“Spectral study of options based on CEV model with multidimensional volatility”

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This article studies the derivatives pricing using a method of spectral analysis, a theory of singular and regular perturbations. Using a risk-neutral assessment, the authors obtain the Cauchy problem, which allows to calculate the approximate price of derivative assets and their volatility based on the diffusion equation with fast and slow variables of nonlocal volatility, and they obtain a model with multidimensional stochastic volatility. Applying a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular perturbations, an analytic formula for approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on \( l \)-fast variables and \( r \)-slowly variables, \( l \geq 1, r \geq 1, l \in \mathbb{N}, r \in \mathbb{N} \) and a local variable. Applying the Sturm-Liouville theory, Fredholm’s alternatives, as well as the analysis of singular and regular perturbations at different time scales, the authors obtained explicit formulas for derivatives price approximations. To obtain explicit formulae, it is necessary to solve \( 2l \) Poisson equations.

Keywords
CEV model, stochastic multidimensional volatility, spectral theory, singular perturbation theory, regular perturbation theory

JEL Classification
G11, G13, G32

INTRODUCTION

The constant elasticity of variance (CEV) model is a generalization of geometric Brownian motion models. This model has been introduced by Cox and Ross (1976) for pricing European call options. The CEV model is based on the assumption that the risk-neutral process that describes the stock price \( S \) has the form: \( dS = (r - q) S dt + \sigma S^\alpha dz \), where \( r \) – risk interest rate, \( q \) – dividend yield, \( \sigma \) – parameter of volatility, \( \alpha \) – positive constant, \( dz \) – Wiener process. For \( \alpha = 1 \), the CEV model coincides with the model of the geometric Brownian motion, if \( \alpha < 1 \), then, with the reduction of asset value, its volatility increases, if \( \alpha > 1 \), then, with the increase in the asset value, its volatility also increases. This corresponds to volatility smile and means that volatility is an increasing function depending on the asset price.

Comparing with the models of the geometric Brownian motion, the advantages of a CEV model lie in the fact that the instability ratio correlates with the price of risky assets and may explain empirical bias, such as the volatility smile (Schroder, 1989). The CEV model is usually applied to calculate the theoretical price, sensitivity and expected volatility of options (Emanuel & MacBeth, 1982; Fouque et al., 2000). In recent years, the problem of a pension fund investment is very urgent, it turns out that the CEV model has been successfully applied to study the effective investment strategy (Davydov & Linetsky, 2001; Davydov & Linetsky, 2003).

At the end of the last century, the attention of financial scientists was drawn to the problem of relationship between the asset price and its
1. RESEARCH METHODOLOGY

The purpose of the article is to establish the approximate derivative prices, which are defined by CEV model with stochastic multidimensional volatility and depend on many factors of a spectral theory and a theory of perturbations.

Thus, when a CEV model with stochastic multidimensional volatility is adequate for describing the dynamics of an underlying, the spectral method outlined above serves as a powerful tool for analytically pricing derivatives on that underlying. Among the topics that have been addressed by applying spectral methods with multidimensional diffusions are option pricing (both vanilla and exotic), mortgages valuation, interest rate modeling, volatility modeling, and credit risk.

The necessity to have stock price diffusions that don’t jump to zero in order to default and still have a non-zero probability of falling to zero leads us to naturally consider CEV processes. Moreover, CEV models have the advantage to provide closed-form formulae for European vanilla options and for the probability of default.

In particular, many problems related to the pricing of derivative assets have been solved analytically by using methods from spectral theory. An overview of the spectral method applied to derivative pricing is as follows.

We consider the probability space \((Q, F, P)\) with systems of Wiener motion \((B^0, B^1, ..., B^n, B^o, B^1, ..., B^o)\) and an exponential random variable \(\eta \sim \text{Exp}(1)\), which is independent of \((B^0, B^1, ..., B^n, B^o, B^1, ..., B^o)\).

