“Optimal behavior strategy in the GMIB product”

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ARTICLE INFO	Aymeric Kalife, Gabriela López Ruiz, Saad Mouti and Xiaolu Tan (2018). Optimal behavior strategy in the GMIB product. *Insurance Markets and Companies*, *9*(1), 41-69. doi:10.21511/ins.09(1).2018.05
DOI	http://dx.doi.org/10.21511/ins.09(1).2018.05
RELEASED ON	Wednesday, 26 September 2018
RECEIVED ON	Tuesday, 27 February 2018
ACCEPTED ON	Tuesday, 18 September 2018
LICENSE	This work is licensed under a Creative Commons Attribution 4.0 International License
JOURNAL	"Insurance Markets and Companies"
ISSN PRINT	2616-3551
ISSN ONLINE	2522-9591
PUBLISHER	LLC “Consulting Publishing Company “Business Perspectives”
FOUNDER	LLC “Consulting Publishing Company “Business Perspectives”

NUMBER OF REFERENCES	NUMBER OF FIGURES	NUMBER OF TABLES
22	**10**	**5**

BUSINESS PERSPECTIVES

LLC "CPC "Business Perspectives"
Hryhorii Skovoroda lane, 10, Sumy,
40022, Ukraine

www.businessperspectives.org

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OPTIMAL BEHAVIOR STRATEGY IN THE GMIB PRODUCT

Abstract

Guaranteed Minimum Income Benefit are variable annuities contract, which offer the policyholder the possibility to convert the guarantee level into an annuities income for life. This paper focuses on the optimal customer behavior assuming the maximization of the discounted expected future cash flows over the full life of the contract duration. Using convenient scaling properties of the contract value enables to reduce the complexity (dimension) of the problem and to characterize the policyholder's decision as a function of the contract moneyness across four main choices: zero withdrawals, guaranteed withdrawals, lapse and the income period election. Sensitivities to key drivers such as the market volatility, the interest rate and the roll-up rate illustrate how crucial are not only the environment, but also the product design features, in order to ensure a fair and robust pricing for both customer and life insurer. In particular, the authors find that most empirical contracts are usually underpriced compared to mean optimal behavior pricing, which empirically translated into multiple updates of behavior assumptions and re-reserving by life insurers in the recent years.

Received on: 27th of February, 2018
Accepted on: 18th of September, 2018

Keywords GMIB, variable annuities, rational behavior, optimal withdrawals, PDE, dynamic programming

JEL Classification C61, G22, G41

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INTRODUCTION

The world's older population is growing rapidly. According to data published in 2015 by the United Nations, there was a substantial increase of 48% (from 607 to 901 million) of people aged 60 or over between 2000 and 2015. And by 2050, the population aged 60 and over might reach nearly 2.1 billion.

Moreover, the "oldest-old" (aged 80 or over) population accounted for 14% of old population (aged 60 or older) in 2015, and is expected to triple 2015's value by 2050. As a result of these demographic shifts, longer life expectancy, increasing lifestyle and health-care costs, the idea that individuals and households need to plan for their own retirement is gaining a lot of attention. On the other hand, low interest rates are putting pressure on the insurance sector, pushing providers and consumers alike to look for ways to make themost of their assets.

This situation has led insurance companies to offer a range of savings products indexed on financial assets (stocks, funds, government bonds...), so-called unit linked products, exposing policyholders to financial markets and providing them with different ways to consolidate investment performance over time, as well as protection against mortality-related risks. However, in order to make these products more attractive, insurers started to offer complex guarantees. The most famous examples of such contracts are variable annuities.

Variable annuities are unit linked or managed fund vehicles, which offer optional guarantee benefits against investment and mortality/longevity risks for the customer. These guarantees are usually referred to as GMxB, where x stands for the class of benefits involved. Specifically, the Guaranteed Minimum Income Benefits (GMIB), launched in the 1990's, provide policyholders the right to convert the benefit base at the end of a so-called "deferral period" into annuities for life, with a constant rate fixed at inception. Generally speaking, the value of the benefit base is not less than the initial account value paid by policyholders. Due to enduring competition, most insurers have added some "features" for these guarantees: for example, the benefit base can be reset to the high-water mark of the account value on anniversary dates (step-up or ratchet) when the market has performed well, or can roll up with a fixed percentage (known as roll-up rate, e.g. 2%), regardless of the market conditions.

In these variable annuities, policyholders' behavior is a major risk for the insurer, and a complex issue that affects life insurance industry in almost every aspect: product design, pricing, marketing and distribution, financial reporting and risk management. The recent Quantitative Impact Study (QIS 5) of the Solvency II framework showed that behavior risk is the most important risk among life underwriting risks for variable annuities, as illustrated by solvency issues experienced by the policyholder run in the late 1980's. As the behavior risk assumptions may significantly impact the profitability, a rigorous modeling framework of the behavior risks is necessary.

Traditionally, the customers' behavior has been modeled by historical or backward looking statistical regressions, which have empirically underestimated the risk due to the scarcity of extreme scenario samples for these new products and the inability to dynamically extrapolate the observed behavior to various market conditions. In contrast, a "rational" behavior strategy valuation is a prudent forward-looking approach, where policyholders lapse in a way that maximizes the net present value of the future cash flows, depending on key drivers such as market conditions, as illustrated by the past observations, especially during the sub-prime crisis since 2008: when the interest rate goes up, the discounted value of the insurer's annuities decreases, so if the portfolio value rises at the same time, then the guarantees become worthless and more customers should surrender their policies to get back the account value; in the opposite situation, where both interest rate and account value decrease, the contract becomes much more attractive, thus leading to a reduced lapse rate.

Naturally, customers are certainly not able to calculate the rational exercise times associated to such very complex products. Still, rational behavior reflects a potential extreme "herd" behavior, as experienced in the last 2008 market crash, with an initial immediate and sustained fall in lapses right after the crash, before an abrupt recovery consistent with the interest rates. In contrast, dynamic lapses modeling are usually unable to provide such empirical dynamics. Besides, competitors can take advantage of new market conditions to propose more attractive products, thus influencing their behavior indirectly, as mostly illustrated since 2010.

Regarding the GMIB variable annuity on which we focus in this paper, rational behavior is not limited only to lapsing the contract, but also to withdrawing money without necessarily terminating the contract, i.e. partial withdrawals at each anniversary date. The solution of the optimal withdrawals problem allows the insurer to mitigate policyholders' behavior risk related to GMIB contracts. Our analysis points out the following observations:

- the rational withdrawal strategy by the policyholder entails 4 specific behaviors only: zero withdrawal, guaranteed withdrawal, income benefit election and lapse. Specifically, withdrawals of medium size do not seem to be among rational decisions;

- given the significant fees levels, the contract seems to be underpriced for most cases. Either increasing the fees or adjusting the roll-up rate can be a solution to overcome this issue.

http://dx.doi.org/10.21511/ins.09(1).2018.05

1. LITERATURE REVIEW

There is a large literature on pricing and hedging variable annuities guarantees. Most of it addresses the individual variable annuities contracts. Milevsky and Posner (2001) price a GMDB contract using the usual risk-neutral valuation theory. Gerber and Shiu (2003) exploit the closed-form solution of European look-back options to price complex guarantees embedded in some equity-linked annuities. Milevsky and Salisbury (2005) study the impact of policyholder behavior on the cost and value of the GMWB rider and argue that the current pricing is not sustainable. An analysis of the design of gen- eral equity-indexed annuities from the investor's perspective and a generalization of the conventional design are proposed in the paper by Boyle and Tian (2008).

The optimal behavior approach in a GMWB valuation was formalized in the work by Dai et al. (2008). They develop a singular stochastic control problem in a continuous framework, and also construct discrete pricing formulation that models withdrawals on discrete dates. In Bauer, Kling, and Russ (2008), the authors develop an extensive and comprehensive framework to price any of the common guarantees available with VAs, using Monte Carlo simulations in deterministic withdrawals scenarios. On the other hand, Chen, Vetzal, and Forsyth (2008), in their work explore the effect of various modeling assumptions on the optimal withdrawals strategy of the policyholder, and examine the impact on the guarantee value under sub-optimal withdrawals behavior. Shah and Bertsimas (2008) analyze the GLWB option in a time continuous framework considering simplified assumptions on population mortality, and adopting different asset pricing models. In the paper by Bacinello et al. (2011), a number of guarantees under a more general financial model with stochastic interest rates, volatility, and mortality are considered. A utility-based approach (see Gao & Ulm, 2012) is used to study the valuation of the GMDB rider.

Holz et al. (2012) price GLWB contracts for different product designs and model parameters under the geometric Brownian Motion dynamic. They consider various policyholders behaviors assumptions including deterministic, probabilistic and stochastic models. The GMIB is studied in the work by Deelstra and Rayée (2013) under a local volatility framework. The authors argue that an appropriate volatility modeling is important to the long-dated guarantees. Finally, Dai and Yang (2013) develop a tree model to price the GMWB rider embedded in deferred life annuity contracts. Other papers investigate the impact of volatility risk, or assess the mortality risk in GLWB, or analyze equity and systematic mortality risks, see for example in the work by Fung et al. (2014), Graf et al. (2011), Piscopo and Haberman (2011). Recently, the work by Shevchenko and Luo (2016) provides a useful general framework to price different living and death guarantees. They use a direct integration method to solve the problem and compare it to PDE-based methods. In the following, we will focus on the PDE method for the GMIB product. Our goal is to be able to analyze the impact of different market drivers and product design on policy holders' behaviors and the value of the contract. The remainder of this paper is organized as follows. Section 2 details the GMIB product. Section 3 introduces the modeling assumptions, while Section 4 provides the valuation of GMIB product from a rational framework perspective, where some convenient scaling properties enable to reduce the dimension of the pricing problem. Section 5 focuses on the results depending on specific product designs and market drivers (interest rates, volatility). The last section is the conclusion.

2. PRODUCT DESCRIPTION

Guaranteed Minimum Income Benefit (GMIB) product appeared in the market in 1996. This guarantee enables policyholders to make annual partial withdrawals (typically from 4% to 7%) of their guaranteed protection amount and ensures an analogous percentage of the GMIB benefit base for their entire lifetime, no matter how the investments in the sub-accounts perform. It combines longevity protection with withdrawals flexibility, hence, it is seen as a "second generation" guarantee. The guarantee can concern one or two lives (typically spouses). Each annual withdrawal does not exceed some maximum value, but it is evident that the total amount of withdrawals is limited only to the exhaustion of the client's account value. Annual withdrawals of about 5% of the (single initial) premium are commonly guaranteed for insured aged 60+. In case of death, any remaining fund value is paid to the insured dependents. To satisfy the new needs of an ageing population, insurance companies have started offering a lifetime benefit feature with GMIB. We will illustrate in the following section a typical commercialized GMIB product.

2.1. Beneftt base and partial withdrawals

To describe the benefit features of this GMIB clearly, some key terms must be addressed: GMIB benefit base and guaranteed withdrawals amount (GWA).

- GMIB benefit base: The GMIB benefit base is an amount used to determine the guaranteed annual withdrawals amount and lifetime payments. The GMIB benefit base is created and increased by allocations and transfers to the account value, as well as annual withdrawn amounts. This percentage is know as roll-up rate and deferral roll-up rate. It must be noticed that the GMIB benefit base is a "fictive" amount, i.e., it can not be considered as an account nor cash value, only as a reference in the calculation of lifetime payments and guaranteed withdrawals amounts.

- (Annual) guaranteed withdrawals amount (GWA): The "annual guaranteed withdrawals amount" is the withdrawals amount suggested by the insurer. It is equal to the annual roll-up rate in effect on the first day of the contract year, multiplied by the current benefit base. It is also the maximum amount upon which the benefit base is reduced without penalty, in contrast to excess withdrawals.

In Table 1, the authors illustrate a concrete example of the calculations of a GMIB benefit. BB designates the benefit base and AV refers to the account value. The contract initial premium is \$100,000. The policyholder does not make any withdrawals till the 6th contract year and once he/she does, all withdrawals stay within the boundaries of the GWA. Therefore, the GMIB benefit base does not diminish. The effect of excess withdrawals will be discussed later. Till the 5th contract anniversary, the deferral roll-up rate is used to calculate the amount, which is credited to the benefit base each year, since no withdrawals has been made. For example, the GMIB benefit base in the 5th year is calculated by taking the value of the previous year and adding the corresponding 5%, i.e.

$$\$120,510+5.4\%\cdot\$120,510=$$
$$=\$120,510+6,507.54=127,017.54.$$

Once the client proceeded to his/her first withdrawal, the roll-up rate determines the evolution of the GMIB benefit base and the annual GWA. If the maximum guaranteed quantity is withdrawn, the benefit base remains unchanged, as shown in years 6 to 9, i.e. the same quantity withdrawn from the benefit base is added by the roll-up amount. In case of withdrawing less than the annual GWA, a greater value for the benefit base is obtained. In the 10th year, the policyholder only withdraws 4%

Table 1. GMIB benefit base evolution for an allocation of \$100,000 with partial withdrawals that does not affect the value of the guaranteed account

