of dataset 1, where we have negative Sharpe ratio, the comparison based on reward-to-variability ratios using asymmetric measures of risk is extremely appropriate.

different pairs of parameters p and q, the ranking of the μ*-free strategies* on the basis of their financial efficiency is not revolutionized. But there is an "exception" when the parameter setting is $p = 0.5$ and $q = 2$ that seems particularly proper for a conservative risk profile. Actually, under this setting the risk parity strategy dominates both equally weighting and minimum-variance that takes the last position.

We now turn the attention to the issue of diversification (Table 9, 10). From the perspective of portfolio weights, the equally weighted strategy is, as expected, at the top with higher Shannon Entropy measure and lower Herfindhal index and Gini coefficient. The risk parity strategy tends to have a middle positioning while at the bottom we find the minimum-variance strategy. Actually, its lowest standard deviation comes, especially for the European emerging markets dataset, from concentration in few asset classes[1]. However, if we consider each strategy's power in reducing total (ex-ante) portfolio risk relative to the weighted average of individual risks, the competition restricts to risk parity and minimum-variance strategies. The risk parity strategy has diversification ratios of 1.442 and 1.292, respectively, with reference to the first and second dataset, the minimum-variance strategy has diversification ratios of 1.677 and 1.220.

Table 9. Levels of diversification in portfolio weights for μ*-free strategies* (average values)

	RISK PARITY	NAIVE RISK PARITY	MINIMUM VARIANCE	EQUALLY WEIGHTED
	DATASET 1: MSCI EMU EQUITY INDICES			
Shannon entropy measure	2.374	2.382	1.041	2.398
Normalized Herfindhal index	0.504%	0.315%	35.762%	0.000%
Gini coefficient	11.311%	9.995%	78.676%	9.291%
Diversification ratio	1.442	1.148	1.677	1.156
	DATASET 2: MSCI EMERGING MARKETS EUROPE EQUITY INDICES			
Shannon Entropy measure	1.582	1.593	0.531	1.609
Normalized Herfindhal index	13.171%	12.778%	67.325%	12.000%
Gini coefficient	11.976%	9.382%	9.211%	8.940%
Diversification ratio	1.292	1.165	1.220	1.167

We can argue that the risk contraction is based on different conditions for the two strategies. For the risk parity strategy, it relies on the maximum dissemination of risk among asset classes that contribute equally to the total risk. In the case of minimum-variance strategy, it is pursued by concentrating the risk load on a small number of asset classes selected according to the inverse of their individual risk and the inverse of correlation with other investment alternatives. We note that, in the case of the first dataset, generally 4 of the 11 asset classes are in the solution for the minimum-variance portfolio and there isn't a rebalancing experiment including more than 6 asset classes. For the second dataset, we generally have 3 of the 5 asset classes involved in the minimum-variance strategy and never more than 4. The comparisons based on the Herfindahl index and on the Gini coefficient confirm our reflections: they are at the extreme for risk parity strategy (with the lowest values) and the minimum-variance strategies (with the highest values). The same indication comes out from the Shannon Entropy measure that achieves its highest value for the risk parity strategy[2].

Table 10. Levels of diversification in risk allocations for μ*-free strategies* (average values)

	RISK PARITY	NAIVE RISK PARITY	MINIMUM VARIANCE	EQUALLY WEIGHTED
	DATASET 1: MSCI EMU EQUITY INDICES			
Shannon entropy measure	2.398	2.395	1.041	2.382
Normalized Herfindhal index	0.000%	0.057%	35.762%	0.327%
Gini coefficient	0.013%	4.219%	78.676%	8.890%
	DATASET 2: MSCI EMERGING MARKETS EUROPE EQUITY INDICES			
Shannon entropy measure	1.609	1.607	0.531	1.595
Normalized Herfindhal index	12.000%	12.112%	67.325%	12.626%
Gini coefficient	0.000%	3.863%	70.032%	4.187%

[1] The minimum-variance portfolio on the basis of dataset 2 is strongly invested in one asset class, that is MSCI Czech Republic, the sub-index with the lowest standard deviation.

[2] It is ln (11) = 2.398 for the first dataset, and ln(5) = 1.609 for the second dataset.

The last evaluation dimension is the asset allocation stability. We report in Table 11 both the average turnover and the maximum turnover. Compared to the minimum-variance strategy, risk parity portfolios dominate in both terms (with lower values). We observe, however, that the average turnover and, in the case of dataset 2, also the maximum turnover can be lower for the naive risk parity strategy.