We consider a homogeneous Markov process, which depends on \(l+r+1\) factors \(\chi = X, Y_1, \ldots, Y_l, Z_l, \ldots, Z_r\), which is defined in some state space \(R = R^l \cdot R^r\), where \((Y_1, \ldots, Y_l) \in R^l, (Z_l, \ldots, Z_r) \in R^r\) is the interval in \(R\) with points \(a_1\) and \(a_2\), such that \(-\infty < a_1 < a_2 < \infty\). Let that \(\chi\) has a beginning at \(R\) and instantly disappears as soon as \(X\) goes beyond \(H\). In particular, the dynamics of \(\chi\) with physical measure \(P\) is as follows:

\[
\chi_t = \begin{cases} X_t, & \text{if } t > 0; \ X_t \not\in H, \\ \Delta, & \text{if } t \leq 0, \\end{cases}
\]

where \(X, Y_1, \ldots, Y_l, Z_l, \ldots, Z_r\) are set

\[
dX_t = \nu(X_t)dt + a(X_t)\times f(Y_1, \ldots, Y_l, Z_l, \ldots, Z_r)dB_t^0,
\]

\[
dY_j = \frac{1}{\eta_j} \cdot \alpha_j(Y_j)dt + \frac{1}{\sqrt{\eta_j}} \cdot \beta_j(Y_j)dB_t^j, \quad j = 1, l.
\]

\[
dZ_i = \gamma_i k_i(Z_i)dt + \sqrt{\gamma_i g_i(Z_i)} dB_t^i, \quad i = 1, r.
\]

\[
d(B^0, B^l) = \rho_{0l} dt, \quad j = 1, l.
\]

\[
d(B^0, B^l) = \rho_{0l} dt, \quad i = 1, r.
\]

\[
d(B^j, B^k) = \rho_{jk} dt, \quad j = 1, l,  \quad i = 1, r.
\]

\[
d(B^j, B^k) = \rho_{jk} dt, \quad j = 1, l,  \quad s = 1, l.
\]

\[
d(B^j, B^k) = \rho_{jk} dt, \quad i = 1, n,  \quad k = 1, r.
\]
\[(X_0, Y_{i0}, \ldots, Y_{i0}, Z_{i0}, \ldots, Z_{i0}) = (x, y_{10}, \ldots, y_{10}, z_{10}, \ldots, z_{10}) \in \mathbb{R},\]

where \( \rho_{j,j} = 0, j \neq r, \rho_{i,i} = 0, \)

\( i \neq k \rho_{ij}, \rho_{ik}, \rho_{j,k}, \) meet the condition

\[|\rho_{ij}|, |\rho_{ik}|, |\rho_{j,k}| \leq 1, \]

and a correlation matrices of a form:

\[
\begin{pmatrix}
1 & \rho_{ij} & \rho_{ik} \\
\rho_{ij} & 1 & \rho_{jk} \\
\rho_{ik} & \rho_{jk} & 1
\end{pmatrix}
\]

are semi positively defined, i.e.

\[1 + 2 \cdot \rho_{ij} \cdot \rho_{ik} \cdot \rho_{jk} - \rho_{ij}^2 - \rho_{ik}^2 - \rho_{jk}^2 \geq 0, \]

\( j = \overline{1, l}, i = \overline{1, r}. \)

In changing from the physical probability measure to the risk-neutral pricing measure, we consider a class of market prices of risk that is general enough to treat credit, equity, and interest rate derivatives in a single framework.

Process \( X \) can represent many economic phenomena and processes, which describe the optimal investment strategies. For example, the stockpiles, the index price, a risk-free short-term interest rate, etc. Even more broadly, \( X \) is an external factor that characterizes the cost of any of the abovementioned processes. We are considering the process \( X \) with stochastic volatility \( a(X, Y) f(Y_{i1}, \ldots, Y_{i1}, Z_{i1}, \ldots, Z_{i1}) > 0 \), which contains both components: local \( a(X, Y) \) and nonlocal \( f(Y_{i1}, \ldots, Y_{i1}, Z_{i1}, \ldots, Z_{i1}) \). Note that the infinitesimal generators (Infiniti zed) for \( Y_j \) and \( Z_j \) have the form \( \forall i, j \).