Year	Deferral/Roll-up rate	GMIB BB	WA	Percentage of GMIB BB	GMIB BB after withdrawals
0	–	\$100,000	\$0	0%	\$100,000
1	4.8%	\$104,800	\$0	0%	\$104,800
2	4.3%	\$109,830	\$0	0%	\$109,830.40
3	5.2%	\$114,553	\$0	0%	\$114,553.11
4	5.4%	\$120,510	\$0	0%	\$120,509.87
5	5.0%	\$127,017	\$0	0%	\$127,017.40
6	4.7%	\$133,368	\$6,268.31	4.7%	\$127,099.96
7	5.2%	\$133,368	\$6,935.15	5.2%	\$126,433.12
8	5.4%	\$133,368	\$7,201.89	5.4%	\$126,166.39
9	6.0%	\$133,368	\$5,334.73	4.0%	\$128,033.54
10	7.3%	\$136,036	\$5,441.43	4.0%	\$130,594.21

http://dx.doi.org/10.21511/ins.09(1).2018.05

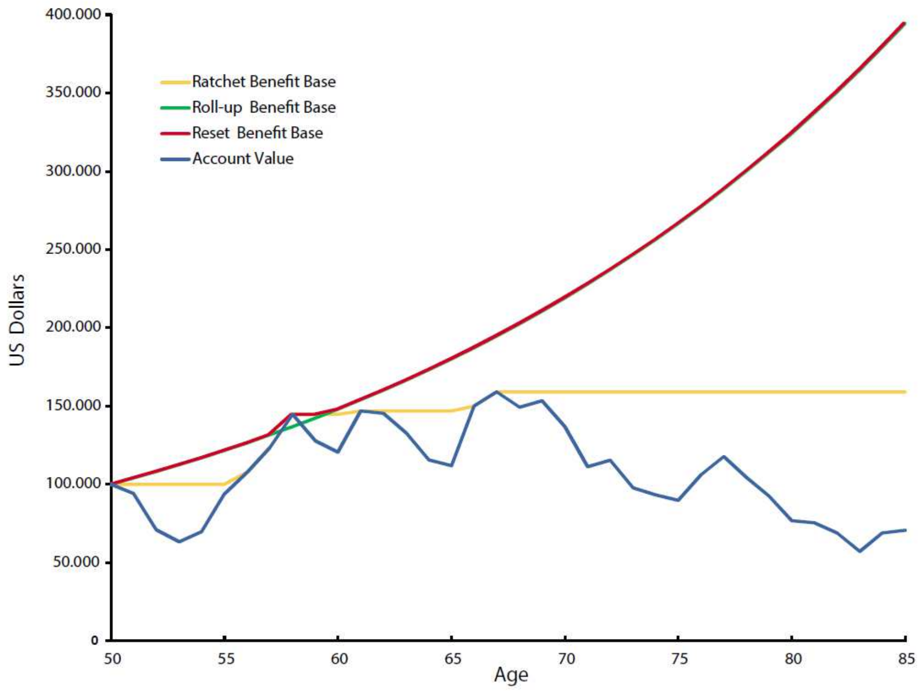

Figure 1. Behavior of the account value and the different options for the GMIB benefit base when no withdrawals have been performed

of the GMIB guaranteed value. Consequently, the benefit base becomes

$$\$133,368 - \$5,334.73 + \$133,368 \cdot 6.0\% =$$
$$= \$128,033.54 + \$8002.08 = \$136,035.62.$$

The step-up option enables the policyholder to reset the guaranteed withdrawals balance to the current higher account value when investment performance is strong. By choosing to reset the benefit base, the policyholder is able to increase the total benefit amount and the annual guaranteed withdrawals amount. The option may reduce the inflation effect on incomes when the account value goes up and the step-up option is available. Accordingly, the period over which lifetime payments can begin is extended of 10 years. Nevertheless, at policyholder's 95th birthday, the lifetime payments are set to automatically begin no matter how many times the reset option has been chosen. In Figure 1, which illustrates the evolution of the protected account value, and the different options for the GMIB benefit base, the contract is issued at the policyholder's 50th birthday. Partial withdrawals from this annuity contract are taxable as ordinary income and, if made prior to age 591/2, may be subject to an additional 10% federal tax and withdrawals charges. All amounts invested in the annuity's portfolios are subject to fluctuation in value and market risk, including loss of principal. The account value may be reduced due to fees and charges such as operations and sales charges, administrative fees, and optional benefits additional charges.

2.2. Fees and charges

The fee structure has an impact on the GMIB price. The GMIB charge is deducted from the contract value periodically. It is usually presented as a percentage of the current account value, although it can also be a percentage of the initial premium, a percentage of the remaining guaranteed benefit amount, or the greater of these two. The annual charge ranges for a GMIB product are comprised between 20 and 75 basis points depending on the nature of the benefit.

2.3. Lifetime payments

Lifetime GMIBs provide guaranteed annual income until death. Policyholders are also able to access potentially increased account values, and control the asset allocation in ways that the traditional variable annuitization normally does not allow. Lifetime GMIBs usually have two options: single life or joint spousal life. For the single life option, the benefit payments end at the death of the person covered. For the joint spousal life option, the benefit payments end when the remaining spouse dies. The fee rate for the single life option ranges from 25 to 55 basis points, while the spousal life option tends to be 10-20 basis points higher.

For the single life option, a spouse continuation option is available upon the first death with the same charge, but the account value and the benefit amount may be adjusted. For the joint spousal life, there will be no recalculation of the benefit amount when the first death occurs. The annual benefit payment amount is a percentage of the initial guaranteed benefit amount. The older the policyholder is, the larger the value of the lifetime payments will be. For example, the guaranteed factor to calculate the lifetime payment for the legacy product Accumulator 7 is 5.3% if the annuity starts at age 73 and 7.1% if the attained age is 83. The factor used to calculate the lifetime payments is given by the insurer and depends on the policyholder's age at inception.

Annual lifetime payments in GMIB products begin as follows:

- the next contract year following the date the account value falls to zero;
- the contract date anniversary following the policyholder's 95th birthday;
- the policyholder's election to exercise the GMIB.

Similarly to GLWB product, GMIB is subject to a waiting period, which begins on the date when the account value is first found, and it ranges from 10 to 15 years depending on the policyholder's age.

If an excess withdrawal, i.e. withdrawals superior to the guaranteed withdrawals amount, reduces the account value to zero, the GMIB will be terminated. Even if an excess withdrawal does not cause the contract to terminate, it can greatly reduce the GMIB benefit base and the value of the benefit, since it is done on a pro rata basis.

In Table 2, cash flows for a maximum annual guaranteed withdrawals strategy are shown. The current and guaranteed factors1 are illustrated for the case of a 60 year old male who acquires the contract in 2016. These factors take into account the age of the policyholder when the contract is issued, his/her gender and probability of survival. In the particular case of the current annuity factor, the market interest rates play a key role. A constant net return of 3% and fees of 4.5% have been considered to facilitate comprehension. The factors need to be recalculated when they are faced with changing market interest rates.

In the case illustrated in Table 2, the policyholder takes the annual guaranteed withdrawals amount till his account value turns to zero at age 76. At that moment, lifetime payments begin and the owner of the policy faces two options: to annuitize the GMIB benefit base or the account value. Annual payments will be based on the guaranteed or current factor depending on the policyholder's choice. This is possible till the client's 85th anniversary, otherwise he will lose the possibility to transform the contracts benefit base into annual income payments. In the described scenario and taking the discount factors into account, the policyholder will obtain $86,981.33, meaning he will not recover the initial premium invested in the contract.

2.4. Death benefit

Insurance companies also offer the possibility of combining the GMIB with Guaranteed Minimum Death Benefit (GMDB) when the contract is purchased. This is not unusual, since variable annuities typically provide a guarantee if the policyholder dies before receiving any income.

The death benefit often equals the greater of the account value and total premiums paid less than any withdrawals. For example, a person had paid premiums totaling $100,000, and had made withdrawals equaling $15,000. The account value stands at $80,000 because of these withdrawals and investment losses. If he were to die, his beneficiary would receive the aforementioned quantity.

http://dx.doi.org/10.21511/ins.09(1).2018.05

Table 2. Protected account value and GMIB behavior given a static withdrawals strategy and lifetime payments[1]

Contract year	AV	Roll-up rate	WA	GMIB BB	GMIB factor	Current factor	Annuitization BB	Annuitization AV	Lifetime payments
0	$100,000		$0	$100,000	4.2%	5.4%	$4,204.79	$5,367.16	–
1	$99,500	6%	$6,000	$100,000	4.3%	5.5%	$4,269.15	$5,444.66	–
2	$93,032.50	6%	$6,000	$100,000	4.3%	5.6%	$4,336.57	$5,193.53	–
3	$86,597.34	6%	$6,000	$100,000	4.4%	5.7%	$4,407.24	$4,935.07	–
4	$80,194.35	6%	$6,000	$100,000	4.5%	5.8%	$4,481.31	$4,668.45	–
5	$73,823.38	6%	$6,000	$100,000	4.6%	6.0%	$4,558.97	$4,392.83	–
6	$67,484.26	6%	$6,000	$100,000	4.6%	6.1%	$4,640.42	$4,107.29	–
7	$61,176.84	6%	$6,000	$100,000	4.7%	6.2%	$4,725.87	$3,810.86	–
8	$54,900.96	6%	$6,000	$100,000	4.8%	6.4%	$4,815.53	$3,502.52	–
9	$48,656.45	6%	$6,000	$100,000	4.9%	6.5%	$4,909.63	$3,181.14	–
10	$42,443.17	6%	$6,000	$100,000	5.0%	6.7%	$5,008.38	$2,845.48	–
11	$36,260.95	6%	$6,000	$100,000	5.1%	6.9%	$5,111.95	$2,494.23	–
12	$30,109.65	6%	$6,000	$100,000	5.2%	7.1%	$5,220.49	$2,126	–
13	$23,989.10	6%	$6,000	$100,000	5.3%	7.3%	$5,334.10	$1,739.42	–
14	$17,899.16	6%	$6,000	$100,000	5.5%	7.4%	$5,452.86	$1,333.15	–
15	$11,839.66	6%	$6,000	$100,000	5.6%	7.7%	$5,576.81	$905.97	–
16	$5,810.46	6%	$5,810.46	$100,189.54	5.7%	7.9%	$5,716.80	$456.79	–
17	$0	6%	$0	$0	5.8%	8.1%	$0	$0,00	$5,716.80
18	$0	6%	$0	$0	6.0%	8.3%	$0	$0,00	$5,716.80
19	$0	6%	$0	$0	6.1%	8.5%	$0	$0,00	$5,716.80
20	$0	6%	$0	$0	6.3%	8.7%	$0	$0	$5,716.80

The combo variable annuities products GMI DB also offer optional death benefits in the form of roll-up or annual ratchet and reset with extra charges. These options are:

- Highest anniversary value death benefit – The "highest anniversary value death benefit" is an optional guaranteed minimum death benefit in connection with the account value. The death benefit is calculated using the highest value of the account on the contract date anniversary.

- Roll-up to age 85 benefit base – The "roll-up to age 85 benefit base" is equal to the GMIB benefit base, i.e. it is reduced dollar-for-dollar in the case of partial withdrawals being done within the limits of the guaranteed withdrawals amount and pro rata, when an excess withdrawal has been made by the policyholder. It is favored by the roll-up amount till the policyholder's 85th birthday. This option is tied only to "Greater of" death benefit, i.e. it can not be chosen individually.

- "Greater of" death benefit – The "greater of" death benefit is an optional guaranteed minimum death benefit in connection with the protected benefit account value only. The death benefit is calculated using the greater of two benefit bases- the greater of the roll-up to age 85 benefit base and the highest anniversary value benefit base. There is an additional charge for the "Greater of" death benefit under the contract.

Once the lifetime payments corresponding to the GMIB start, the policyholder loses the possibility of keeping the GMDB. This right is lost at policyholder's 95th anniversary since the lifetime payments start automatically. The return of principal, highest anniversary value, and "Greater of" guaranteed minimum death benefits will terminate without value if the account value falls to zero as a result of withdrawals or payment of any applicable charges. This will happen whether the policyholder elects the GMIB or receive lifetime GMIB payments or not.

Policyholders can elect the optional death benefit guarantees between age 20 and 68, implying

1 The current (resp. guaranteed) factor is the annual income rate when the income benefit is based on the account value (resp. benefit base).

that this product targets a "younger" sector of the population compared to the "return of principal", which can be chosen till age 80.

Some numerical examples will be presented to illustrate the evolution of the GMIB and GMDB under partial withdrawals. A premium of $100,000 is considered for a policyholder aged 60, with no additional contributions, and no transfers. Throughout these examples, no charges are deducted from the account value and there is a fixed roll-up rate of 4%. The assumed returns do not follow any market trends and were chosen to serve the purposes of illustrating two types of scenarios.

We define the following notation:

- GMIB BB: Guaranteed minimum income benefit base;
- RP BB: Return of principal benefit base;
- RU BB: Roll-up to age 85 benefit base;
- HA BB: Highest anniversary value benefit base;
- GO BB: "Greater of" benefit base.

Table 3 shows that the account value is reduced dollar-for-dollar by the withdrawals amount before considering market behavior no matter the size of withdrawals. In alternative 1, when the owner with- draws the annual guaranteed withdrawals amount $[4\% \left(\text{roll-up rate}\right) \cdot \$128,785$ (the roll-up benefit bases as of the 6th contract anniversary), the GMIB and roll-up to age 85 benefit bases neither decrease nor increase. The return of principal benefit base is reduced pro rata as follows: since the withdrawals amount of $5,151 equals 4.21% of the account value $\left(\$5,151 = 4.21\% \cdot \$122,346\right)$, the return of principle (RP) benefit base is also reduced by 4.21%, while the highest anniversary value (HAV) benefit base is reduced dollar-for-dollar, i.e. $128,785 (HA BB as of the last contract date anniversary) – $5,151 = $123,634 for the 6th contract year.

