“Spectral study of options based on CEV model with multidimensional volatility”

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SPECTRAL STUDY OF OPTIONS BASED ON CEV MODEL WITH MULTIDIMENSIONAL VOLATILITY

Abstract

This article studies the derivatives pricing using a method of spectral analysis, a theory of singular and regular perturbations. Using a risk-neutral assessment, the authors obtain the Cauchy problem, which allows to calculate the approximate price of derivative assets and their volatility based on the diffusion equation with fast and slow variables of nonlocal volatility, and they obtain a model with multidimensional stochastic volatility. Applying a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular perturbations, an analytic formula for approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on l-fast variables and r-slowly variables, $l \geq 1, r \geq 1$, $l \in N, r \in N$ and a local variable. Applying the Sturm-Liouville theory, Fredholm's alternatives, as well as the analysis of singular and regular perturbations at different time scales, the authors obtained explicit formulas for derivatives price approximations. To obtain explicit formulae, it is necessary to solve 2l Poisson equations.

Keywords CEV model, stochastic multidimensional volatility, spectral theory, singular perturbation theory, regular perturbation theory

JEL Classification G11, G13, G32

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INTRODUCTION

The constant elasticity of variance (CEV) model is a generalization of geometric Brownian motion models. This model has been introduced by Cox and Ross (1976) for pricing European call options. The CEV model is based on the assumption that the risk-neutral process that describes the stock price S has the form: $dS = (r - q)Sdt + \sigma S^{\alpha} dz$, where r – risk interest rate, q – dividend yield, σ – parameter of volatility, α – positive constant, dz – Wiener process. For $\alpha = 1$, the CEV model coincides with the model of the geometric Brownian motion, if $\alpha < 1$, then, with the reduction of asset value, its volatility increases, if $\alpha > 1$, then, with the increase in the asset value, its volatility also increases. This corresponds to volatility smile and means that volatility is an increasing function depending on the asset price.

Comparing with the models of the geometric Brownian motion, the advantages of a CEV model lie in the fact that the instability ratio correlates with the price of risky assets and may explain empirical bias, such as the volatility smile (Schroder, 1989). The CEV model is usually applied to calculate the theoretical price, sensitivity and expected volatility of options (Emanuel & MacBeth, 1982; Fouqueet at al., 2000). In recent years, the problem of a pension fund investment is very urgent, it turns out that the CEV model has been successfully applied to study the effective investment strategy (Davydov & Linetsky, 2001; Davydov & Linetsky, 2003).

At the end of the last century, the attention of financial scientists was drawn to the problem of relationship between the asset price and its volatility (Schroder, 1989). It was found that the asset price behaves like volatility. According to a Black-Scholes model, volatility is constant (Black & Scholes, 1973). As a result, this led to a series of works to expand this model. Empirical studies have found that volatility is a random variable dependent on time. Hull and White (1989), Stein and Stein (1991), Heston (1993) introduced analytical models with stochastic volatility. Carr and Linetsky (2006), Aboulaich et al. (2013) investigated the stochastic volatility model with jumps. The CEV model is a generalization of dynamic volatility models. In particular, it provides an opportunity to examine the asset price that changes continuously over time. There is a series of works dedicated to this problem (Andersen & Piterbarg, 2007; Lindsay & Brecher, 2012). In this article, we consider the CEV model with multidimensional stochastic volatility. Using a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular disturbances, an analytic formula for the approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on l-fast variables and r-slowly variables, $l \geq 1$, $\mathrm{r} \geq 1$, $l \in N, r \in N$ and a local variable. The theorem of closeness in estimation of financial instruments prices approximation is proved.

1. RESEARCH METHODOLOGY

The purpose of the article is to establish the approximate derivative prices, which are defined by CEV model with stochastic multidimensional volatility and depend on many factors of a spectral theory and a theory of perturbations.