\[
\begin{align*}
P_{Y_j}^{(0)} & = \frac{1}{\eta_j} \left( \frac{1}{2} \beta_j^2 (y_j) \partial^2 \eta_j + \alpha_j (y_j) \partial \eta_j \right), \\
P_{Z_i}^{(0)} & = \gamma_i \left( \frac{1}{2} g_i^2 (z_i) \partial^2 \gamma_i + c_i (z_i) \partial \gamma_i \right),
\end{align*}
\]

are characterized by the values \( 1 / \eta_j \) and \( \gamma_i \), respectively. Thus, \( Y_1, \ldots, Y_k \) and \( Z_1, \ldots, Z_k \) have an internal time scale \( \eta_j > 0 \) and \( 1 / \gamma_i > 0 \). We consider \( \eta_j < 1 \) and \( \gamma_i < 1 \), so that the internal time scale \( Y_j \) is small, and the internal time scale \( Z_i \) is large. Consequently, \( Y_j, j = \overline{1, l} \) are fast variables, and \( Z_i, i = \overline{1, n} \) are slowly variables. Note that \( P_{Y_j}^{(0)} \) and \( P_{Z_i}^{(0)} \) have the form

\[
L = \frac{1}{2} a^2 (x) \partial^2 x + b(x) \partial x - k(x),
\]

\( x \in (a_1, a_2), \ k(x) = 0, \)

for all \( x \in I \), are always self-conjugated in a Hilbert space \( H = L^2 (h, p) \), where \( H \in \mathbb{R} \) is the interval terminating at \( a_1 \) and \( a_2 \) and \( p \) is a diffusion density rate.

The boundary conditions for \( a_1 \) and \( a_2 \) are implemented on the output, input and regular bounds.

We evaluate the derivative security with payoff at time \( t > 0 \), which may depend on trajectory \( X \).

In particular, we will consider the forms of payoff:

\[
\text{Payoff} = H (X, t, X) \cdot I_{t, \phi, \Gamma},
\]

where \( \theta \) is a random moment of time during which there is a failure to make a payment of premium. Since we are interested in derivatives estimates, we must determine the dynamics \( (X, Y_1, \ldots, Y_k, Z_1, \ldots, Z_k) \) with risk-neutral measure estimate, which we denote as \( P \). We have the following dynamics:

\[
\begin{align*}
&dX = \left( \frac{b(X) - a(X)}{x} f(Y_1, \ldots, Y_k, Z_1, \ldots, Z_k) \times Q(Y_1, \ldots, Y_k, Z_1, \ldots, Z_k) \right) dt + \\
&+ a(X) \cdot f(Y_1, \ldots, Y_k, Z_1, \ldots, Z_k) d\tilde{B}^{\gamma_i},
\end{align*}
\]

\[
\begin{align*}
&dY = \left( \frac{1}{\eta_j} \alpha_j (Y_0) - \frac{1}{\sqrt{\eta_j}} \beta_j (Y_0) \times Q(X_0, \ldots, X_l, Y_0, Z_1, \ldots, Z_k) \right) dt + \frac{1}{\sqrt{\eta_j}} \beta_j (Y_0) d\tilde{B}^{\gamma_i},
\end{align*}
\]

\[
\begin{align*}
&dZ = \left( \frac{1}{\gamma_i} k_i (Z_0) - \sqrt{\gamma_i} g_i (Z_0) \times Q(X_0, \ldots, X_l, Y_0, Z_1, \ldots, Z_k) \right) dt + \sqrt{\gamma_i} g_i (Z_0) d\tilde{B}^{\gamma_i},
\end{align*}
\]

\[
\begin{align*}
&d \left( B^{\gamma_i}, \tilde{B}^{\gamma_i} \right) = \rho_{\alpha, \gamma} dt, j = \overline{1, l},
\end{align*}
\]

\[
\begin{align*}
&d \left( B^{\gamma_i}, \tilde{B}^{\gamma_i} \right) = \rho_{\gamma, \gamma} dt, i = \overline{1, r},
\end{align*}
\]

\[
\begin{align*}
&d \left( B^{\gamma_i}, \tilde{B}^{\gamma_i} \right) = \rho_{\gamma, \gamma} dt, j = \overline{1, l}, i = \overline{1, r},
\end{align*}
\]

\[
\begin{align*}
&d \left( B^{\gamma_i}, \tilde{B}^{\gamma_i} \right) = \rho_{\gamma, \gamma} dt, j = \overline{1, l}, s = \overline{1, \ell},
\end{align*}
\]

\[
\begin{align*}
&d \left( B^{\gamma_i}, \tilde{B}^{\gamma_i} \right) = \rho_{\gamma, \gamma} dt, i = \overline{1, n}, k = \overline{1, n},
\end{align*}
\]