In the case of an excess withdrawal, as it is the case of contract years 6 and 7 of the second scenario, the return of principal is reduced in the same way: since the withdrawals amount of $7,000 equals 5.721% of the account value in 6th year ($7,000 divided by $122,346 = 5,721%), the RP benefit base is reduced by 5.721%. The pro rata reduction of the roll-up benefit bases is as follows: $7,000 (the amount of the withdrawal, including any applicable withdrawals charge) – $5,151(GWA) = $1,849

Table 3. GMIB and GMDB behavior given policyholder's withdrawals when the account value is less than the GMIB benefit base at the time of the first withdrawal

Guaranteed minimum death benefit									
Year	**Net return**	**AV**	**WA**	**Roll-up rate**	**GMIB BB**	**RP BB**	**HAV BB**	**RU BB**	**GO BB**
0	–	$100,000	–	–	$100,000	$100,000	$100,000	$100,000	$100,000
1	3%	$103,000	$0	4%	$104,000	$100,000	$103,000	$104,000	$104,000
2	4%	$107,120	$0	4%	$108,160	$100,000	$107,120	$108,160	$108,160
3	6%	$113,547.20	$0	4%	$113,547.20	$100,000	$113,547.20	$113,547.20	$113,547.20
4	6%	$120,360.03	$0	4%	$120,360.03	$100,000	$120,360.03	$120,360.03	$120,360.03
5	7%	$128,785.23	$0	4%	$128,785.23	$100,000	$128,785.23	$128,785.23	$128,785.23
Alternative 1: annual guaranteed withdrawal amount (dollar-for-dollar)									
6	–5%	$122,345.97	$5,151.41	4%	$128,785.23	$95,789.47	$123,633.82	$128,785.23	$128,785.23
7	1%	$118,418.02	$5,151.41	4%	$128,785.23	$91,622.45	$118,482.42	$128,785.23	$128,785.23
8	–2%	$116,049.66	$5,151.41	4%	$128,785.23	$87,555.36	$113,331.01	$128,785.23	$128,785.23
9	2%	$118,370.66	$5,151.41	4%	$128,785.23	$83,745.01	$108,179.60	$128,785.23	$128,785.23
10	2%	$120,738.07	$5,151.41	4%	$128,785.23	$80,171?94	$103,028.19	$128,785.23	$128,785.23
Alternative 2: excess withdrawals (pro-rata)									
6	–5%	$122,345.97	$7,000	4%	$126,839.35	$94,278.52	$121,765.78	$126,839.35	$126,839.35
7	3%	$119,016.35	$7,000	4%	$124,786.30	$88,733.49	$114,803.39	$126,839.35	$126,839.35
8	–2%	$109.636,02	$4,991.45	4%	$124,786.30	$83,742.03	$109,811.94	$126,839.35	$126,839.35
9	2%	$106,837.29	$4,991.45	4%	$124,786.30	$78,750.58	$104,820.49	$126,839.35	$126,839.35
10	2%	$103,982.59	$4,991.45	4%	$124,786.30	$73,759.13	$ 99,829.04	$126,839.35	$126,839.35

http://dx.doi.org/10.21511/ins.09(1).2018.05

("excess"), which represents 1.511% of the account value, there is a decrease of 1.511% in the roll-up benefit bases. The highest anniversary value benefit base is reduced dollar-for-dollar and pro rata as follows: $128,785 (HAV BB as of the last contract date anniversary) –

$$\$5{,}151(\text{GWA}) = \\ = \$1{,}868\left[(\$128{,}785 - \$5{,}151)\cdot 1.511\%\right] = \\ = \$121{,}766.$$

In Table 4, the account value is reduced dollar-for-dollar by the withdrawn amount no matter the size of withdrawal: in year 7, the AV is calculated as $135,224.50 [AV as of the last contract date anniversary] – $7,000 (the amount of the withdrawal, including any applicable withdrawals charge)×(1+0.03 [assumed net return for the 7th contract anniversary]) = $128,224.50×1.03 = $132,071.23. When the owner limits himself to making only annual guaranteed withdrawals [4% (roll-up rate)×$128,785 (the roll-up benefit bases as of the 6th contract anniversary)] the GMIB and roll-up to age 85 benefit are reduced dollar-for-dollar, but since AV after withdrawals is greater than the aforementioned quantity, they are automatically set to $130,073.

As a result of the GMIB benefit base increase in contract year 6, the annual withdrawals amount in contract year 7 is $5,203 [4% (roll-up rate)×$130,073 (the roll-up benefit bases as of the sixth contract anniversary)]. The return of principal benefit base is reduced pro rata as shown in the previous exam- ples and the highest anniversary value benefit base is reduced dollar-for-dollar as follows: $128,785 (highest anniversary value benefit base as of the 5th contract date anniversary) – $5,151 = $123,634. The highest anniversary value benefit base is reset to the protected benefit account value after withdrawals ($130,073).

In the case of an excess withdrawal and similar to the previous example, the roll-up bases will be reduced in the same percentage as the excess. Taking the 6th contract anniversary as an example, it is reduced by 5.177% ($7,000 divided by $135,224), which gives $121,944. The roll-up to age 85 benefit base and GMIB benefit base are then set to the protected account value after withdrawals $128,224, since this value is clearly higher than that of the benefit bases after the pro rata reduction. The RP benefit base continues to be reduced on a pro rata basis. The highest anniversary value benefit base is reduced dollar-for-dollar and pro rata, as follows: $128,785 (highest anniversary

Table 4. GMIB and GMDB behavior given policyholder's withdrawals when the account value is greater than the GMIB benefit base at the time of the first withdrawal

Guaranteed minimum death benefit									
Year	**Net return**	**AV**	**WA**	**Roll-up rate**	**GMIB BB**	**RP BB**	**HAV BB**	**RU BB**	**GO BB**
0	–	$100,000	–	–	$100,000	$100,000	$100,000	$100,000	$100,000
1	3%	$103,000	$0	4%	$104,000	$100,000	$103,000	$104,000	$104,000
2	4%	$107,120	$0	4%	$108,160	$100,000	$107,120	$108,160	$108,160
3	6%	$113,547.20	$0	4%	$113,547.20	$100,000	$113,547.20	$113,547.20	$113,547.20
4	6%	$120,360.03	$0	4%	$120,360.03	$100,000	$120,360.03	$120,360.03	$120,360.03
5	7%	$128,785.23	$0	4%	$128.785.23	$100,000	$128,785.23	$128,785.23	$128,785.23
Alternative 1: annual withdrawals amount									
6	5%	$135,224.50	$5,151.41	4,00%	$130,073.09	$96,190.48	$130,073.09	$130,073.09	$130,073.09
7	3%	$133,975.28	$ 5,202.92	4%	$130,073.09	$92,454.92	$128,772.36	$130,073.09	$130,073.09
8	-2%	$126,196.91	$ 5,202.92	4%	$130.073,09	$88,643.14	$123,569.43	$130,073.09	$130,073.09
9	2%	$123,413.86	$ 5,202.92	4%	$130,073.09	$84,906.09	$118,366.51	$130,073.09	$130,073.09
10	2%	$120,575.16	$ 5,202.92	4%	$130,073.09	$81,242.32	$115,372.24	$130,073.09	$130,073.09
Alternative 2: excess withdrawal									
6	5%	$135,224.50	$7,000	4%	$128,224.50	$94,823.42	$128,224.50	$128,224.50	$128,224.50
7	3%	$132,071.23	$7,000	4%	$125,071.23	$89,797.62	$125,071.23	$125,071.23	$125,071.23
8	-2%	$122,569.81	$5,002.85	4%	$117,566.96	$86,132.41	$120,068.38	$117,566.96	$120,068.38
9	2%	$119,918.30	$4,802.74	4,00%	$115,115.56	$82,682.80	$115,265.65	$115,115.56	$115,265.65
10	2%	$117,417.87	$4,610.63	4%	$112,807.25	$79,436.11	$112,807.25	$112,807.25	$112,807.25

value benefit base as of the 5th contract anniversary) – \$5,151 (annual withdrawals amount) – \$1,690[(\$128,785 – \$5,151)×1.367%] = \$121,944. Here 1.367% represents the percentage of the excess in withdrawals with respect to the GWA (\$1,849 divided by \$135,224 = 1.367%). The highest anniversary value benefit base is also reset to the protected account value after withdrawals (\$128,224).

In the following section, we will set up the mathematical formulation of the product for the purpose of studying its valuation. For the sake of simplicity, we will limit our study to a single benefit base.

3. FORMULATION AND BASIC NOTATIONS

In summary, GMIB contracts promise a policyholder an income stream at maturity for the rest of his life. Before the contract maturity, the insured is allowed to withdraw a certain amount on a yearly basis, called a withdrawal. If the GMIB contract contains a death benefit (GMIB DB), then a certain amount is paid to the beneficiaries in case the policyholder dies during the term of the contract.

To formulate our problem, we consider an x-year old policyholder possessing a GMIB contract. At inception, an initial endowment is invested in a risky asset S_t. The specifications of the contract include a set of dates $0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$, where $t_0 = 0$ is the contact inception and $t_N = T$ its maturity. These so-called contract anniversaries are the dates in which events can take place, i.e. bonuses, withdrawals, payments, etc...

3.1. The contract assumptions

3.1.1. The financial market

Variable annuities pricing is based on the common pricing literature which assumes the existence of a risk neutral measure Q under which future cash flows can be valued as their expected discounted values. The existence of such measure implies an arbitrage-free financial market. Moreover, the derivative's payoff can be replicated by a self-financing strategy, which allows the insurer to hedge the liabilities.

We assume that the risky asset S_t, which serves as an underlying mutual fund for the variable annuity, follows a Geometric Brownian motion with constant coefficients under Q:

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where σ is the volatility or the risky asset, r the risk-free rate and W a standard Brownian motion under Q.

The money market evolves with risk-free interest rate, and the numeraire process B_t is given by:

$$dB_t = rB_t dt.$$

Under the risk neutral probability measure, the discounted asset process $B_t^{-1}S_t$ is a martingale.

3.1.2. The mortality assumption

It is common practice among insurers to use deterministic mortality rate to evaluate and replicate their policy pool. We also use this assumption in this chapter by considering future mortality rates as a deterministic curve. Moreover, we make the common assumption that financial markets and biometric events are independent. Let us introduce the mortality notations as:

- x_0: the policyholder's age at the contract inception;
- q_n: the probability that the policyholder, aged x_0 at inception, dies between time t_{n-1} and t_n;
- p_n: the probability that the policyholder, aged x_0 at inception, is alive at time t_n;
- ω: the limiting age beyond which survival is impossible.

According to the definition, we have $p_n = (1 - q_n)\, p_{n-1}$, where $n \in \{1, 2, \ldots, N\}$. From the insurer's perspective, the percentage of active contracts in a large policy pool of policyholders aged $x = x_0 + t_n$ at a given time t_n is thus given by p_n.

http://dx.doi.org/10.21511/ins.09(1).2018.05

3.1.3. *The contract state variables*

At a given anniversary date t_n, the value of a GMIB contract, purchased by an x_0 – year old policyholder at inception, is determined by three main state variables: the account value, the benefit base, and the a two-states variable determining if he is alive or dead at time t_n :

- account value A_t : the value of the investment account, which is indexed on the asset value S_t, and reduced by withdrawals and fees;
- benefit base G_t : also referred as the guarantee account, is an "imaginary" wealth upon which annuities, guaranteed withdrawals and benefits are calculated. However, if the insured wants to lapse the contract, he will not be able to get this wealth;
- death process I_n : a two-states variable in $\{0,1\}$ informing if the policyholder died during $(t_{n-1}, t_n]$, or is still alive at t_n. The death probability in the interval $(t_{n-1}, t_n]$ is given by $q_n = \mathrm{P}\left(I_n = 0 | I_{n-1} = 1\right) = 1$, which depends on the policyholder's age at inception.

More state variables need to be included if one needs to incorporate stochastic interest rate and/or volatility, take into account taxation or consider different benefit bases, i.e. evolving differently or for different riders.

We restrict our analysis to single premium contracts $A_0 = G_0$, i.e. one premium at inception with no additional contributions. The policyholder can either withdraw money or exercise the income benefit.

Withdrawals include "zero" withdrawals, guaranteed ones, i.e. up to a limited amount fixed by the insurer, excess withdrawals, i.e. withdrawals that exceed the guaranteed withdrawals amount, or completely surrender the contract, i.e. lapse.

For the sake of simplicity, we assume that the policyholder can take withdrawals each policy anniversary t_n, and denote by γ_n the withdrawals amount. The income benefit also starts at anniversary years and, in case of a death benefit, the latter is paid out at these dates as well. Thus, the state variables described above may have discontinuities at times $t_1, \ldots, t_n$. Therefore, for a state variable Y, we distinguish between its value $Y_{t_n^-}$ before and $Y_{t_n^+}$ after events take place at the anniversary date t_n.

3.1.4. *Development between two policy years $(t_{n-1}, t_n]$*

Assuming that an annual guarantee fee α is continuously charged by the issuer, the value of the account value A_{t_n} evolves as:

$$A_{t_n^-} = A_{t_{n-1}^+} \cdot \frac{S_{t_n}}{S_{t_{n-1}}} exp\left(-\alpha \Delta t\right), \quad n = 1,\ 2,\ \ldots,\ N,$$

where $\Delta t = t_n - t_{n-1}$ and S_t follow a Geometric Brownian and has the closed formula:

$$S_{t_n} = S_{t_{n-1}} exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} z_n\right),$$

where $z_1, \ldots, z_n$ are independent and identically distributed standard normal random variables.

In practice, the guaranteed fee is charged discretely and proportional to the account value that can easily be incorporated into the wealth process. Denoting the discretely charged fee with the annual basis as $\bar{\alpha}$, the wealth process becomes:

$$A_{t_n^-} = A_{t_{n-1}^+} \frac{S_{t_{n-1}}}{S_{t_n}} \left(1 - \bar{\alpha}\right) \Delta t.$$

The benefit base remains constant between two policy years, i.e.:

$$G_{t_n^-} = G_{t_{n-1}^+}.$$

Remark 1. For continuously charged fees, the evolution of the account value can be rewritten in the form of an SDE:

$$dA_t = \left(r - \alpha\right) A_t dt + \sigma A_t dW_t, \tag{1}$$

where W_t is the risky asset Brownian motion, σ is its volatility, and r is the risk-free rate.

In GMIBs, some of the fees are actually proportional to the benefit base. We denote by $\alpha^A \left(resp.\ \alpha^G\right)$ fees proportional to the account value (resp. benefit base). In this case, we rewrite the account dynamic between t_{n-1}^{+} and t_n^{-} as

$$dA_t = \left(r - \alpha^A\right) A_t dt - \alpha^G G_{t_{n-1}^{+}} dt + \sigma A_t dW_t.$$

3.1.5. Transition at a policy year t_n

As mentioned earlier, the contract events take place at the discrete policy years. In the following, we denote by γ_n^{gua} the guaranteed withdrawals amount, and f_n is the cash flow at time t_n. The guaranteed withdrawals amount at t_n is typically proportional to the benefit base at time t_{n-1}^{+} $\left(\text{or } t_n^{-}\right)$, by a rate η fixed by the insurer at inception, i.e. $\gamma_n^{gua} = \eta G_{t_n^{-}}$. Each policy year can exhibit the following scenarios:

1. *The insured has died within the previous year* $\left(t_{n-1}, t_n\right]$:

If the insured has died within the previous year and no death benefit has been set in place we have

$$A_{t_n^{+}} = 0, \;\; G_{t_n^{+}} = 0, \;\; \gamma_n^{gua} = 0 \;\text{ and }\; f_n = 0.$$

2. *The insured has survived the previous policy year and does not withdraw any money from the account at time* t_n:

Different ratchet and roll-upmechanisms can be applied to the benefit base at t_n, thus changing the value of the guaranteed withdrawals amount. The different parameters develop as follows:

- roll-up only: $G_{t_n^{+}} = \left(1+\eta\right) G_{t_n^{-}}$;
- ratchet: $G_{t_n^{+}} = \max\left(G_{t_n^{-}}, A_{t_n^{-}}\right)$;
- reset: $G_{t_n^{+}} = \max\left(\left(1+\eta\right) G_{t_n^{-}}, A_{t_n^{-}}\right)$.