Thus, when a CEV model with stochastic multidimensional volatility is adequate for describing the dynamics of an underlying, the spectral method outlined above serves as a powerful tool for analytically pricing derivatives on that underlying. Among the topics that have been addressed by applying spectral methods with multidimensional diffusions are option pricing (both vanilla and exotic), mortgages valuation, interest rate modeling, volatility modeling, and credit risk.

The necessity to have stock price diffusions that don't jump to zero in order to default and still have a non-zero probability of falling to zero leads us to naturally consider CEV processes. Moreover, CEV models have the advantage to provide closed-form formulae for European vanilla options and for the probability of default.

In particular, many problems related to the pricing of derivative assets have been solved analytically by using methods from spectral theory. An overview of the spectral method applied to derivative pricing is as follows.

We consider the probability space (Q, F, P) with systems of Wiener motion $\left(B^x,\ B^{y_1},\ldots,\ B^{y_l},\ B^{z_1},\ldots,B^{z_r}\right)$ and an exponential random variable $\eta \sim Exp(1)$, which is independent of $\left(B^x,\ B^{y_1},\ldots,\ B^{y_l},\ B^{z_1},\ldots,B^{z_r}\right)$. We consider a homogeneous Markov process, which depends on $l+r+1$ factors $\chi = X,\ Y_1,\ \ldots,\ Y_l,\ Z_1,\ldots,\ Z_r$, which is defined in some state space $R = H\cdot R^l\cdot R^r$, where $\left(Y_1,\ \ldots,\ Y_l\right)\in R^l$, $\left(Z_1,\ldots,Z_r\right)\in R^r I$ is the interval in R with points a_1, and a_2, such that $-\infty < a_1 < a_2 < \infty$. Let that χ has a beginning at R and instantly disappears as soon as X goes beyond H. In particular, the dynamics of χ with physical measure P is as follows:

$$\chi_t = \begin{cases} X_t,\ Y_{1t},\ \ldots,\ Y_{lt},\ Z_{1t},\ldots,Z_{rt}, & \theta_H > t, \\ \Delta, & \theta_H \leq t, \end{cases}$$

$$\theta_H = inf\left(t>0\text{: } X_t \notin H\right),$$

where $X,\ Y_1,\ \ldots,\ Y_l,\ Z_1,\ldots,Z_r$ are set

$$dX_t = v\left(X_t\right)dt + a\left(X_t\right)\times$$
$$\times f\left(Y_{1t},\ \ldots,\ Y_{lt},\ Z_{1t},\ldots,Z_{rt}\right)dB_t^x,$$

$$dY_{jt} = \frac{1}{\eta_j}\cdot\alpha_j\left(Y_{jt}\right)dt + \frac{1}{\sqrt{\eta_j}}\cdot\beta_j\left(Y_{jt}\right)dB_t^{y_j},\ j=\overline{1,l}.$$

$$dZ_{it} = \gamma_i k_i\left(Z_{it}\right)dt + \sqrt{\gamma_i}\,g_i\left(Z_{it}\right)dB_t^{z_i},\ \mathrm{i}=\overline{1,r}.$$

$$d\left(B^x, B^{y_j}\right)_t = \rho_{xy_j}dt,\ j=\overline{1,l}.$$

$$d\left(B^x, B^{z_i}\right)_t = \rho_{xz_i}dt,\ i=\overline{1,r}.$$

$$d\left(B^{y_j}, B^{z_i}\right)_t = \rho_{y_jz_i}dt,\ \ j=\overline{1,l},\ \ i=\overline{1,r}.$$

$$d\left(B^{y_j}, B^{y_r}\right)_t = \rho_{y_jy_s}dt,\ \ j=\overline{1,l}, s=\overline{1,l}.$$

$$d\left(B^{z_i}, B^{z_k}\right)_t = \rho_{z_iz_k}dt,\ \ i=\overline{1,n},\ \ k=\overline{1,r}.$$