\[
(X_0, Y_{i0}, \ldots, Y_{i0}, Z_{i0}, \ldots, Z_{i0}) \in E.
\]
where
\[ d\tilde{B}_t := dB_t + \frac{v(X_t) - b(X_t)}{a(X_t) f(Y_t, \ldots, Y_n, Z_t, \ldots, Z_n)} dt, \]
\[ d\tilde{B}_{t}^{i} := dB_{t}^{i} + \Phi_{j} (Y_{t}, \ldots, Y_{n}, Z_{t}, \ldots, Z_{n}) dt, \]
\[ d\tilde{B}_{t}^{i} := dB_{t}^{i} + \Psi_{j} (Y_{t}, \ldots, Y_{n}, Z_{t}, \ldots, Z_{n}) dt, \]
where \( \rho_{ij} = 0, j \neq s, \rho_{zz} = 0, i \neq k. \)

We implement such conditions so that the system (1) has the only strong solution.

Random time \( \tau \) is the time of a derivative asset. In our case, a default can occur in neither of two ways:

when \( X \) is beyond the interval \( H; \)

at random time \( \theta_{i} \), which is managed by risk level \( h(X_{i}) \geq 0. \)

This can be expressed as follows:
\[
\theta = \tau_{H} \cup \tau_{s},
\theta_{H} = \inf \{ t \geq 0; X \notin H \},
\theta_{s} = \inf \left\{ t \geq 0; h(X_{i}) ds \geq \zeta e(X, Y_{i}, \ldots, Y_{n}, Z_{i}, \ldots, Z_{n}) \right\}
\]

Note that the random variable \( \eta \) is independent of \( X, Y_{i}, \ldots, Y_{n}, Z_{i}, \ldots, Z_{n}. \)

We will calculate the derivative asset of some payoff using risk-free pricing and Markovian chain \( X \), the price \( w^{\pi_{j}}(t, x, y_{i}, \ldots, y_{n}, z_{i}, \ldots, z_{n}) \) of certain derivative assets at the initial moment of time has the form:

\[ w^{\pi_{j}}(t, x, y_{i}, \ldots, y_{n}, z_{i}, \ldots, z_{n}) = \]
\[ = \tilde{E}_{x, y_{i}, \ldots, y_{n}, z_{i}, \ldots, z_{n}} \left[ \exp \left( - \int_{0}^{t} \frac{v(X_{u}) - b(X_{u})}{a(X_{u}) f(Y_{u}, \ldots, Y_{n}, Z_{u}, \ldots, Z_{n})} du \right) H(X_{t} | _{t \geq \theta_{j}}) \right], \]

where \( \overline{\eta} = (\eta_{1}, \ldots, \eta_{r}) \), \( \overline{\gamma} = (\gamma_{1}, \ldots, \gamma_{s}) \), and \( (x, y_{i}, \ldots, y_{n}, z_{i}, \ldots, z_{n}) \in E \) is a starting point of the process \( (X, Y_{1}, \ldots, Y_{n}, Z_{1}, \ldots, Z_{n}) \) applying the Feynman-Katsel formula, we can show that \( w^{\pi_{j}}(t, x, y_{i}, \ldots, y_{n}, z_{i}, \ldots, z_{n}) \) satisfies the following Cauchy problem (Linetsky, 2007):

\[
(-\partial_{1} + P^{\pi_{j}}) w^{\pi_{j}} = 0,
\]
\[
(y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \in R, \quad t \in R^{+},
\]
\[ w^{\pi_{j}}(0, x, y_{i}, \ldots, y_{n}, z_{i}, \ldots, z_{n}) = H(x), \]