Here η represents the roll-up rate, which determines the quantity credited annually to the benefit base. This quantity is also used to calculate the guaranteed withdrawals amount each year by $\gamma_n^{gua} = \eta . G_{t_n^{-}}$. If no withdrawals are made from the contract, i.e $W_{t_n} = 0$, we have $A_{t_n^{+}} = A_{t_n^{-}}$ and the cash flows $f_n = 0$.

3. *The insured has survived the previous policy year and at the policy anniversary withdraws an amount within the limits of the guaranteed withdrawals amount:*

Any withdrawals up to the guaranteed annual withdrawals amount reduce the account value by the withdrawn amount. Of course, we do not allow for negative policyholder account values and thus get $A_{t_n^{+}} = \max\left(0, A_{t_n^{-}} - \gamma_n\right)$ and $f_n = \gamma_n$. The transformations discussed in the 2nd scenario) occur simultaneously with the withdrawals resulting in:

- roll-up only: $G_{t_n^{+}} = \left(1+\eta\right) G_{t_n^{-}} - \gamma_n$;
- ratchet: $G_{t_n^{+}} = \max\left(G_{t_n^{-}} - \gamma_n, A_{t_n^{-}}\right)$;
- reset: $G_{t_n^{+}} = \max\left(\left(1+\eta\right) G_{t_n^{-}} - \gamma_n, A_{t_n^{-}}\right)$.

The guaranteed annual amount γ_n^{gua} needs to be recalculated using the formula presented immediately above. Note that if the annuity owner withdraws the maximum quantity γ_n^{gua}, the level of the benefit base remains stable when the roll-up is taken into account:

$$G_{t_n^{+}} = \left(1+\eta\right) G_{t_n^{-}} - \gamma_n^{gua} = \\ = \left(1+\eta\right) G_{t_n^{-}} - \eta . G_{t_n^{-}} = G_{t_n^{-}}.$$

4. *The insured has survived the previous policy year and the policy anniversary, and withdraws an amount exceeding the limit of the withdrawals guarantee:*

In this case, the account value is again reduced by the withdrawals amount $A_{t_n^{+}} = \max\left(0, A_{t_n^{-}} - \gamma_n\right)$. The benefit base as of the last contract anniversary date is reduced pro rata by the percentage of the excess withdrawals w.r.t the account value, i.e. $G_{t_n^{-}} \cdot (\gamma_n - \gamma_n^{gua}) / A_{t_n^{-}}$. Therefore, we have:

$$G_{t_n^{+}} = G_{t_n^{-}} \left(1 - \frac{\gamma_n - \gamma_n^{gua}}{A_{t_n^{-}}}\right).$$

We then apply the ratchet if there is any, i.e $G_{t_n^{+}} = \max\left(G_{t_n^{+}}, A_{t_n^{-}} - \gamma_n\right)$.

5. *The insured has survived the previous policy year and decides to activate the GMIB rider:*

In this case, the contract matures and lifetime payments begin by the following policy anniver-

http://dx.doi.org/10.21511/ins.09(1).2018.05

sary date taking into account the state of the variables at time t_n. Details on annuitization are given in the following section.

We summarize the previous cases into the following:

- roll-up only case:

$$A_{t_n^+} = h^A\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) := \max\left(0, A_{t_n^-} - \gamma_n\right),$$

$$G_{t_n^+} = h^G\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) :=$$

$$:= \begin{cases} \max\left(0, (1+\eta) G_{t_n^-} - \gamma_n\right) \text{ if } \gamma_n \le \gamma_n^{gua} \\ G_{t_n^-}\left(1 - \dfrac{\gamma_n - \gamma_n^{gua}}{A_{t_n^-}}\right) \text{ if } \gamma_n > \gamma_n^{gua} \end{cases}. \quad (2)$$

- ratchet only case:

$$A_{t_n^+} = h^A\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) := \max\left(0, A_{t_n^-} - \gamma_n\right),$$

$$G_{t_n^+} = h^G\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) :=$$

$$:= \begin{cases} \max\left(A_{t_n^-} - \gamma_n, G_{t_n^-} - \gamma_n\right) \text{ if } \gamma_n \le \gamma_n^{gua} \\ \max\left(A_{t_n^-} - \gamma_n, G_{t_n^-}\left(1 - \dfrac{\gamma_n}{A_{t_n^-}}\right)\right) \text{ if } \gamma_n > \gamma_n^{gua} \end{cases}.$$

- reset (roll-up + ratchet) case:

$$A_{t_n^+} = h^A\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) := \max\left(0, A_{t_n^-} - \gamma_n\right),$$

$$G_{t_n^+} = h^G\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) :=$$

$$:= \begin{cases} \max\left(A_{t_n^-} - \gamma_n, (1+\eta) G_{t_n^-} - \gamma_n\right) \text{ if } \gamma_n \le \gamma_n^{gua} \\ \max\left(A_{t_n^-} - \gamma_n, G_{t_n^-}\left(1 - \dfrac{\gamma_n - \gamma_n^{gua}}{A_{t_n^-}}\right)\right) \text{ if } \gamma_n > \gamma_n^{gua} \end{cases}.$$

Remark 2. The guaranteed rate is usually set equal to the ratchet rate, i.e $\gamma_n^{gua} = \eta G_{t_n^-}$ at time t_n (ii). The ratchet case can easily be deduced from the reset case by setting the ratchet and guaranteed rate to 0.

3.1.6. The income and death benefit

At maturity, the holder of a GMIB contract can select to take a lump sum of the account value A_{t_N}, annuitize this amount at an "actual" annuitization rate or annuitize the benefit base at pre-specified guaranteed annuitization rate. Annuity factors, which give the annuitization rates, denoted by $\ddot{a}_{t_N}^{act}$ for the actual, and $\ddot{a}_{t_N}^{gua}$ for the guaranteed, are defined as the price of an annuity paying one dollar each year with either a the market's rates curve, or an internal guaranteed rates defined by the insurer. The calculations of the annuity factors take into account the probability that the insurer is alive in the future. They are given by:

$$\ddot{a}_{t_N}^{(.)} = \sum_{t_i = t_N}^{\omega - x_0} p_i e^{-r_{t_i}^{(.)}(t_i - t_N)},$$

where $r^{(.)}$ is risk-free interest rate in case of annuitizing the account value, and based on hypothesis fixed by the insurer in case of annuitizing the benefit base. Therefore, annuitizing the account value is equivalent to a lump sum, and annuitizing a benefit base G is equivalent to the amount

$$G\frac{\ddot{a}_{t_N}^{gua}}{\ddot{a}_{t_N}^{act}}.$$

For GMIB contracts analyzed in this thesis, annuitization is not restricted to the maturity t_N. Indeed, t_N is actually the last anniversary date in which the insured is allowed to annuitize. Typically, the policyholder can exercise his income benefit starting from the 10th year of the contract. An annuity factor is then defined for each date $t_n \in \{t_{10}, t_N\}$. These factors are increasing, since an older insurer will likely to have less annuities than a younger one.

Thus, the cash flow of the income benefit, based on a financially rational acting customer, is given by:

$$P\left(t_n, A_{t_n^-}, G_{t_n^-}\right) = \max\left(A_{t_n^-}, G_{t_n^-}\frac{\ddot{a}_{t_n}^{gua}}{\ddot{a}_{t_n}^{act}}\right).$$

Otherwise $P\left(t_n, A_{t_n^-}, G_{t_n^-}\right) = 0$, where P denotes the income benefit, $A_{t_n^-}$ is the level of the account value, and $G_{t_n^-}t$ is he level of the benefit base.

The policyholder can subscribe to a GMDB along with the GMIB. In this case, if the policyholder's death occurs before or at the contract maturity, a death benefit is provided to the beneficiaries. Assuming the policyholder dies during $\left(t_{n-1}, t_n\right]$, the beneficiaries will receive the amount $D\left(t_n, \ldots\right)$ at t_n.

There are several types of death benefits. The most famous one is the so-called return of premium death benefit (return of principle for the GMIB product). The death benefit can also consist of the greater of the annual ratchet benefit base and the current account value. Insurers typically charge tenth of potential market growth for this additional rider. In the case of the combo GMIB DB, this additional charge is 0.35% for this particular death benefit.

In any case, we resume the death benefit cash flow $D\left(t_n, \ldots\right)$ at t_n by:

$$D(t_n, A_{t_n^-}, G_{t_n^-}) = \begin{cases} A_0, & \text{return of principal death benfit (type 0)}, \\ A_{t_n^-}, & \text{account value death benfit (type 1)}, \\ G_{t_n^-}, & \text{benefit base the death benefit (type 2)}, \\ \max(A_{t_n^-}, G_{t_n^-}), & \text{greater of the two death benefit (type 3)}. \end{cases}$$

Our numerical analyses will be based on type 2, where the benefit base evolves as described in previous section.

4. CONTRACT VALUATION

To model mortality, the standard way is to use official life tables to estimate the death probability $q_n = \mathrm{P}\left(I_n = 0 \middle| I_{n-1} = 1\right)$ during $\left(t_{n-1}, t_n\right]$. They provide annual death probabilities for each age and gender in a given country. Some adjustments can be applied to these tables. In addition to life tables, other approaches can be considered such as the stochastic benchmark Lee-Carter model Lee and Carter (1992), which forecasts the required death probabilities accounting for systematic mortality risk.

For pricing purposes, we consider a pool of policyholders who hold identical contracts and in which each insured has the same age, gender and thus the same probabilities of life and death[2]. We assume the number of policyholders to be large enough such that the assumption that deaths occur exactly according to probability q_n is justified. Given this set of conditions, mortality risk is fully diversified.

In the following, we set up the pricing framework of the GMIB contract. In particular, we are interested in the rational policyholder behavior, which maximizes the expected value of his future cash flows. We will address a stochastic control problemas formulated in the work by Shevchenko and Luo (2016).

4.1. The stochastic control problem

Let $\gamma = \left(\gamma_1, \ldots, \gamma_N\right)$ be a withdrawals strategy, $G = \left(G_{t_0}, \ldots, G_{t_N}\right)$ – the state variable corresponding to the benefit base, $A = \left(A_{t_0}, \ldots, A_{t_N}\right)$ – the account value, and $I = \left(I_0, \ldots, I_N\right)$ – the death state. We introduce the state vector before the withdrawals as $X_n = \left(A_{t_n^-}, G_{t_n^-}, I_n\right)$, at time t_n^- and $X = \left(X_1, \ldots, X_N\right)$. The present value of the overall payoff of the GMIB contract is defined as:

$$H_0\left(X, \gamma\right) = B_{0,N} H_N\left(X_N\right) + \sum_{n=1}^{N-1} B_{0,n} f_n\left(X_n, \gamma_n\right), \quad (3)$$

where

$$\begin{aligned} H_N\left(X_N\right) = {} & P\left(t_N, A_{t_N^-}, G_{t_N^-}\right) \cdot 1_{I_{N=1}} + \\ & + D\left(t_N, A_{t_N^-}, G_{t_N^-}\right) \cdot 1_{I_{N=0}}, \end{aligned} \quad (4)$$

is the cash flow at maturity and

$$\begin{aligned} f_n\left(X_n, \gamma_n\right) = {} & \overline{f}_n\left(A_{t_n^-}, G_{t_n^-}, \gamma_n\right) \cdot 1_{I_{n=1}} + \\ & + D\left(t_n, A_{t_n^-}, G_{t_n^-}\right) \cdot 1_{I_{n=0}} \end{aligned} \quad (5)$$

is the cash flow at time t_n. Here $1_{(.)}$ is the indicator function, and $B_{i,j}$ is the discount factor from t_j to t_i $B_{i,j} = exp\left(-r\left(t_j - t_i\right)\right)$, $t_j > t_i$.

To simplify notations, we drop the mortality state variable I_n when the policyholder is alive, i.e. $I_n = 1$, in the function argument. We define

2 Other criterion can be taken into account to compose an "homogeneous pool". Some life tables consider smoking, the policy value, etc... However, these criteria are specific to other insurance products. To our best knowledge, the mortality table of variable annuities is restricted to age. The resulting death probability is a weighted average of the same age male and female holding the policy.

http://dx.doi.org/10.21511/ins.09(1).2018.05

$V(t_n, A, G)$ as the price of the contract with a guarantee at the policy year t_n when $A_{t_n^-} = A$, $G_{t_n^-} = G$. We assume that the financial risk can be eliminated via continuous hedging, i.e. complete and frictionless market, and that mortality risk is fully diversified via selling the contract to a large number of insured of the same age. Thus, the average of the contract payoffs of M policy holders $H_0(X,\gamma)$ converges to $\mathrm{E}_{t_0}^{\mathrm{I}}\left[H_0(X,\gamma)\right]$, as $M \to \infty$, where I is the real probability measure corresponding to the mortality process $I_1, I_2, \ldots, I_N$. Then the price under the given withdrawals strategy γ can be calculated as:

$$V\left(t_0, A_{t_0}, G_{t_0}\right) = \mathrm{E}_{t_0}^{\mathrm{I},Q}\left[H_0(X,\gamma)\right], \tag{6}$$

where $\mathrm{E}_{t_0}^{\mathrm{I},Q}\left[H_0(X,\gamma)\right]$ denotes the expectation w.r.t the state vector X, conditional on information available at time t_0, i.e w.r.t both the financial risky asset process under Q, and the mortality process underthe real probability measure .

Remark 3. The fair fee $\bar{\alpha} = \alpha^*$ is defined as the fees charged so that the value of the contract at time t_0 is equal to the premium, i.e. $V(0, A_0, G_0) = A_0$. It is important to note that the strategy γ can change for different realizations of underlying wealth process and the control variable γ_n at t_n affects the transition law of the underlying wealth process from t_n to t_{n+1}, i.e calculating the contract price in this case is reduced to solving an optimal stochastic control problem.

The withdrawals strategy γ can depend on the information available at time tn through the state variable X at t_n and is assumed to be given when the price of the contract is calculated in the Equation (6). Withdrawals strategies are classified into three categories : static, optimal and suboptimal.