$$\left(X_0,\ Y_{10},\ldots,Y_{l0},Z_{10},\ldots,Z_{r0}\right)= =\left(x,y_{10},\ldots,y_{l0},z_{10},\ldots,z_{r0}\right)\in R,$$

where $\rho_{y_jy_s}=0$, $j\neq r$, $\rho_{z_iz_k}=0$, $i\neq k\rho_{xy_j}$, ρ_{xz_i}, $\rho_{y_jz_i}$, meet the condition $\left|\rho_{xy_j}\right|,\left|\rho_{xz_i}\right|,\left|\rho_{y_jz_i}\right|\leq 1,$ and a correlation matrices

of a form: $\begin{pmatrix} 1 & \rho_{xy_j} & \rho_{xz_i} \\ \rho_{y_jx} & 1 & \rho_{y_jz_i} \\ \rho_{z_ix} & \rho_{z_iy_j} & 1 \end{pmatrix}$

semi positively defined, i.e.
$1+2\cdot\rho_{xy_j}\cdot\rho_{xz_i}\cdot\rho_{y_jz_i}-\rho^2_{xy_j}-\rho^2_{xz_i}-\rho^2_{y_jz_i}\geq 0,$
$j=\overline{1,l},\ i=\overline{1,r}.$

In changing from the physical probability measure to the risk-neutral pricing measure, we consider a class of market prices of risk that is general enough to treat credit, equity, and interest rate derivatives in a single framework.

Process X can represent many economic phenomena and processes, which describe the optimal investment strategies. For example, the stockpiles, the index price, a risk-free short-term interest rate, etc. Even more broadly, X is an external factor that characterizes the cost of any of the abovementioned processes. We are considering the process X with stochastic volatility $a\left(X_t\right)f\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{nt}\right)>0,$ which contains both components: local $a\left(X_t\right)$ and nonlocal $f\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{nt}\right)$. Note that the infinitesimal generators (Infiniti zed) for Y_j and Z_i have the form $\forall i,\ j.$

$$P^{\eta_j}_{Y_j}=\frac{1}{\eta_j}\left(\frac{1}{2}\cdot\beta_j^{\,2}\left(y_j\right)\partial^2_{y_jy_j}+\alpha_j\left(y_j\right)\partial_{y_j}\right),$$

$$P^{\delta_i}_{Z_i}=\gamma_i\left(\frac{1}{2}\cdot g_i^{\,2}\left(z_i\right)\partial^2_{z_iz_i}+c_i\left(z_i\right)\partial_{z_i}\right),$$

are characterized by the values $1/\eta_j$ and γ_i, respectively. Thus, $Y_1,\ldots,Y_l$ and $Z_1,\ldots,Z_n$ have an internal time scale $\eta_j>0$ and $1/\gamma_i>0$. We consider $\eta_j<1$ and $\gamma_i<1$, so that the internal time scale Y_j is small, and the internal time scale Z_i is large. Consequently, Y_j, $j=\overline{1,l}$ are fast variables, and Z_i, $i=\overline{1,n}$ are slowly variables. Note that $P^{\eta_j}_{Y_j}$ and $P^{\delta_i}_{Z_i}$ have the form

$$L=\frac{1}{2}\cdot a^2\left(x\right)\partial^2_{xx}+b\left(x\right)\partial_x-k\left(x\right),$$

$$x\in\left(a_1,a_2\right),\ k\left(x\right)=0,$$

for all $x\in I$, are always self-conjugated in a Hilbert space $H=L^2\left(h,p\right)$, where $H\in R$ is the interval terminating at a_1 and a_2 and p is a diffusion density rate.

The boundary conditions for a_1 and a_2 are implemented on the output, input and regular bounds.