Table 11. Average and maximum turnover for *μ-free strategies*

	RISK PARITY	NAIVE RISK PARITY	MINIMUM VARIANCE	EQUALLY WEIGHTED
	DATASET 1: MSCI EMU EQUITY INDICES			
Average turnover	7.633%	6.688%	34.130%	7.265%
Maximum turnover	15.883%	17.776%	88.960%	20.014%
	DATASET 2: MSCI EMERGING MARKETS EUROPE EQUITY INDICES			
Average turnover	8.886%	7.988%	20.441%	8.216%
Maximum turnover	18.237%	17.113%	56.061%	17.493%

Conclusion

In recent years, risk parity strategy has increased in popularity. In this article, we have proposed an empirical investigation in order to compare this approach of investing with other asset allocation strategies that have in common the fact they don't require expected returns as input for the portfolio construction process; namely naive risk parity, minimum-variance and equally weighting. This comparison has been undertaken using two different datasets composed by MSCI Emu and MSCI EM Europe sub-indices. Given this choice, the empirical investigation has been performed under different market conditions, over periods of slightly different length and considering a different number of asset classes included in the investment universe.

Differently from the existing literature, in our comparison we have distinguished three evaluation dimensions of the out-of-sample return series from the selected strategies: financial efficiency, level of diversification and asset allocation stability. Further, for each evaluation dimension we have computed a broad and comprehensive set of measures. We are not aware of any other contribution on the subject with a similar expansion of the evaluation tools.

Generally speaking, our findings are in line, more robust and more well-proven than those achieved by Maillard, Roncalli e Teiletche (2010), while tend to contradict with Chaves et al. (2011, 2012). In terms of financial efficiency, we cannot consistently acknowledge the superiority of risk parity strategy, rather we can say it has a middle positioning whatever the risk-adjusted performance measure we use, in the sense that it persistently prevails over naive risk parity and equally weighting but is dominated by minimum-variance. However, when we put risk parity strategy to the test on the basis of the other evaluation dimensions, it shows comparatively various points of strength: the proportions allocated to asset classes are not extremely fluctuating across the rebalancing dates as it happens with minimum-variance; at the same time risk parity can overcome the rival in terms of diversification ratio. What's more, it can achieve this result in a sound way that is with the maximum dissemination of risk among asset classes and with a saving on transaction costs coming from low turnover.

We believe that the evidence we have provided in this article can be of practical utility. From the point of view of the investment industry, it should impact the selection of methodologies for building alternative-weighted equity indices other than the common capitalization-weighted indices. For institutional investors (like pension funds), our findings can legitimate risk parity as an option to consider for the strategic asset allocation if they decide to pursue an approach different from the classical Markowitz's approach.