where the operator \( P^{\pi_{j}} \) has the form:
\[ P^{\pi_{j}} = \sum_{j=1}^{l} \frac{1}{P_{0j}} P_{0j} + \sum_{j=1}^{l} \frac{1}{P_{jj}} P_{jj} + \]
\[ + P_{0j} + \sum_{j=1}^{l} \sum_{i=1}^{m} \frac{U_{ij} \eta_{i}}{P_{jj}} + \sum_{i=1}^{m} \sum_{i=1}^{m} \gamma_{i} U_{ii} + \gamma_{1} U_{11},
\]
\[
P_{0j} = \frac{1}{2} \beta_{j}^{2} (y_{j}) \theta_{j}^{2} y_{j},
\]
\[
P_{jj} = \beta_{j} (y_{j}) \left( \rho_{ij} a(x) f(y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \theta_{j}^{2} - \right. \]
\[ \left. \cdot \Phi_{j} (y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \theta_{j}^{2} \right),
\]
\[
P_{ki} = \frac{1}{2} \beta_{j}^{2} (y_{j}) f(y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \theta_{j}^{2} + \]
\[ + \left( 2 a(x) f(y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \times \right. \]
\[ \left. \cdot \Psi_{j} (y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \theta_{j}^{2} \right),
\]
\[ U_{ij} = g_{i} (z_{i}) \theta_{j}^{2} y_{j},
\]
\[ U_{ii} = g_{i} (z_{i}) \theta_{j}^{2} - \Phi_{j} (y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \theta_{j}^{2},
\]
\[ U_{ij} = \frac{1}{2} g_{i}^{2} (z_{i}) \theta_{j}^{2} + c_{i} (z_{i}) \theta_{j}^{2},
\]
\[ k(x) = r(x) + h(x) \quad P_{0j} = P_{ij}.
\]

We assume that the diffusion with the infinitesimal generator \( P_{ij} \) has an invariant distribution \( H \) with density \( \pi_{j} (y_{j}) \).

\[ \pi_{j} (y_{j}) = \frac{2}{\beta_{j}^{2} (y_{j})} \exp \left[ \int_{y_{j}}^{\infty} \frac{2a(x)}{\beta_{j}^{2} (x)} d\theta \right], \quad \forall j = \overline{1, l}.
\]

Besides the initial condition (3), function \( w^{\pi_{j}}(t, x, y_{1}, \ldots, y_{n}, z_{1}, \ldots, z_{n}) \) must meet boundary conditions at the points \( a_{1} \) and \( a_{2} \) of interval \( H. \) The boundary conditions at points \( a_{1} \) and \( a_{2} \) belong to domain \( P^{\pi_{j}} \) and will depend on the nature of process \( X \) on the points of \( H \) and are classified as natural, output, input or regular (Borodin & Salminen, 2002). The Cauchy problem (1)-(2) for
\[
\begin{align*}
\{ f, a_{1}, \ldots, a_{1}, b_{1}, \ldots, b_{1}, \Phi_{1}, \ldots, \Phi_{1}, c_{1}, \ldots, c_{1}, \\
g_{1}, \ldots, g_{1}, \Psi_{1}, \ldots, \Psi_{1}
\end{align*}
\]
has no analytical solution. However, for fixed $\gamma$, conditions containing $\eta$ and deviate arbitrarily,

$$
(\quad,\quad,\quad,\quad)
$$

is natural for axis, which leads to singular perturbations.

For fixed $\eta$, conditions containing $\gamma$, are small for some small $\gamma$-axis, which causes regular perturbations. Thus, the $\eta$-axis and $\gamma$-axis initiate a combined singular-regular perturbation of the operator $P$. In order to find the asymptotic solution of the Cauchy problem (2)-(3), we develop $w_{\eta,\gamma}$ in orders $\sqrt{\eta}$ and $\sqrt{\gamma}$:

$$
w_{\eta,\gamma} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sqrt{\eta}^i \sqrt{\gamma}^j w_{h, j, k, h, k, l, k, l} \ldots
$$

where

$$
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sqrt{\eta}^i \sqrt{\gamma}^j \sqrt{h}^i \sqrt{j}^j \ldots
$$

Approximate price is calculated as follows:

$$
w_{\eta,\gamma} \approx w_{\eta,\gamma} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sqrt{\eta}^i \sqrt{\gamma}^j \sqrt{h}^i \sqrt{j}^j \ldots
$$

The choice of development in half-integer orders $\eta$ and $\gamma$ is natural for $P_{\eta,\gamma}$.

By carrying out an analysis of singular perturbations at corresponding levels, we obtain that $w_{\eta,\gamma}$, $w_{\eta,\gamma}$, $w_{\eta,\gamma}$ does not depend on $y_1, \ldots, y_r$. The main results of asymptotic analysis are given using the following formulæ:

$$
\sum_{j=0}^{\infty} P_{ij} w_{\eta,\gamma} + (-\partial_j + \langle P \rangle_j) w_{\eta,\gamma} = 0,
$$

$$
w_{\eta,\gamma} (0, x, z_1, \ldots, z_r) = H (x),
$$

$$
P_{ij} w_{\eta,\gamma} + P_{ij} w_{\eta,\gamma} + (-\partial_j + \langle P \rangle_j) w_{\eta,\gamma} +
+ \sum_{i=0}^{\infty} P_{i} w_{\eta,\gamma} + \sum_{i=0}^{\infty} P_{i} =
\Rightarrow w_{\eta,\gamma} (0, x, z_1, \ldots, z_r) = 0,
$$