We evaluate the derivative security with payoff at time $t>0$, which may depend on trajectory X. In particular, we will consider the forms of payoff:

$$Payoff=H\left(X_t\right)\cdot I_{\theta>t},$$

where θ is a random moment of time during which there is a failure to make a payment of premium. Since we are interested in derivatives estimates, we must determine the dynamics $\left(X,Y_1,\ldots,Y_l,Z_1,\ldots,Z_r\right)$ with risk-neutral measure estimate, which we denote as $\widetilde{P}$. We have the following dynamics:

$$\begin{cases}
dX_t=\begin{pmatrix} b\left(X_t\right)-a\left(X_t\right)\cdot f\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{rt}\right)\times \\ \times Q\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{rt}\right)\end{pmatrix}dt+ \\
+a\left(X_t\right)\cdot f\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{rt}\right)d\widetilde{B}^x_t, \\
dY_{jt}=\begin{pmatrix} \frac{1}{\eta_j}\cdot\alpha_j\left(Y_{jt}\right)-\frac{1}{\sqrt{\eta_j}}\cdot\beta_j\left(Y_{jt}\right)\times \\ \times\Phi_j\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{rt}\right)\end{pmatrix}dt+\frac{1}{\sqrt{\eta_j}}\cdot\beta_j\left(Y_{jt}\right)d\widetilde{B}^{y_j}_t, \\
dZ_{it}=\begin{pmatrix} \gamma_ik_i\left(Z_{it}\right)-\sqrt{\gamma_i}g_i\left(Z_{it}\right)\times \\ \times\Psi_i\left(Y_{1t},\ldots,Y_{lt},Z_{1t},\ldots,Z_{rt}\right)\end{pmatrix}dt+\sqrt{\gamma_i}g_i\left(Z_{it}\right)d\widetilde{B}^{z_i}_t, \\
d\left\langle\widetilde{B}^x,\widetilde{B}^{y_j}\right\rangle_t=\rho_{xy_j}dt, j=\overline{1,l}. \\
d\left\langle\widetilde{B}^x,\widetilde{B}^{z_i}\right\rangle_t=\rho_{xz_i}dt, i=\overline{1,r}. \\
d\left\langle\widetilde{B}^{y_j},\widetilde{B}^{z_i}\right\rangle_t=\rho_{y_jz_i}dt, j=\overline{1,l},\ i=\overline{1,r}. \\
d\left\langle\widetilde{B}^{y_j},\widetilde{B}^{y_s}\right\rangle_t=\rho_{y_jy_s}dt, j=\overline{1,l}, s=\overline{1,l}. \\
d\left\langle\widetilde{B}^{z_i},\widetilde{B}^{z_k}\right\rangle_t=\rho_{z_iz_k}dt, i=\overline{1,n}, k=\overline{1,n}. \\
\left(X_0,Y_{10},\ldots,Y_{l0},Z_{10},\ldots,Z_{r0}\right)= \\
=\left(x,y_{10},\ldots,y_{l0},z_{10},\ldots,z_{r0}\right)\in E,
\end{cases}$$

where

$$d\widetilde{B}_t^x := dB_t^x + \left(\begin{array}{l} \dfrac{v(X_t) - b(X_t)}{a(X_t)\cdot f(Y_{1t},\ldots, Y_{lt}, Z_{1t},\ldots, Z_{nt})} + \\ + Q(Y_{1t},\ldots, Y_{lt}, Z_{1t},\ldots, Z_{rt}) \end{array} \right) dt,$$

$$d\widetilde{B}_t^{y_j} := dB_t^{y_j} + \Phi_j(Y_{1t},\ldots, Y_{lt}, Z_{1t},\ldots, Z_{rt})dt,$$

$$d\widetilde{B}_t^{z_i} := dB_t^{z_i} + \Psi_i(Y_{1t},\ldots, Y_{lt}, Z_{1t},\ldots, Z_{rt})dt,$$

where $\rho_{y_j y_s} = 0,\ j \neq s,\ \rho_{z_i z_k} = 0,\ i \neq k.$

We implement such conditions so that the system (1) has the only strong solution.