- *Static case.* Under a static strategy γ, the policyholder's decisions are deterministic, fixed at the beginning of the contract, and independent the state variable value. Under this strategy, the price of the contract can be calculated as:

$$V\left(t_0, A_{t_0}, G_{t_0}\right) = \mathrm{E}^{\mathrm{I},Q}\left[H_0(X,\gamma)\right].$$

- *Optimal case.* Under the optimal withdrawals strategy, the withdrawals amount γ_n depends on the information available at time t_n through the state variable X_n. The optimal strategy is the strategy γ under which the contract price is maximized, i.e. worst case scenario for the insurer best case scenario for the insured:

$$\gamma^*(X) = \underset{\gamma \in A}{argsup}\, \mathrm{E}^{\mathrm{I},Q}\left[H_0(X,\gamma)\right], \tag{7}$$

where the supremum is taken over all admissible strategies γ and denoted by the set A. That is, for each time t_n, we have $\gamma_n \in \left[0, A_{t_n^-}\right]$.

- *Suboptimal case.* Any other strategy γ different from γ^* is called suboptimal. It can also depend on the state variable.

In the following, we will be interested in the optimal case. Given that the state variable $X = (X_1, \ldots, X_N)$ is a Markov process, and the contract payoff is represented by Formula (3), the calculation of the contract value under optimal strategy given by Equation (7) is brought to a more general problem whereby the policyholder starts at an arbitrary time t_n. This falls within the framework of standard optimal stochastic control problems for a controlled Markov process. Note that the control variable γ_n depends on the account value A and benefit base G.

Finding the contract value $V(t_n, x)$ at time t_n when $X_n = x$ for $n = N-1, \ldots, 0$ is done via a backward Bellman equation. Since the account value A evolves between two anniversary dates, and the benefit base is a constant piecewise function (i.e. changes at anniversary dates only), the required backward recursion is written between t_{n+1}^- and t_n^+ as:

$$\begin{aligned}
&V\left(t_n^+, A, G\right) = \\
&\mathrm{E}^{\mathrm{I}}\left[B_{n,n+1}\left(1_{I_{n+1}=1} \cdot \mathrm{E}_{t_n^+}^{Q}\left[V\left(t_{n+1}^-, A_{t_{n+1}^-}, G_{t_{n+1}^-}\right) \middle| A, G\right] + \right.\right. \\
&\left.\left. +1_{I_{n+1}=0} \cdot \mathrm{E}_{t_n^+}^{Q}\left[D\left(t_{n+1}^-, A_{t_{n+1}^-}, G_{t_{n+1}^-}\right) \middle| A, G\right]\right)\right] = \\
&= \left(1 - q_{n+1}\right)\mathrm{E}_{t_n^+}^{Q}\left[V\left(t_{n+1}^-, A_{t_{n+1}^-}, G_{t_{n+1}^-}\right) \middle| A, G\right] + \\
&+ q_{n+1}\mathrm{E}_{t_n^+}^{Q}\left[D\left(t_{n+1}^-, A_{t_{n+1}^-}, G_{t_{n+1}^-}\right) \middle| A, G\right],
\end{aligned}$$

with jump condition

$$V\left(t_n^-,A,G\right)=\max_{\gamma_n\in A_n}\left(\overline{f}_n\left(A,G,\gamma_n\right)+\right.$$
$$\left.+V\left(t_n^+,h^A\left(A,G,\gamma_n\right),h^G\left(A,G,\gamma_n\right)\right)\right).$$

The recursion starts from the maturity condition $V\left(t_N^-,A,G\right)=P\left(t_N^-,A,G\right)$ goes backwards for $n=N-1,N-2,\ldots,0.$

Remark 4. Given that the mortality and financial asset processes are assumed independent, and the withdrawals decision does not affect the mortality process, we have:

$$\sup_\gamma \mathrm{E}_{t_0}^{Q,\mathrm{I}}\left[H_0\left(X,\gamma\right)\right]=\sup_\gamma \mathrm{E}_{t_0}^{Q}\left[\mathrm{E}_{t_0}^{\mathrm{I}}\left[H_0\left(X,\gamma\right)\right]\right].$$

One can calculate the expected value of the payoff Boyle and Tian w.r.t the mortality process:

$$\tilde{H}_0\left(A,G\right)=\mathrm{E}_{t_0}^{\mathrm{I}}\left[H_0\left(X,\gamma\right)\right],$$

and then calculate the price under the given strategy $\mathrm{E}_{t_0}^{Q}\left[\tilde{H}_0\left(A,G\right)\right]$, or under the optimal strategy $\sup_\gamma \mathrm{E}_{t_0}^{Q}\left[\tilde{H}_0\left(A,G\right)\right]$.

Therefore we have:

$$\tilde{H}_0\left(A,G\right)=B_{0,N}\left(P\left(t_N,A_{t_N^-},G_{t_N^-}\right)\cdot \mathrm{E}_{t_0}^{\mathrm{I}}\left[1_{I_N=1}\right]+\right.$$
$$+D\left(t_N,A_{t_N^-},Gt_N^-\right)\cdot \mathrm{E}_{t_0}^{\mathrm{I}}\left[1_{I_N=0}\right]\Big)+$$
$$\left.+\sum_{n=1}^{N-1}B_{0,n}\left(\overline{f}_n\left(A_{t_n^-},G_{t_n^-},\gamma_n\right)\cdot \mathrm{E}_{t_0}^{\mathrm{I}}\left[1_{I_n=1}\right]+D\left(t_n,A_{t_n^-},G_{t_n^-}\right)\right)\right).$$

Moreover, since $\mathrm{E}_{t_0}^{\mathrm{I}}\left[1_{I_n}=1\right]=\mathrm{P}\left(\tau>t_n\,|\,\tau>t_0\right)=p_n$ and $\mathrm{E}_{t_0}^{I}\left[1_{I_n}=0\right]=\mathrm{P}\left(t_{n-1}<\tau<t_n\,|\,\tau>t_0\right)=p_{n-1}q_n$

for random death time τ i.e. $p_n=p_{n-1}\left(1-q_n\right)$, we can rewrite $\tilde{H}_0\left(A,G\right)$ as follows:

$$\tilde{H}_0\left(A,G\right)=$$
$$=B_{0,N}\left(p_N P\left(t_N,A_{t_N^-},G_{t_N^-}\right)+q_N p_{N-1} D\left(t_N,A_{t_N^-},G_{t_N^-}\right)\right)+ \quad (8)$$
$$+\sum_{n=1}^{N-1}\left(p_n\overline{f}_n\left(A_{t_n^-},G_{t_n^-},\gamma_n\right)+p_{n-1}q_nD\left(t_n,A_{t_n^-},G_{t_n^-}\right)\right).$$

Note that previously we defined $q_n=\mathrm{P}\left(t_{n-1}<\tau\le t_n\,|\,\tau>t_{n-1}\right)$.

The payoff (3) has the same general form as the payoff (8). Thus, the optimal stochastic control problem $\Phi\left(t_0,A_0,G_0\right)=\sup \mathrm{E}\left[\tilde{H}_0\left(A,G\right)\right]$ can be solved using Bellman equation.

We describe the optimization problem at each policy anniversary date recursively by the two following equations

$$\left(t_n^+,A,G\right)=\mathrm{E}_{t_n^+}^{Q}\left[B_{n,n+1}\Phi\left(t_{n+1}^-,A_{t_{n+1}^-},G_{t_{n+1}^-}\right)|A,G\right], \quad (9)$$

and

$$\Phi\left(t_n^-,A,G\right)=\max_{\gamma_n\in A_n}\left(p_n\overline{f}_n\left(A,G,\gamma_n\right)+\right.$$
$$+p_{n-1}q_nD\left(t_n,A,G\right)+ \quad (10)$$
$$\left.+\Phi\left(t_n^+,h^A\left(A,G,\gamma_n\right),h^G\left(A,G,\gamma_n\right)\right)\right),$$

for $n=N-1,N-2,\ldots,0$ starting from the final condition:

$$\Phi\left(t_N^-,A,G\right)=p_NP\left(t_N^-,A,G\right)+$$
$$+p_{N-1}q_ND\left(t_N^-,A,G\right). \quad (11)$$

As a consequence, the recursion leads to the same solution $\Phi\left(t_0,A,G\right)=V\left(t_0,A,G\right)$, and the same optimal strategy γ. Moreover, for each tn we have t_n we have $\Phi\left(t_n,A,G\right)=p_nV\left(t_n,A,G\right)+p_{n-1}q_nD\left(t_n,A,G\right)$.

4.2. Numerical scheme for the discrete withdrawals model

Realistic VA riders with discrete events such as ratchets, bonuses as set-up options and optimal withdrawals have no closed form solutions. Their fair price needs to be calculated numerically, even for a standard Brownian motion with constant interest rates and volatility.

The numerical solution of the backward recursion (9)-(10) is accomplished using PDEs, direct integration or regression type Monte Carlo methods. Under the static strategy, one can always use standard Monte-Carlo to simulate state variables forward in time till the contract maturity or the policyholder death and average the payoff cash flows over a large number of independent realizations.

In the case of discrete withdrawal, following the procedure of deriving the Hamilton-Jacobi-Bellman (HJB) equations in stochastic control problems, the value of the annuity under optimal

http://dx.doi.org/10.21511/ins.09(1).2018.05

withdrawals is found to be governed by a one-dimensional PDE, similar to the Black-Scholes equation, with jump conditions at each withdrawing date to link the prices at the adjacent periods.

In the following, we provide detailed description of the algorithm used to calculate the fair value of he VA riders and the optimal strategy.

4.2.1. General algorithm

The algorithmstarts from a final condition for the contract value at t_N^-. Subsequently, solving the PDE gives solution for the contract value at t_{N-1}^+. The PDE used to calculate the expected value (9) under the assumed risk-neutral process for the risky asset S_t is easily derived using Feynman-Kac theorem. When the risky asset follows a geometric Brownian motion process, the governing PDE right after a withdrawals decision t_n^+ to right before the following one t_{n+1}^- for $n = N-1, N-2, \ldots, 0$ is expressed as the follows:

$$\partial_t \Phi + \frac{1}{2}\sigma^2 A^2 \partial_{AA}\Phi + \left(r - \alpha^A\right) A_t \partial_A \Phi - \\ -\alpha^G G_{t_n^+} \partial_A \Phi - r\Phi = 0, \quad (12)$$

to which we add boundary conditions given in the next section.

Note that the benefit base changes only at the anniversary dates and is a constant parameter between two anniversary dates. PDE (12) is solved using the Crank-Nicolson finite differences methods. Dai, Kuen Kwok, and Zong (2008), Huang, Forsyth, and Labahn (2012) used the scheme for pricing GMWB with discrete optimal withdrawals. Of course, if the volatility and/or interest rates are stochastic, then one needs to add extra dimensions to the PDE.

Applying the jump condition (10) to the solution at t_{N-1}^+, we obtain the solution at t_{N-1}^- from which further backward time stepping gives us solution at t_{N-2}^+, and so on. The numerical algorithm takes the following key steps:

1. Generate a finite grid for the account value A and benefit base G, i.e. $A_0 < A_1 < \ldots < A_J$ and $$0 = G_0 < G_1 < \ldots < G_K.$$
2. At t_N, define the final condition for each note point $\left(A_j, G_k\right)$, $j = 1,2,\ldots,J$ and $k = 1,2,\ldots,K$ to get $\Phi\left(t_n^-, A, G\right)$ and the boundary conditions (17)-(18) for $A_{\min}$ and $A_{\max}$ for each potential $G_{k\in\{1,2,\ldots,K\}}$.
3. For each potential benefit G_k, $k = 1,2,\ldots,K$, solve the PDE using the Crank-Nicolson finite differences scheme to obtain $\Phi\left(t_{N-1}^+, A, G\right)$.
4. Apply the jump condition (10) by performing a linear search of the withdrawals amount γ_{N-1}^* that gives the maximum $\Phi\left(t_{N-1}^-, A, G\right)$. In general, this involves a two-dimensional interpolation in $\left(A, G\right)$, since the $h^G\left(A, G\right)$ and $h^G\left(A, G\right)$ do not necessarily fall in the grid nodes.
5. Repeat (3) and (4) for $t = t_{N-2}, t_{N-3}, \ldots, t_1$.
6. Evaluate Equation (12) for the backward time step t_1 to t_0 to obtain solution $\Phi\left(t_0, A, G\right)$ at A_0 and G_0.

We can add more complexity to the model, for example, by incorporating stochastic interest rates or stochastic volatility, in this case, the dimension of the pricing PDE (12). We can also add more constant path wise state variables that evolve only at the anniversary (tax base, extra benefit base, etc...), which will affect only the jump condition, i.e. the search of the optimum will have to be performed based on the new variables as well.

The finite difference scheme will be discussed in more detail in the next section.

4.2.2. Description of the finite differences scheme

Recall that the value function Φ satisfies the following recursion

$$\Phi\left(t_n^+, A, G\right) = \mathrm{E}_{t_n}^{Q}\left[B_{n,n+1}\Phi\left(t_{n+1}^-, A_{t_{n+1}^-}, G_{t_{n+1}^-}\right) \middle| A, G\right], \quad (13)$$

$$\Phi\left(t_n^+, A, G\right) = \max_{\gamma_n \in A_n}\left(p_n \overline{f}_n\left(A, G, \gamma_n\right) + \right. \\ + p_{n-1} q_n D\left(t_n^+, A, G\right) + \\ \left. + \Phi\left(t_n^+, h^A\left(A, G, \gamma_n\right), h^G\left(A, G, \gamma_n\right)\right)\right). \quad (14)$$

Within each time interval $\left(t_{n-1}, t_n\right)$, only the account value varies, since all the benefit bases, death and life, remain constant. Thus, for $t \in \left(t_n^+, t_{n+1}^-\right]$, the annuity value $\Phi\left(t, A, G\right)$ solves the following linear PDE for each fixed value of the benefit base G

$$\partial_t \Phi + L\Phi = 0, \tag{15}$$

where the operator α is

$$L\Phi = \frac{1}{2}\sigma^2 A^2 \partial_{AA}\Phi + \\ +\left(r - \alpha^A\right) A \partial_A \Phi - \alpha^G G \partial_A \Phi - r\Phi. \tag{16}$$

4.2.3. Localization and boundary conditions

Equation (15) is originally posed on the domain $\left(t, A\right) \in \left[0, T\right] \cdot \left[0, \infty\right)$. For calculation purposes, and because asset prices are finite and so is the account value, one needs to localize this domain to $\left[0, T\right] \cdot \left[0, A_{\max}\right]$, where $A_{\max}$ is large enough not to be attained by the account value during the lifetime of the annuity. Thus, we need to add complementary boundary conditions. We consider that we are between two anniversary dates t_n^+ and t_{n+1}^- backwards. When $A = 0$, the policyholder has no longer the possibility to make any withdrawals from his account. However, if the IB election is possible, then the income period begins, given the policyholder is alive, and the death benefit is activated if he is dead at t_{n+1}. Since the account value is equal to zero, then the annuitization will be indexed on the benefit base. Therefore, we have:

$$\Phi\left(t, 0, G\right) = e^{-r\left(t_{n+1} - t\right)} \times \\ \times \left(p_{n+1} \frac{\ddot{a}_{t_{n+1}}^{gua}}{\ddot{a}_{t_{n+1}}^{act}} G + p_n q_{n+1} D\left(t_{n+1}^-, 0, G\right) \right). \tag{17}$$

When $A = A_{\max}$, we consider retrieving all the cash more interesting than any other strategy if the policyholder is alive. If he dies, the death benefit will be activated. Therefore, the Dirichlet boundary condition for this case is

$$\Phi\left(t, A_{\max}, G\right) = e^{-r\left(t_{n+1} - t\right)}\left(p_{n+1} A_{\max} + p_n q_{n+1} A_{\max}\right) = \\ = e^{-r\left(t_{n+1} - t\right)} p_n A_{\max}. \tag{18}$$

Let us define the solution domains

$$\bar{\Omega}_n = \left[t_{n-1}^+, t_n^-\right] \cdot \left[0, A_{\max}\right],$$

$$\bar{\Omega} = \bigcup_{t_n} \left[t_{n-1}^+, t_n^-\right] \cdot \left[0, A_{\max}\right].$$

The pricing problem for the GMIB variable annuity combined withDB under the discrete withdrawals scenario is then achieved in $\bar{\Omega}$ as follows: within each set $\bar{\Omega}$, $n = 1, \ldots, N-1$, the solution to the problem is the viscosity solution of a decoupled set of linear PDEs (12) with final condition (11) and boundary conditions (17)-(18) computed from the nonlinear algebraic Equation (14).