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Appendix

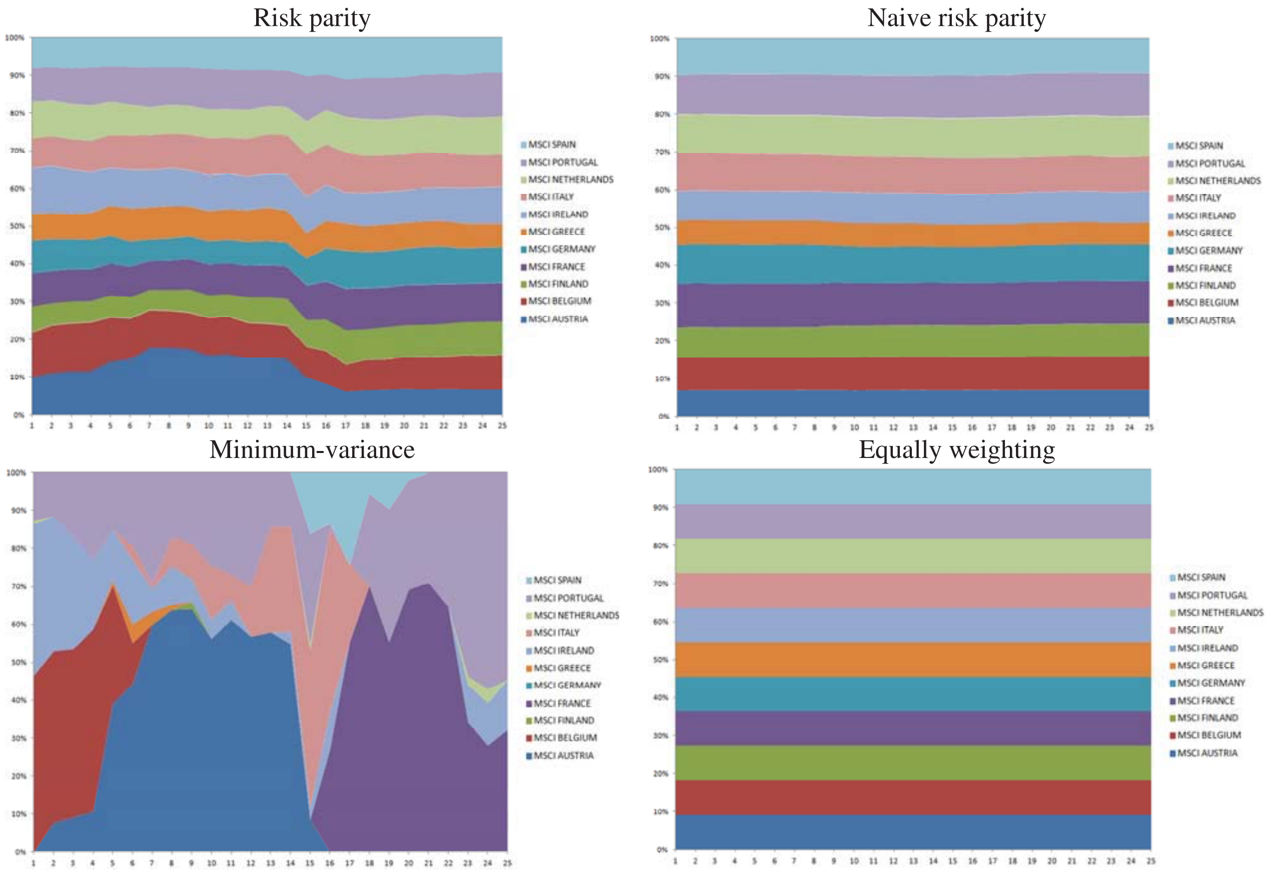

Fig. 1. Portfolio weights for *μ-free strategies* (dataset 1)

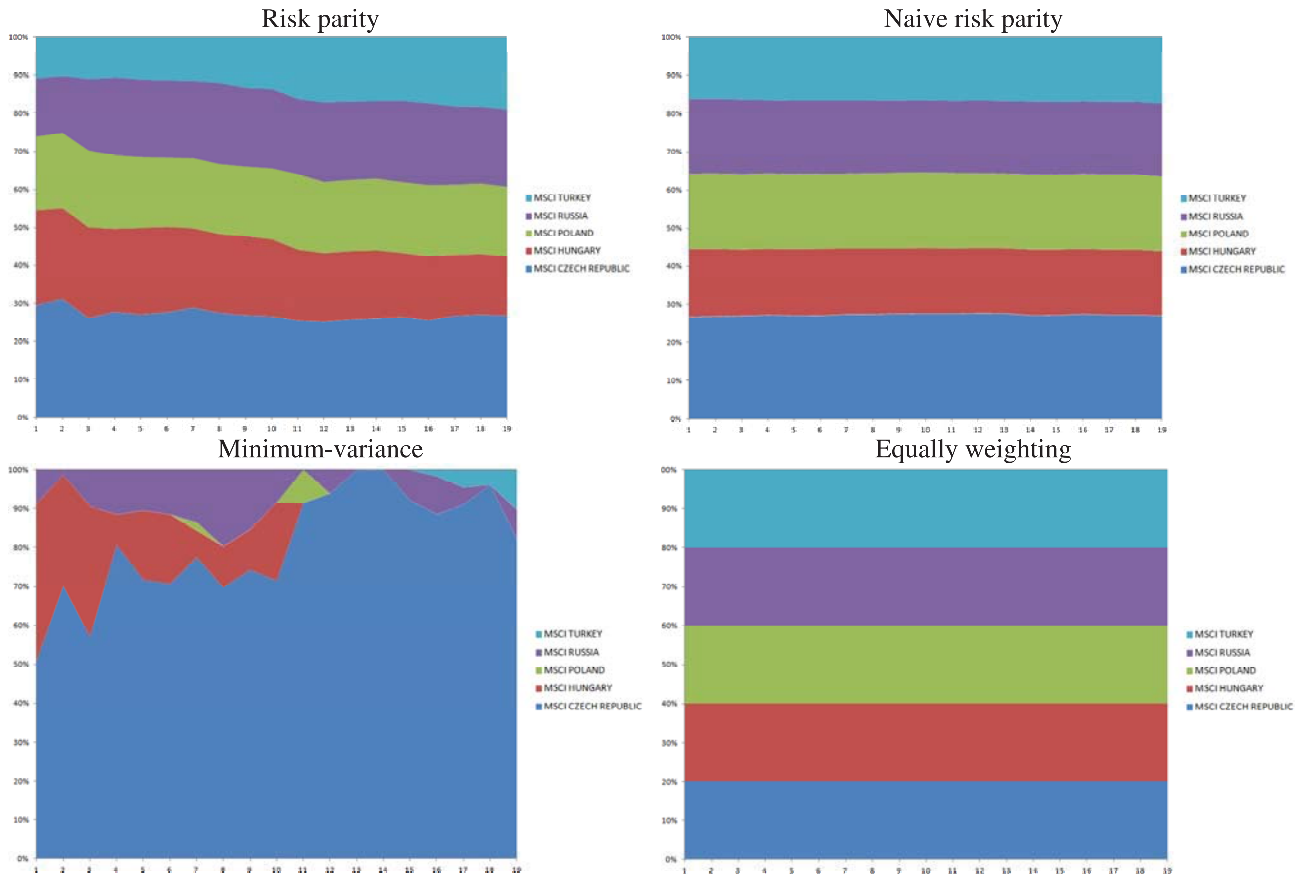

Fig. 2. Portfolio weights for *μ-free strategies* (dataset 2)