Random time τ is the time of a derivative asset. In our case, a default can occur in neither of two ways:

when X is beyond the interval H;

at random time θ_h, which is managed by risk level $h(X_t) \geq 0.$

This can be expressed as follows:

$$\begin{cases} \theta = \tau_H \cup \tau_h, \\ \theta_H = inf\{t \geq 0: X_t \notin H\}, \\ \theta_H = inf\left\{ \begin{array}{l} t \geq 0: \int_0^t h(X_s)ds \geq \\ \geq \varepsilon(X, Y_1, \ldots, Y_l, Z_1,\ldots, Z_n) \end{array} \right\} \end{cases}$$

Note that the random variable η is independent of $X, Y_1, \ldots, Y_l, Z_1,\ldots, Z_n$.

We will calculate the derivative asset of some payoff using risk-free pricing and Markovian chain X, the price $w^{\overline{\eta},\overline{\gamma'}}(t, x, y_1,\ldots, y_l, z_1,\ldots, z_r)$ of some derivative assets at the initial moment of time has the form:

$$w^{\overline{\eta},\overline{\gamma'}}(t, x, y_1,\ldots, y_l, z_1,\ldots, z_r) =$$
$$= \widetilde{E}_{x,y_1,\ldots,y_l,z_1,\ldots,z_r}\left[exp\left(-\int_0^t r(X_m)dm \right) H\left(X_t I_{\{t>\theta\}}\right) \right],$$

where $\overline{\eta} = (\eta_1,\ldots, \eta_l)$, $\overline{\gamma'} = (\gamma_1,\ldots, \gamma_r)$, and $(x, y_1,\ldots, y_l, z_1,\ldots, z_r) \in E$ is a starting point of the process $(X, Y_1, \ldots, Y_l, Z_1,\ldots, Z_r)$ applying the Feynman-Kats formula, we can show that $w^{\overline{\eta},\overline{\gamma'}}(t, x, y_1,\ldots, y_l, z_1,\ldots, z_r)$ satisfies the following Cauchy problem (Linetsky, 2007):

$$\left(-\partial_t + P^{\overline{\eta},\overline{\gamma'}}\right) w^{\overline{\eta},\overline{\gamma'}} = 0,$$
$$(y_1,\ldots, y_l, z_1,\ldots, z_r) \in R,\quad t \in R^+, \tag{1}$$

$$w^{\overline{\eta},\overline{\gamma'}}(0, x, y_1,\ldots, y_l, z_1,\ldots, z_r) = H(x), \tag{2}$$

where the operator $P^{\overline{\eta},\overline{\gamma'}}$ has the form:

$$P^{\overline{\eta},\overline{\gamma'}} = \sum_{j=1}^{l} \frac{1}{\eta_j} P_{0j} + \sum_{j=1}^{l} \frac{1}{\sqrt{\gamma_j}} P_{1j} +$$

$$+ P_{2j} + \sum_{i,j} \sqrt{\frac{\gamma_i}{\eta_j}} U_{3ij} + \sum_i \sqrt{\gamma_i} U_{1i} + \sum_i \gamma_i U_{2i},$$

$$P_{0j} = \frac{1}{2}\cdot \beta_j^2(y_j)\partial^2_{y_j y_j} + \alpha_j(y_j)\partial_{y_j},\quad j = \overline{1, l},$$

$$P_{1j} = \beta_j(y_j)\left(\begin{array}{l} \rho_{xy_j} a(x) f(y_1,\ldots, y_l, z_1,\ldots, z_r)\partial_x - \\ -\Phi_j(y_1,\ldots, y_l, z_1,\ldots, z_r) \end{array} \right)\partial_{y_j},$$

$$P_{2j} = \frac{1}{2}\cdot a^2(x) f^2(y_1,\ldots, y_l, z_1,\ldots, z_r)\partial^2_{xx} +$$
$$+ \left(\begin{array}{l} b(x) - a(x) Q(y_1,\ldots, y_l, z_1,\ldots, z_r)\times \\ \times f(y_1,\ldots, y_l, z_1,\ldots, z_r) \end{array} \right)\partial_x - k(x),$$