4.3. Construction of the scheme

Let $\left(A_0, A_1, \ldots, A_J\right)$ be the equally spaced grid in the direction of the account value with $A_0 = 0$ and $A_J = A^{\max}$. Analogously, $\left(G_0, \ldots, G_K\right)$ is an equally spaced grid for the benefit base with $G_0 = 0$ and $G_K = G^{\max} = A^{\max}$. The spacial steps for both variables are considered to be equal. That is:

$$\Delta A = \Delta G,$$

where

$$\Delta A = \frac{A_{\max} - A_0}{J} \text{ and } \Delta G = \frac{G_{\max} - G_0}{K}.$$

Hence, $A_j = j\Delta A$ and $G_k = k\Delta G$, $\forall j, k$. The discrete time steps are denoted by $n\Delta t$ for $n = 1, \ldots, N$, where $T = N\Delta t$. Since, in our analysis, we consider that events occur only at anniversary dates, which are yearly, $\Delta t = 1$ and each time t_n coincides with the discrete time step $t_n = n$.

The numerical procedure to solve the approximation in (15) is the standard finite difference approach. We use the two-level implicit finite difference scheme to discretize the differential term $L\Phi$ as given in (16). Let $L_h \Phi_{j,k}^n$ denote the discrete value of the differential operator at the node $\left(A_j, G_k, t_n\right)$.

The approximation is then given by:

$$L_h \Phi_{j,k}^n = \frac{\sigma^2}{2} A_j^2 \frac{\Phi_{j+1,k}^n - \Phi_{j,k}^n - \Phi_{j-1,k}^n}{\Delta A^2} + \\ + \left\{\left(r - \alpha^A\right) A_j - \alpha^G G_k\right\} \frac{\Phi_{j+1,k}^n - \Phi_{j-1,k}^n}{2\Delta A} - r\Phi_{j,k}^n.$$

http://dx.doi.org/10.21511/ins.09(1).2018.05

The general theta-scheme for solving Equation (15) is given by:

$$\frac{\Phi_{j,k}^{n+1}-\Phi_{j,k}^{n}}{\Delta t}=\theta L\Phi_{j,k}^{n+1}+\left(1-\theta\right)L_h\Phi_{j,k}^{n},$$

where θ is a weighting factor, $0<\theta\leq 0$. For $\theta=0$, this scheme is the explicit scheme, whereas $\theta=1$ corresponds to the implicit one. The error of the previous cases is of $O\left(\Delta A^2,\Delta t\right)$. The explicit scheme has stability issues, while the implicit scheme is absolutely stable. The most popular scheme for approximating the solution of the Black-Scholes equation is the Crank-Nicolson scheme obtained for $\theta=1/2$. The latter is shown to be unconditionally stable and $O\left(\Delta A^2,\Delta t^2\right)$ convergent (see Duffy, 2004). In particular, this scheme is used in Dai et al., (2008) to solve the optimal pricing problem of the GMWB rider with rational behavior.

The discretization w.r.t the benefit base is not important here. However, since the PDE is solved backwards between t_{n+1}^- and t_n^+, we need to divide this time period, i.e. the period between two consecutive withdrawals dates, into finer time steps for a good accuracy due to the finite difference approximation to the partial derivatives.

4.3.1. Applying the jump condition

Recall that changes in the benefit base only occur at withdrawals dates. After withdrawing the amount n at time t_n, the account value changes from $A_{t_n^-}$ to $A_{t_n^+}=h^A\left(A,G,\gamma_n\right)$, and the benefit base drops from $G_{t_n^-}$ to $G_{t_n^+}=h^G\left(A,G,\gamma_n\right)$. The jump condition of $\Phi\left(t_n^-,A,G\right)$ across t_n is given by:

$$\begin{aligned}&\Phi\left(t_n^-,A,G\right)=\\&=\max_{0\leq\gamma_n\leq A}\Big(\Phi\left(t_n^+,h^A\left(A,G,\gamma_n\right),h^G\left(A,G,\gamma_n\right)\right)+\\&+p_n\bar{f}\left(A,G,\gamma_n\right)+p_{n-1}q_nD\left(t_n^-,A,G\right)\Big),\end{aligned}\tag{19}$$

For the optimal strategy, the withdrawals amount γ_n is chosen under the restriction $0\leq\gamma_n\leq A$ to maximize the value of $\Phi\left(t_n^-,A,G\right)$ in Equation (19).

The application of the jump condition decreases the account value and benefit base. For each G_j, a continuous solution from PDE (15) is associated. We can restrict the possible values for the withdrawals amount to multiples of ΔA. This implies, for a given account value A_j at time t_n^-, the withdrawals amount γ takes j possible values: $\gamma=A_j-A_l,\ l=1,2,\ldots,j$.

However, numerical tests showed that a finer grid is preferable for the withdrawals amount. Therefore, it is not guaranteed that the account value, nor the benefit base after the withdrawal, $A_{t_n^+}$ and $G_{t_n^+}$, fall within their respective grid nodes. To solve this issue, a two-dimensional interpolation is required. In this work, we adopted a bi-linear interpolation.

Suppose the jump condition requires the value $\Phi\left(A,G\right)$ at the point $\left(A,G\right)$ located inside a grid $A_i\leq A\leq A_{i+1}$ and $G_j\leq G\leq G_{j+1}$, then the interpolation is performed as follows:

$$\begin{aligned}\Phi\left(A,G\right)\approx&\frac{G_{i+1}-G}{G_{i+1}-G_i}\Phi\left(A,G_j\right)+\\&+\frac{G-G_i}{G_{i+1}-G_i}\Phi\left(A,G\right),\end{aligned}\tag{20}$$

where

$$\begin{aligned}\Phi\left(A,G_j\right)\approx&\frac{A_{i+1}-A}{A_{i+1}-A_i}\Phi\left(A_i,G_j\right)+\\&+\frac{A-A_i}{A_{i+1}-A_i}\Phi\left(A_{i+1},G_j\right),\end{aligned}$$

$$\begin{aligned}\Phi\left(A,G_j\right)\approx&\frac{A_{i+1}-A}{A_{i+1}-A_i}\Phi\left(A_i,G_{j+1}\right)+\\&+\frac{A-A_i}{A_{i+1}-A_i}\Phi\left(A_{i+1},G_{j+1}\right).\end{aligned}$$

At last, the jump condition is achieved through combining (19) and (20) to find the optimal withdrawals and maximize the function Φ.

4.3.2. Similarity and dimension reduction

An important feature of the contract value is that it exhibits good scaling properties in the Black-Scholes case. We can easily verify that the solution $\Phi\left(t,A,G\right)$ of PDE (15) with boundary conditions (17) and event conditions (18) verifies:

$$\Phi(t,\xi A,\xi G)=\xi\Phi(t,A,G),$$

for any scalar $\xi>0$. Therefore, choosing $\xi=\frac{1}{G}$, we obtain:

$$\Phi(t,A,G)=G\Phi\left(t,\frac{A}{G},1\right)=G\phi\left(t,\tilde{A}\right),$$

where $\tilde{A}=A/G$. It means that we need only solve to the corresponding equations for the one-dimensional function defined in the following:

- between two consecutive withdrawals dates (t_{n-1},t_n), ϕ follows the PDE:

$$\partial_t\phi+\frac{1}{2}\sigma^2\tilde{A}^2\partial_{\tilde{A}\tilde{A}}\phi+\left(r-\alpha^A\right)\tilde{A}\partial_{\tilde{A}}\phi-\\-\alpha^G\partial_{\tilde{A}}\phi-r\phi=0. \quad (21)$$

- at the anniversary dates t_n, the jump condition is explicitly expressed as the following:

$$\phi\left(t_n^-,\tilde{A}_{t_n^-},\tilde{\gamma}_n\right)=\max_{\gamma}\left(h_1(\tilde{\gamma})\phi\left(t_n^+,h_2\left(\tilde{A}_{t_n^+},\tilde{\gamma}\right)\right)+\right.\\\left.+p_n\tilde{\gamma}_n+p_{n-1}q_nD\left(t_n^+,\tilde{A}_{t_n^+},1\right)\right),$$

where the functions h_1 and h_2 are a reduced version of the account value and benefit base evolution at the anniversary dates, and are defined according to different cases:

1. Roll-up only case:

$$h_1(\tilde{\gamma})=\begin{cases}1+\eta-\tilde{\gamma}_n & \text{if } \tilde{\gamma}_n\le\eta;\\ 1-\dfrac{\tilde{\gamma}_n-\eta}{\tilde{A}} & \text{if } \tilde{A}\ge\tilde{\gamma}_n>\eta.\end{cases}$$

$$h_2(\tilde{\gamma})=\begin{cases}\dfrac{\tilde{A}-\tilde{\gamma}_n}{1+\eta-\tilde{\gamma}_n} & \text{if } \tilde{\gamma}_n\le\eta;\\ \dfrac{\tilde{A}-\tilde{\gamma}_n}{1-\dfrac{\tilde{\gamma}-\eta}{\tilde{A}}} & \text{if } \tilde{A}\ge\tilde{\gamma}_n>\eta.\end{cases}$$

2. Ratchet only case:

$$h_1(\tilde{\gamma})=\max\left(\tilde{A}-\tilde{\gamma}_n,1-\frac{\tilde{\gamma}_n}{\tilde{A}}\right);$$

$$h_2(\tilde{\gamma})=\min\left(1,\tilde{A}\right).$$

3. Reset case (roll-up+ratchet):

$$h_1(\tilde{\gamma})\begin{cases}\max\left(\tilde{A}-\tilde{\gamma}_n,1+\eta-\tilde{\gamma}_n\right) & \text{if } \tilde{\gamma}_n\le\eta;\\ \max\left(\tilde{A}-\tilde{\gamma}_n,1-\dfrac{\tilde{\gamma}_n-\eta}{\tilde{A}}\right) & \text{if } \tilde{A}\ge\tilde{\gamma}_n>\eta.\end{cases}$$

$$h_2(\tilde{\gamma})=\begin{cases}\min\left(1,\dfrac{\tilde{A}-\tilde{\gamma}_n}{1+\eta-\tilde{\gamma}_n}\right) & \text{if } \tilde{\gamma}_n\le\eta;\\ \min\left(1,\dfrac{\tilde{A}-\tilde{\gamma}_n}{1-\dfrac{\tilde{\gamma}-\eta}{\tilde{A}}}\right) & \text{if } \tilde{A}\ge\tilde{\gamma}_n>\eta.\end{cases}$$

It can be easily verified that PDE (21) does not depend on the benefit base, since the latter is constant between two consecutive withdrawals dates t_{n-1}^+ and t_n^-. Therefore, the resolution of the two-dimensional problem can be reduced into a one-dimensional problem. This is particularly useful when adding more stochastic variables like stochastic volatility and/or interest rates.

5. RESULTS

The main goal of this study is to assess the behavior risk of a given GMIB product, in case policy- holders maximize their expected cash flows. Through optimal withdrawals amounts, or IB election, the insurer is concerned that his product may lead to "undesirable" policyholders behaviors. Given a state of the universe in a future time, and a set of up-front fixed hypotheses (management and guar- antee fees, interest rates and volatility), the optimal framework allows us to predict these behaviors through the stochastic control problem detailed in previous sections. The product hypotheses usually change to account for a new financial environment or customers needs.

We choose two close variations of the product launched in the last decade, which we call them re- spectively, Product A and Product B. These are two GMIB products to which a death benefit (DB) can be added, i.e. Product A-DB and Product B-DB. The parameters values are given in Table 5.

http://dx.doi.org/10.21511/ins.09(1).2018.05

Table 5. Model parameters

Parameters	Product A	Product B
Policy holders initial age x_0	60	60
First date for IB election	10th	10th
Last age for IB election	85th	95th
Last age for DB if any	95	95
Interest rate r	2% and 4%	2% and 4%
Volatility σ	20%	20%
Roll-up rate η	6%	r + 1%
Initial premium A_0	\$100, 000	\$100, 000
Total fees $\bar{\alpha} = \alpha^A + \alpha^G$	3.5%	3.5%

We compare these variations for roll-up and reset, with and without death benefit. We conduct the experiments to illustrate the following:

- Policyholders rational behavior for Products A, B, A-DB and B-DB based on the two dimension approach, giving the withdrawals surface as a function of the account value A and benefit base G, for two different interest rate values, in four different periods in time. We also give with- drawals as a function of the moneyness A/G based on the dimension reduction approach:
- policyholders rational behavior for different values of the volatility $\sigma = \{10\%, 20\%, 30\%, 40\%\}$;
- the contract initial value as a function of interest rates for Product A-DB;
- the contract initial value as a function of the total fees.