From the analysis of regular perturbations, we obtain

$$
\left(-\partial_j + \langle P \rangle_j \right) w_{\eta,\gamma} = v_j \partial_j w_{\eta,\gamma},
$$

$$
\partial_j w_{\eta,\gamma} (0, x, z_1, \ldots, z_r) = 0, \quad i = 1, r.
$$

Operators $\langle P \rangle_j$, $w_j$, $v_j$ are set by the formulæ:

$$
\langle P \rangle_j = \frac{1}{2} \sigma^2 a^2 (x) \partial_j^2 + \left( b (x) - \bar{f} a (x) \right) \partial_j - k (x),
$$

$$
x \in (a_i, a_2),
$$

$$
\nu_j = -v_j a (x) \partial_j a^2 (x) \partial_j^2 - v_j a^2 (x) \partial_j^2 -
- \mu_i a (x) \partial_j - \mu_i a (x) \partial_j,
$$

$$
\nu_i = g_i \rho_i, \quad \nu_i = g_i \nu_i', \quad v_0 = g_0, \nu_i',
$$

$$
\forall i = 1, n \text{ and the norm function}
$$

$$
\langle X \rangle_j : = \int X (y_1, \ldots, y_r) \pi_j (y_j) dy_j, \quad \forall j = 1, l
$$

$$
\langle X \rangle_{j=1} = \int X (y_1, \ldots, y_r) \pi_j (y_2) \pi_j (y_2) dy_2 \ldots \langle X \rangle_{j=1} =
= \int X (y_1, \ldots, y_r) \pi_j (y_2) \pi_j (y_2) dy_2 \ldots dy_r,
$$

$$
\langle X \rangle_{j=1} = X, \quad \langle f Q \rangle = \langle \overline{f Q} \rangle, \quad \langle f \rangle = \overline{f}.
$$

We find solutions to the equations (3)-(5) on the basis of Eigen functions, eigenvalues of the operator $\langle P \rangle_j$, each of which satisfies a corresponding Poisson equation:

$$
P_0 v_1 = f - \langle f \rangle,
$$

$$
P_{02} v_2 = \langle f \rangle - \langle f \rangle_l \ldots,
$$

$$
P_{02} v_2 = \langle f \rangle_l - \langle f \rangle_{l=1},
$$

$$
P_{j=1} v_1 = \langle f \rangle_l - \langle f \rangle_{l=1},
$$

$$
P_{j=1} v_1 = \langle f \rangle_l - \langle f \rangle_{l=1},
$$

$$
P_{j=1} v_1 = \langle f \rangle_l - \langle f \rangle_{l=1},
$$

$$
P_{j=1} v_1 = \langle f \rangle_l - \langle f \rangle_{l=1}.
Theorem 1: Let the equation be:

$$-\{P_2\}v_n = \mu_n v_n, \quad v_n \in \text{dom}(\{P_2\}),$$

and $h \in H$. Then, the solution $w_{0,\sigma}$ has the form:

$$w_{0,\sigma} = \sum_{n}^{} k_n v_n T_n, \quad k_n = (v_n, h), \quad T_n = e^{-\mu_n t}.$$  \hspace{1cm} (7)

Theorem 2: Let $k_n$, $v_n$, $T_n$ be equations (6)-(7):

$$\mathcal{W}_{k,n} = (v_n, \mathcal{W}_n), \quad M_{k,n} := \frac{T_n - T_m}{\mu_k - \mu_n}.$$  \hspace{1cm} (8)

Then, the solution $w_{1,\sigma}$ of equation (4) has the form:

$$w_{1,\sigma} = \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n M_{k,n} - \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n T_n.$$  \hspace{1cm} (9)

Note that $w_{1,\sigma}$ is linear in the parameter group $(\theta_{1, \sigma}, \theta_{2, \sigma}, u_{1, \sigma}, k_{1, \sigma})$.