$$U_{3ij} = \rho_{xz_i}\beta_j(y_j) g_i(z_i)\partial^2_{y_j z_i},$$

$$U_{1i} = g_i(z_i)\left(\begin{array}{l} \rho_{xz_i} a(x) f(y_1,\ldots, y_l, z_1,\ldots, z_r)\partial_x - \\ -\Psi_i(y_1,\ldots, y_l, z_1,\ldots, z_r) \end{array} \right)\partial_{z_i},$$

$$U_{2i} = \frac{1}{2}\cdot g_i^2(z_i)\partial^2_{z_i z_i} + c_i(z_i)\partial_{z_i},$$

$$k(x) = r(x) + h(x),\quad P_{0j} = P^1_{Y_j}.$$

We assume that the diffusion with the infinitesimal generator $P^1_{Y_j}$ has an invariant distribution H with density $\pi_j(y_j)$.

$$\pi_j(y_j) = \frac{2}{\beta_j^2(y_j)} exp\left\{ \int_{y_{j0}}^{y_j} \frac{2\alpha_j(\theta)}{\beta_j^2(\theta)} d\theta \right\},\quad \forall j = \overline{1, l}.$$

Besides the initial condition (3), function $w^{\overline{\eta},\overline{\gamma'}}(t, x, y_1,\ldots, y_l, z_1,\ldots, z_r)$ must meet boundary conditions at the points a_1 and a_2 of interval H. The boundary conditions at points a_1 and a_2 belong to domain $P^{\overline{\eta},\overline{\gamma'}}$ and will depend on the nature of process X on the points of H and are classified as natural, output, input or regular (Borodin & Salminen, 2002). The Cauchy problem (1)-(2) for

$$\left(\begin{array}{l} f, \alpha_1,\ldots, \alpha_l, \beta_1,\ldots, \beta_r, \Phi_1,\ldots, \Phi_l, c_1,\ldots, c_r, \\ g_1,\ldots, g_r, \Psi_1,\ldots, \Psi_r \end{array} \right)$$

has no analytical solution. However, for fixed $\overline{\gamma'}$, conditions containing $\overline{\eta}$ and deviate arbitrarily small in $\overline{\eta}$ axis, which leads to singular perturbations.

For fixed η_j conditions containing γ_i are small for some small $\overline{\gamma'}$-axis, which causes regular perturbations. Thus, the $\overline{\eta}$-axis and $\overline{\gamma'}$-axis initiate a combined singular-regular perturbation of the operator P_2. In order to find the asymptotic solution of the Cauchy problem (2)-(3), we develop $w^{\overline{\eta},\overline{\gamma'}}$ in orders $\sqrt{\eta_j}$ and $\sqrt{\gamma_i}$:

$$w^{\overline{\eta},\overline{\gamma'}} = \sum_{i_1\ge 0}\cdots\sum_{i_l\ge 0}\sum_{j_1\ge 0}\cdots\sum_{j_r\ge 0}\sqrt{\eta_1}^{\,j_1}\cdots$$
$$\cdots\sqrt{\eta_l}^{\,j_l}\sqrt{\gamma_1}^{\,i_1}\cdots\sqrt{\gamma_r}^{\,i_r}w_{j_1,\dots,j_n,i_1,\dots i_l},$$