5.1. Overview of the policyholder behavior

5.1.1. *Roll-up only case*

First, in Figure A1, we present the withdrawals amount surface as a function of the account value A and benefit base G, for Products A and A-DB with roll-up only for fixed times $t = 3,\ 13,\ 23$ and $r = 2\%,\ 4\%$. As a first remark, the guaranteed account must be higher than the account value for the policyholder to stay in the contract. Depending on the moneyness, he or she can

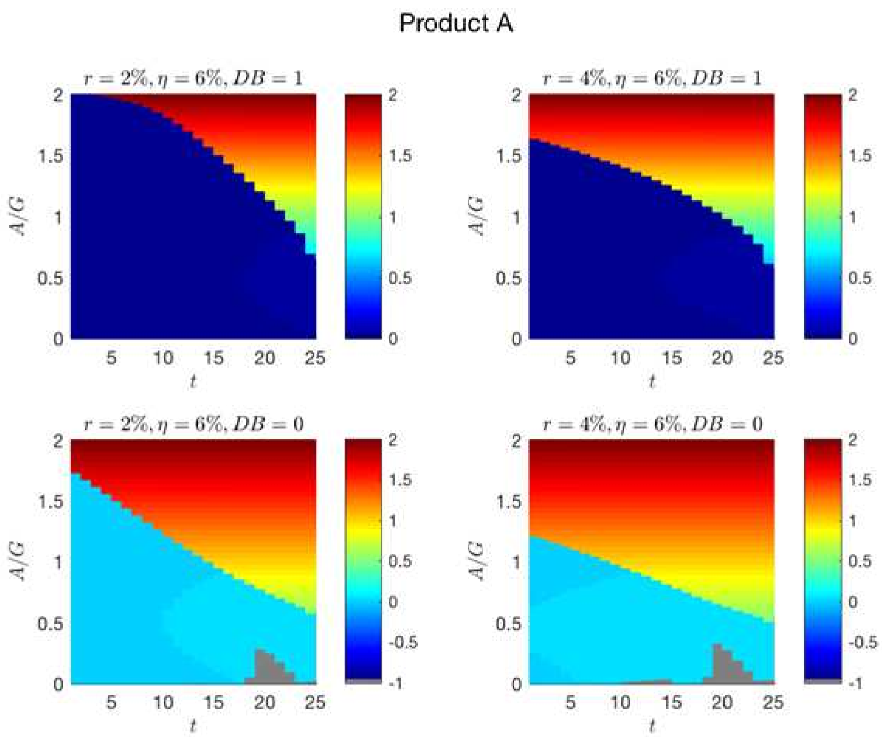

Figure 2. The withdrawals strategy $\tilde{\gamma} = \dfrac{\gamma}{G}$ as a function of time t and moneyness $\dfrac{A}{G}$

choose not to withdraw, or withdraw the guaranteed amount. The IB election (using the guaranteed account) is left at maturity or for very small account values. These findings are actually confirmed in Figure 2 where we give the withdrawals strategy as a function of time t and moneyness A/G based on the dimension reduction scheme.

Based on Figure A1 and Figure 2, we can point out the following remarks:

- there is a wider range of guaranteed withdrawals for $r = 4\%$ compared to $r = 2\%$. It tells us that for a roll-up rate as high as 6%, the policyholder tends to wait for the benefit base to increase at this rate in a low interest rates environment instead of draining the subaccount and guaranteed account;
- the death benefit increases the expected cash flows in the future, which is also a motive for the policyholder to wait;
- the IB election indexed on the guaranteed account is only expected to happen in the absence of a death benefit. Even in such case, it only takes place closer to the maturity of the product and for small account values. Actually, the ratio $\ddot{a}^{act} / \ddot{a}^{gua}$ is not very favorable for the insured, and he or she would rather start withdrawing the guaranteed amount few years earlier;
- actually, fees are quite high so the account value usually drops quickly. This restricts the analysis to A/G relatively small.

Note that reducing the dimension allows to increase the speed of calculations.

Product A was launched in a period when interest rates were around 4%, which justifies the choice of a 6% roll-up rate. Given the behavior expressed in Figure 2, the actual interest rate level may seem to be quite high and the product more interesting for the insured than the insurer.

Later, Product B was launched with reconsidered assumptions. The roll-up rate becomes indexed on the short-term interest rates with a spread of 1%. The insurance company attracted the costumer by setting a longer limiting age to annuitize (until the policyholder's 95th anniversary). In Figure 3, we give the policyholder behavior in time as a function of the moneyness.

While the behavior remains very close, we can however notice that in the absence of the death

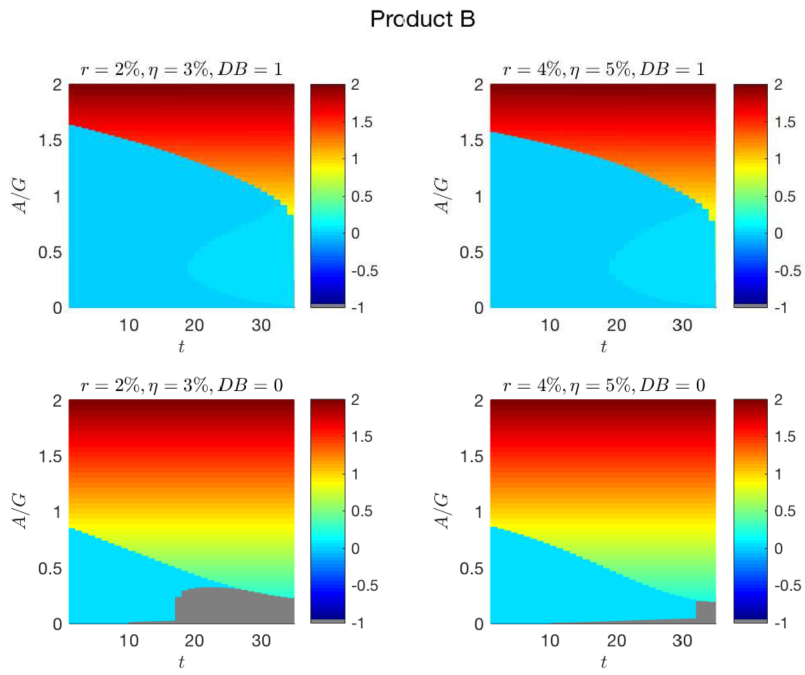

Figure 3. The withdrawals strategy as a function of time and moneyness $\frac{A}{G}$

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benefit, partial withdrawals disappear, the likelihood of the IB election is higher for low interest rate, and left until the last years for higher interest rate. Moreover, the likelihood of recovering the account value is higher than in Product A. In the presence of a death benefit, withdrawing the guarantee becomes interesting around the 20th anniversary of the contract, while an IB election is exercised at maturity.

5.1.2. Reset case

Ratchets allow for the benefit base to be set at the account value level when the latter is higher than the previous benefit base level. Combined with the roll-up, we have the reset, which is a very attractive feature for policyholders who are interested in the stochastic performance of stock markets, but at the same time want to have a guaranteed minimum performance. We present in Figure A2 and A3 the results related to products A and B with and without death benefit for the previous interest rates and roll-up values. In this case, the insured sticks with guaranteed or zero withdrawals for most time, and tends to elect the income benefit in the last anniversary date if the account value is low, and recover it otherwise. The reset is very costly for the insurance company, however, fees are quite high for this product and the ratchet takes place only at early dates, since the account value is brought down by the fees rate.

In what follows, we will focus on the roll-up only case in an attempt to analyze the impact of some key parameters in the pricing and expected policyholders rational behavior for these products.

5.1.3. The impact of volatility

The volatility level assumption is very important for variable annuities in general and the GMIB product in particular. Based on Product A for the hypothesis used above, we compare two levels of volatility (which we can compare to the 20% volatility case given in Figure 2. We can see in Figure A4 and A5 that the lower the volatility, the earlier guaranteed withdrawals start. Moreover, the lapsing likelihood is also higher. This means that the more uncertain are markets, the more the policyholder tends to withdraw money from his account. On the other hand, the IB election does not seem to be affected.

5.1.4. Roll-up rate and fees

There is a trade-off between roll-up rate and fees. The roll-up is the mechanism that ensures the policyholder a minimum return, which can be higher than the money market. However, to be able to provide interesting roll-up rates, insurance companies need to be hedged from uncertain interest rates. In this sense, they use for example swaps. Therefore, they need to collect fees that at least allow for a fair pricing for the contract, i.e. such that the paid cash flows equal the premium. On the other hand, high fees can have a perverse effect. By decreasing the sub-account value, especially in periods of low performance, present collected fees reduce future potential ones. On the long run, combined with guaranteed withdrawals, the income benefit can be elected by bringing the account value to zero. Moreover, when the the account value falls to zero, the insurance company can no longer collect fees and starts to pay the guarantee.

In Figure A6, we compare the value of the contract at inception for different parameters of Product B as a function of total fees. The fair price corresponds to $\phi\left(0, \tilde{A}_0\right) = 1$. We see that the contract is under-priced with death benefit. The fair fees would be as high as 7%. Without the death benefit, they are around 3%, which is close to the rates applied by the insurance company. In Figure A7, we conduct a similar test by varying the roll-up rate for Products A, A-DB, B and B-DB for $r = 2\%$. We notice that $\phi\left(0, \tilde{A}_0\right) = 1$ corresponds to a roll-up rate that is close to the interest rate except one of the products. Indeed, Products A, A-DB and B-DB are under-priced for the features they provide. On the other hand, the insurance company was conservative in the roll-up rate assumption for Product B, which allows it to be profitable even for the worst case scenario. Of course, insurance companies do not expect (and hope not) that all policyholders follow an optimal behavior. However, prudent hypotheses can prevent from important losses. Including a proportion of policyholders that are likely to behave op- timally is one of the solutions. Note that the GMIB product is less risky than the GMWB in that the annuity factor is defined with conservative assumptions.

CONCLUSION

In this work, we analyzed the optimal behavior of a policyholder entering a GMIB contract combined with a death benefit guarantee. The solution is based on a optimal stochastic control framework using the mean criterion in a Black-Scholes framework, and solved numerically using recursive dynamic programming techniques. Considering only the account value varies between two anniversary dates, we used finite differences methods and a linear search for the optimal withdrawals to maximize the expectation of discounted future cash flows. Such calculations give an optimal withdrawals function that depends on time, account value and benefit base. Taking advantage of the good scaling properties provided by the contract payoff and the asset price, we are able to reduce the dimensionality of the problem to time and moneyness, making calculations faster and results interpretation easier.

The policyholder's optimal behavior corresponds to the worst case scenario for insurance companies. Therefore, even though insurers are not expected to behave optimally, a good insight of how they may act in case they do, given a market environment, can allow insurers to be more effective in pricing and hedging their products. We find that the optimality consisted mainly in four choices: zero with- drawals, guaranteed withdrawals, lapse and IB election. We presented these results for two different products before analyzing the impact of some of the contract key parameters.

Finally, these results can be used as a guide to practitioners in the design of new products where a particular client behavior is desired. The model can be used to calculate the fair withdrawals fee with hedging purposes. In particular, we find that these products are under-priced in case of a optimal behaviors, as it was already mention ed by Milevsky and Salisbury (2006) in the case of GMWBs. We believe that understanding policyholder behavior is a critical concern to the future of life insurance business, and due to its importance, further research is required.

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APPENDIX

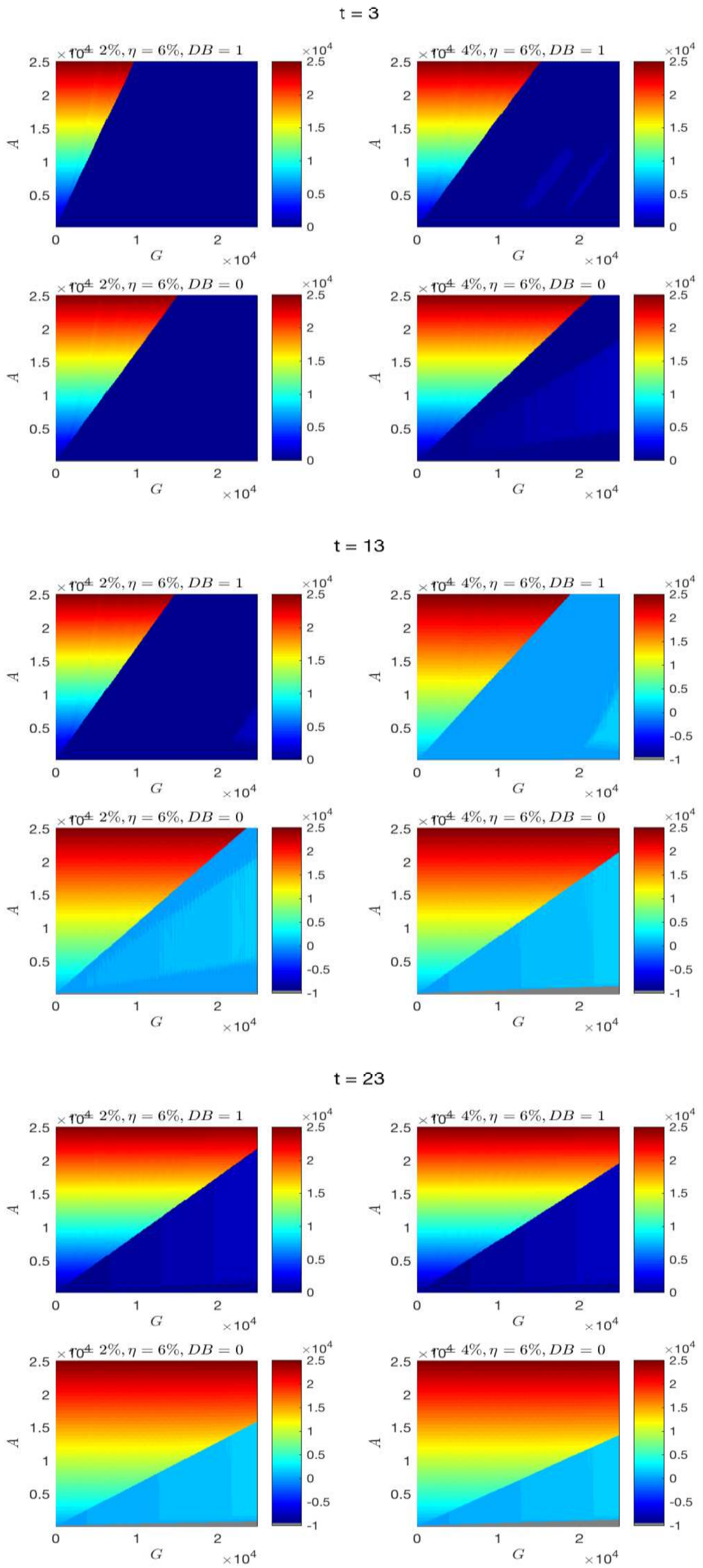

Figure A1. Policyholder optimal withdrawals amount as a function of the account value A and benefit base G for Product A and A-DB