Theorem 3: Let $k_n$, $v_n$ and $T_n$ are set by (6)-(7) and $M_{k,n}$ from (8)-(9), then:

$$w_{1,\sigma} = (v_n, \mathcal{W}_n), \quad M_{k,n} := \frac{T_{1,n} - T_m}{\mu_k - \mu_n}.$$  \hspace{1cm} (10)

Then, the solution $w_{1,\sigma}$ has the form:

$$w_{1,\sigma} = \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n M_{k,n} - \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n T_n +$$

$$+ \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n M_{k,n} - \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n T_n +$$

$$+ \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n \left(\frac{\partial_{x} \mu_n}{2}\right) N_{k,n} -$$

$$- \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n \left(\frac{\partial_{x} \mu_n}{2}\right) \frac{1}{2} T_n.$$  \hspace{1cm} (11)

We proved that $u_{1,\sigma}$ are linear in

$$\left(v_n, \mathcal{W}_n, b, f(q), b, f(q)\right).$$

We have obtained the approximate solution:

$$w_{1,\sigma} \approx w_{1,\sigma} + \sum_{n}^{} \mathcal{W}_{k,n} v_n M_{k,n} - \sum_{n}^{} k_n \mathcal{W}_{k,n} v_n T_n.$$  \hspace{1cm} (12)

2. APPLICATION OF THE DESCRIBED METHODOLOGY

The models developed by these scholars have their advantages and disadvantages, but each of them is used to increase the liquidity of financial markets. The findings are credit spread of credit market instruments, calculating option prices for interest rates, determining the risk and derivatives’ rate of return of the stock market financial instruments.

Using the method of Eigen function expansions, we derive analytical solutions for zero-coupon bonds and bond options under CEV processes for the shadow rate. This class of models can be used to model low interest rate regimes.

Let’s assume that the asset is defined by $S_t = I_{[0,\sigma]} X_t$. Since $S_t$ must be positive, the space of states $X$ will be $(a, b) = (0, \infty)$. A multiscalar diffusion is formed on the default time using a method of continuous variations (Carr & Linetsky, 2006). In particular, $P$ is the dynamics of $X$ before default is set:

$$dX_t = (\nu + bX_t^{2k}) X_t dt +$$

$$+ \left(f(Y_t, \ldots, Z_t) X_t^r\right) X_t dB_t,$$

$$h(X_t) = \nu + bX_t^{2k}.$$  \hspace{1cm} (13)

For ease of calculation, the risk-free interest rate is zero; $r = 0$. Let’s calculate the approximate price of a European option, which is described by $S$. 
We write down the operator \( \langle P_2 \rangle \) and associated with it densities at a rate \( p(x) \):

\[
\langle P_2 \rangle = \frac{1}{2} \sigma^2 x^{2\nu+2} \frac{\partial^2}{\partial x^2} + (v + bx^\nu) x \frac{\partial}{\partial x}, \quad (v + bx^\nu),
\]

\[
p(x) = \frac{2}{\sigma^2} \cdot x^{2\nu-2-2\nu} \exp(Wx^{-\nu}), \quad W = \frac{v}{\sigma^2} |x|
\]

We consider the diffusion operator (10), the end of the interval \( a_2 = \infty \) is a natural boundary. However, the classification of end point \( a_1 = 0 \) depends on the value of \( k \) and \( \frac{b}{\sigma^2} \), i.e.:

1) \( \frac{b}{\sigma^2} \geq \frac{1}{2}, \quad k < 0, \quad a_1 = 0 \) is a trivial case;
2) \( \frac{b}{\sigma^2} \in \left(0, \frac{1}{2}\right), \quad k \in \left[\frac{b}{\sigma^2} - \frac{1}{2}, 0\right), \quad a_1 = 0 \) this number plays a role of a starting point;
3) \( \frac{b}{\sigma^2} \in \left(0, \frac{1}{2}\right), \quad k < \frac{b}{\sigma^2} - \frac{1}{2}, \quad a_1 = 0 \) at such a condition, the start of the interval is constant.