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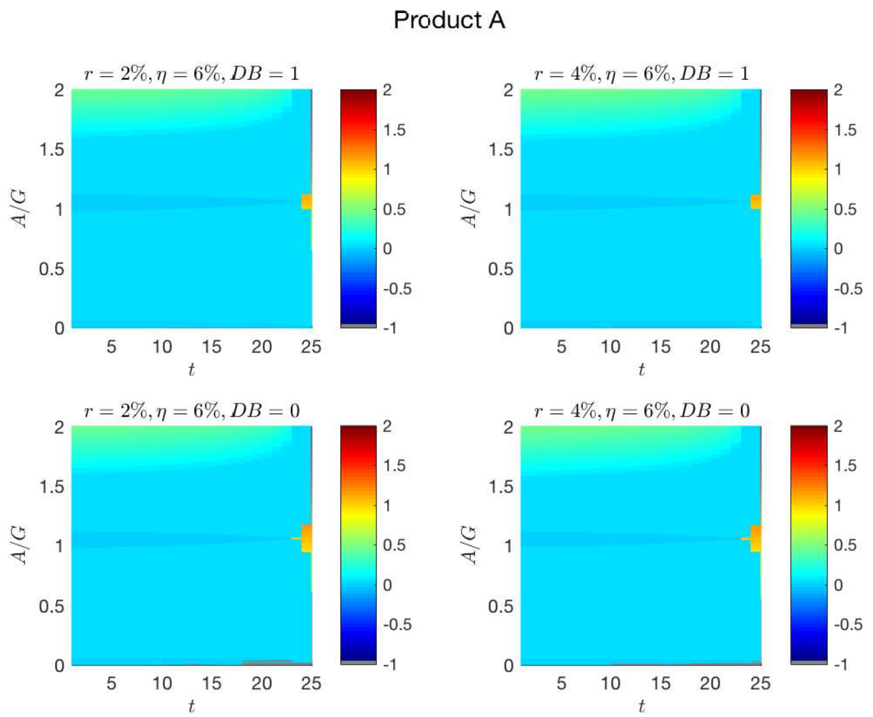

Figure A2. The withdrawals strategy as a function of time t and moneyness $\frac{A}{G}$

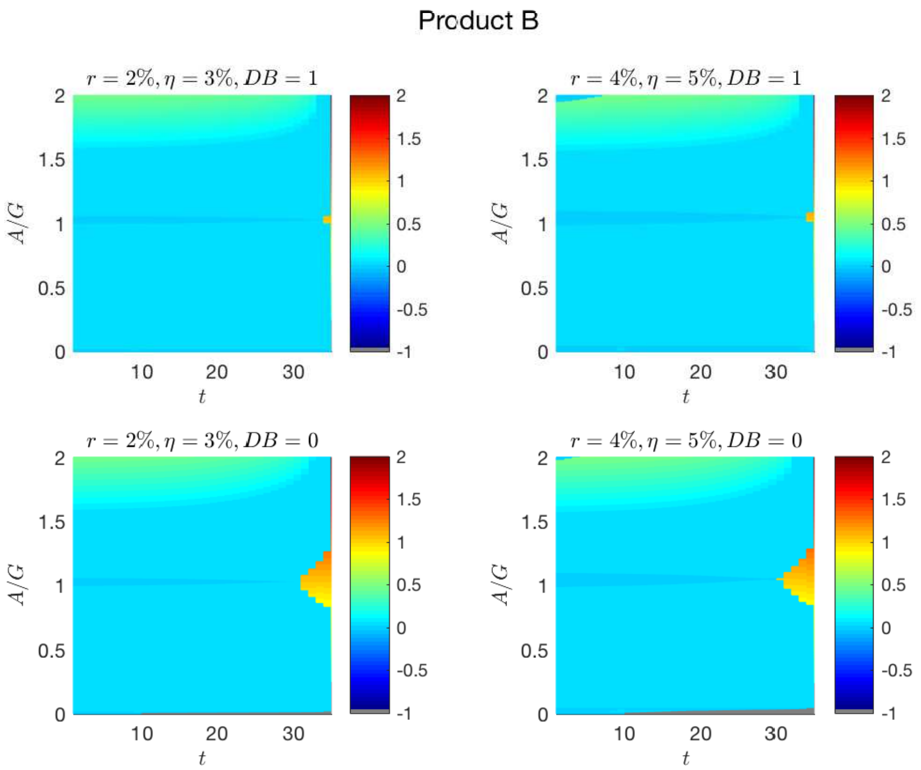

Figure A3. The withdrawals strategy as a function of time t and moneyness $\frac{A}{G}$

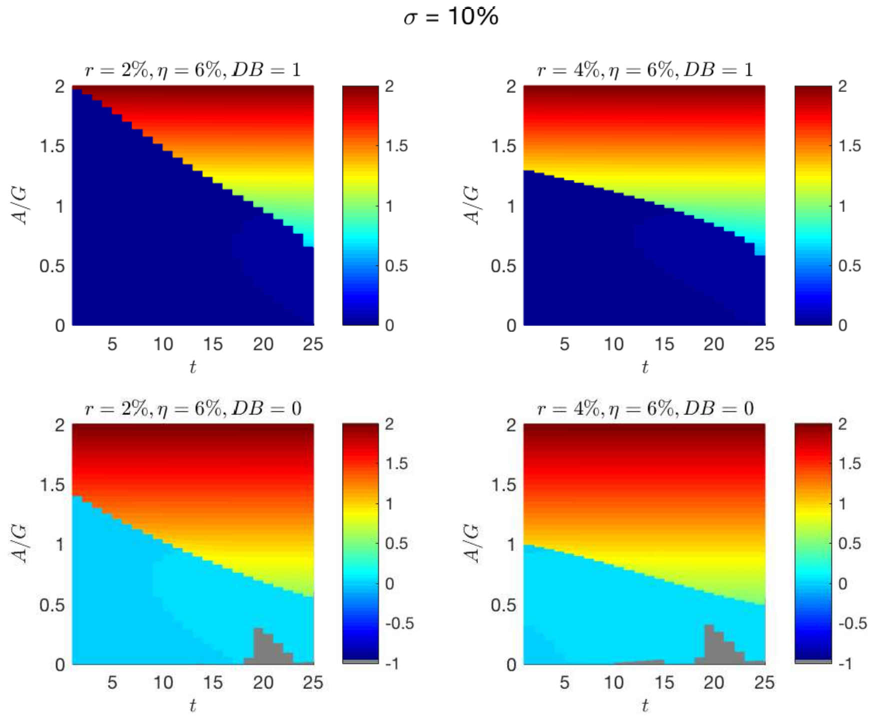

Figure A4. The withdrawals strategy as a function of time t and moneyness $\frac{A}{G}$ for $\sigma = 10\%$

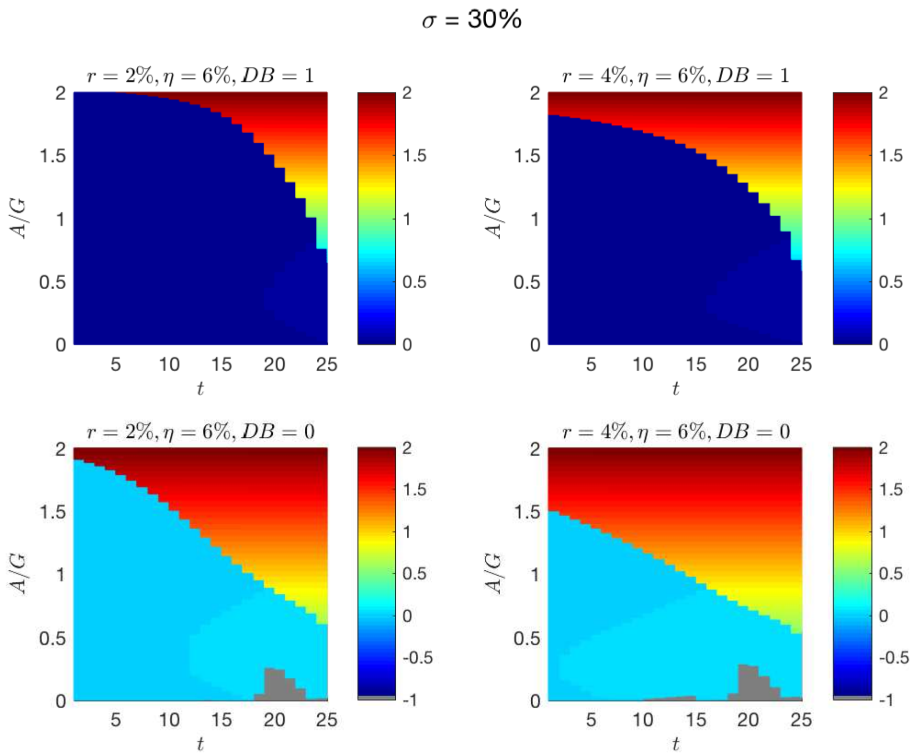

Figure A5. The withdrawals strategy as a function of time t and moneyness $\frac{A}{G}$ for $\sigma = 30\%$

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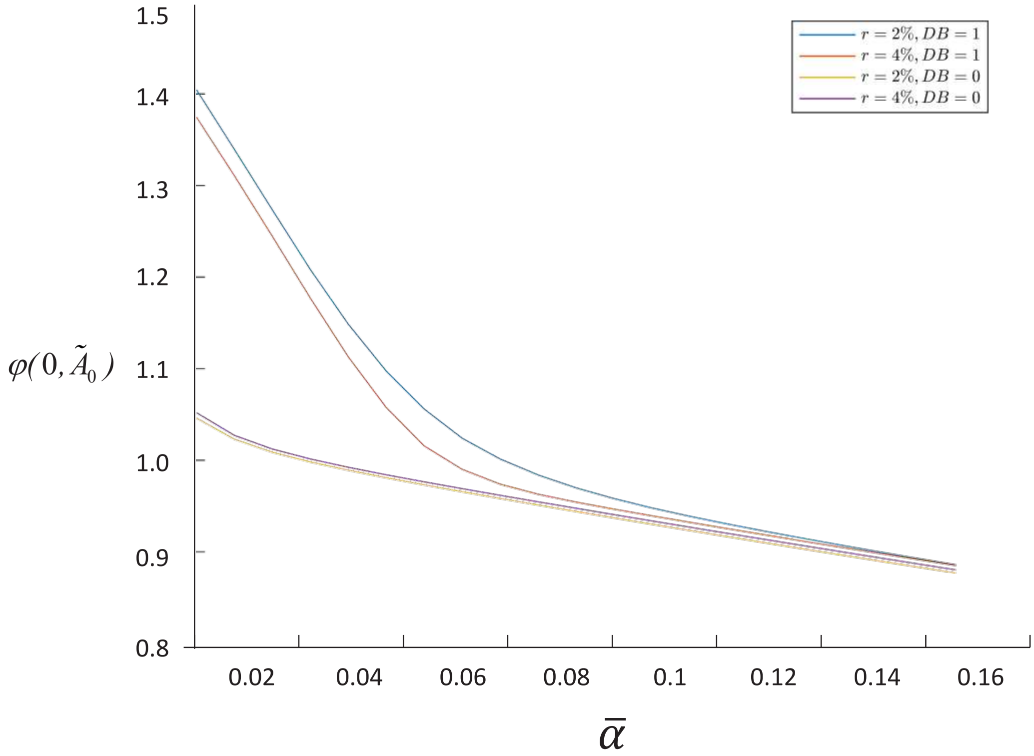

Figure A6. The contract value at inception as a function of the total fees $\bar{\alpha}$ for Product A with and without DB for $r = 2\%,\ 4\%$

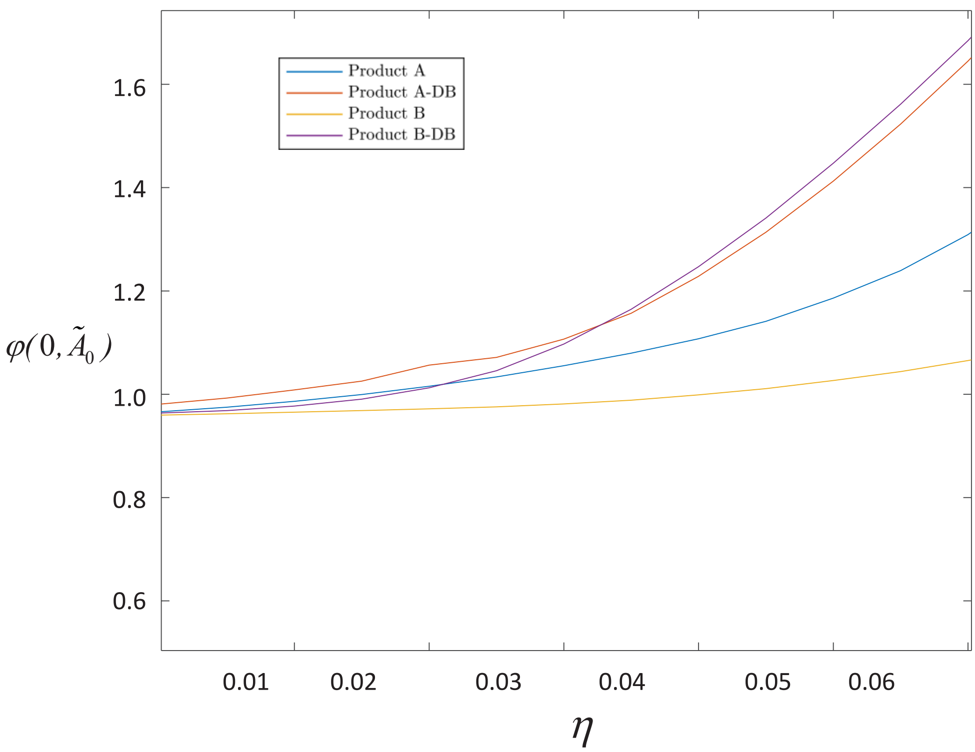

Figure A7. The contract value at inception as a function of the roll-up rate η for Product A and B with and without DB for $r = 2\%$