If \( (b, \sigma, k) \) satisfy \( \frac{b}{\sigma^2} \in \left(0, \frac{1}{2}\right), \) and
\( k \in \left[\frac{b}{\sigma^2} - \frac{1}{2}, 0\right), \) \( a_1 = 0 \) then \( a_i \) is considered as a keeling boundary. In this case, we find Egg functions and eigenvalues (7) boundary conditions:

\[
limit_{x \to 0} v_n = 0, \quad \text{if} \quad \frac{b}{\sigma^2} \in \left(0, \frac{1}{2}\right).
\]

The solution has the following form (Mendoza-Arriaga et al., 2010):

\[
v_n = W_2^\nu \frac{\Gamma((n-1)\nu)}{\Gamma(n+\nu)} x^{n+\nu} \exp(-Wx^{-\nu}) L_{n-1}^{(i)}(Wx^{-\nu}),
\]

\( n = 1, 2, 3, \ldots \),

\[
\mu_n = 2v|x|/(n+\nu), \quad \nu = \frac{1+2\left(\frac{2}{\sigma^2}\right)}{2|k|},
\]

where \( L_n^{(i)} \) are generalized Laguerre polynomials. For the system (10), operators \( W \) and \( V \) have the form:

\[
\mathcal{W}_j = -v_j x^{2i+1} \partial_x x^{2i+1} \partial_x^2 - v_{2j} x^{2i+1} \partial_x^2,
\]

\[
\mathcal{V}_j = -v_{2j} x^{2i+1} \partial_x^2 - v_{2j+1},
\]

Payoff for a European call option with the execution price \( K > 0 \) can be expanded as follows:

\[
(K - S_t)^+ = (K - X_t)^+ I_{(\{\theta\})} + K (1 - I_{\{\theta\}}).
\]

The first item on the right-hand side (11) is the option payoff before the default at time \( t \). The second item represents option payoff after the default, which may occur at time \( t \). Thus, the option value with execution price \( K \) is designated by \( w^{\pi \gamma}(t, x; K) \) can be expressed as a sum of two parts:

\[
w^{\pi \gamma}(t, x; K) = w_0^{\pi \gamma}(t, x; K) + w_D^{\pi \gamma}(t, x; K),
\]

where

\[
w_0^{\pi \gamma}(t, x; K) = \tilde{E}_{\gamma \eta \nu \ldots \gamma \eta \nu} \left[(K - X_t)^+ I_{\{\theta\}}\right],
\]

\[
w_D^{\pi \gamma}(t, x; K) = K - \tilde{E}_{\gamma \eta \nu \ldots \gamma \eta \nu} \left[\delta_{\gamma} (X_t) I_{\{\theta\}}\right] d\gamma' =
\]

\[
= K - \tilde{E}_{\gamma \eta \nu \ldots \gamma \eta \nu} \left[(K - X_t)^+ I_{\{\theta\}}\right].
\]

Note that \( 1 \notin L^*\{R^*, p\} \), we used the fact that the integral of Dirac's function is equation. As payoff functions:

\[
H_0(x) = (K - x)^+ \quad \text{and}
\]

\[
H_1(x) = \delta_{\gamma} (x) 4, \in L^*\{R^*, p\},
\]

we can calculate:

\[
k_{\nu, \eta} = (v^\nu(\cdot). (K_{-\cdot})^+), \quad k_{\nu, \eta} = (v^\nu, \delta_{\gamma}).
\]

The coefficients can be found in Mendoza-Arriaga et al. (2010).

The approximate European option price which are described by system (10) can now be calculated using theorems 1-4.

Note that volatility \( \sigma^{\pi \gamma} \) of the option with price \( w^{\pi \gamma}(t, x; K) \) is solved by the formula:

\[
w^{\pi \gamma}(t, x; K) = w^{RS}(t, x, \sigma^{\pi \gamma}; K),
\]

where \( w^{RS}(t, x, \sigma^{\pi \gamma}; K) \) is Black-Scholes price with volatility \( \sigma^{\pi \gamma} \).

Note that figures are constructed component-wise in each corresponding time scale, similarly to both components in works of Lorig (2014), Burtnyk and Malytska (2016).
CONCLUSION

This paper expands the methodology of approximate pricing for a wide range of derivative assets. Using a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular disturbances, an analytic formula for the approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on \( l \)-fast variables and \( r \)-slowly variables, \( l \geq 1, r \geq 1, l \in N, r \in N \) and a local variable. Applying the analysis of singular and regular perturbations at different time scales, we obtained explicit formulas for derivatives price approximations. The theorem of closeness estimates of financial instruments approximate prices is proved.

The main advantage of our pricing methodology is that by combining methods from spectral theory, regular perturbation theory, and the theory of singular perturbations, we reduce everything to the solution of the equations to find their Eigen functions and eigenvalues.

REFERENCES