where

$$\sum_{i_1\ge 0}\cdots\sum_{i_l\ge 0}\sum_{j_1\ge 0}\cdots\sum_{j_r\ge 0}\sqrt{\eta_1}^{\,j_1}\cdots\sqrt{\eta_l}^{\,j_l}\sqrt{\gamma_1}^{\,i_1}\cdots$$
$$\cdots\sqrt{\gamma_r}^{\,i_r}w_{j_1,\dots,j_n,i_1,\dots i_l} = \lim_{m_1\to\infty,\,\dots,\,m_{l+r}\to\infty}\sum_{i_1\ge 0}^{m_1}\cdots\sum_{i_l\ge 0}^{m_l\ge 0}\sum_{j_1\ge 0}^{m_{l+1}\ge 0}\cdots$$
$$\cdots\sum_{j_n\ge 0}^{m_{l+r}}\cdots\sqrt{\eta_l}^{\,jl}\sqrt{\gamma_1}^{\,i_1}\cdots\sqrt{\gamma_r}^{\,i_r}w_{j_1,\dots,j_r,i_1,\dots i_l},$$

Approximate price is calculated as follows:

$$w^{\overline{\eta},\overline{\gamma'}} \approx w_{\overline{0},\overline{0'}} + \sum_{j=1}^{l}\sqrt{\eta_j}\,w_{\overline{1_j},\overline{0'}} + \sum_{i=1}^{r}\sqrt{\gamma_i}\,w_{\overline{0},\overline{1'_i}}.$$

The choice of development in half-integer orders η_j and γ_i is natural for $P^{\overline{\eta},\overline{\gamma'}}$.

By carrying out an analysis of singular perturbations at corresponding levels, we obtain that $w_{\overline{0},\overline{0'}}$, $w_{\overline{1_j},\overline{0'}}$, $w_{\overline{0},\overline{1'_i}}$ does not depend on $y_1,\dots,y_l$. The main results of asymptotic analysis are given using the following formulae:

$$\sum_{j=1}P_{0j}w_{\overline{2_j},\overline{0'}} + \left(-\partial_t + \langle P_2\rangle\right)w_{\overline{0},\overline{0'}} = 0,$$
$$w_{\overline{0},\overline{0'}}\left(0,x,z_1,\dots,z_r\right) = H(x), \tag{3}$$

$$P_{0j}w_{\overline{3_j},\overline{0'}} + P_{1j}w_{\overline{2_j},\overline{0'}} + \left(-\partial_t + \langle P_2\rangle\right)w_{\overline{1_j},\overline{0'}} +$$
$$+\sum_{k\ne j}P_{1k}w_{\overline{1_{kj}},\overline{0'}} + \sum_{i\ne j}P_{1i} = \tag{4}$$
$$= \mathcal{W}_j w_{\overline{0},\overline{0'}},\ w_{\overline{1_j},\overline{0'}}\left(0,x,z_1,\dots,z_n\right) = 0,$$

$$\overline{1_{kj}} = \left(\underbrace{0,\dots\,1}_{k}\underbrace{0,\,1,\,0,\dots.0}_{j}\right).$$

From the analysis of regular perturbations, we obtain

$$\left(-\partial_t + \langle P\rangle_2\right)w_{\overline{0},\overline{1'_i}} = v_i\partial_{z_i}w_{\overline{0},\overline{0'}},$$
$$w_{\overline{0},\overline{1'_i}}\left(0,x,z_1,\dots,z_r\right) = 0,\ \ i = \overline{1,r}. \tag{5}$$

Operators $\langle P_2\rangle$, w_j, v_i та ∂_{z_i} are set by the formulae:

$$\langle P_2\rangle = \frac{1}{2}\cdot\overline{\sigma}^2a^2(x)\partial_{xx}^2 + \left(b(x) - \overline{fQ}a(x)\right)\partial_x - k(x),$$
$$x\in\left(a_1,a_2\right),$$

$$\mathcal{W}_j = -v_{3j}a(x)\partial_x a^2(x)\partial_{xx}^2 - v_{2j}a^2(x)\partial_{xx}^2 -$$
$$-\mathcal{U}_{2j}a(x)\partial_x a(x)\partial_x - \mathcal{U}_{1j}a(x)\partial_x,$$

$$\mathcal{V}_i = -\acute{\imath}_{1i}a(x)\partial_x - v_{0i}\ \text{ and }\ \partial_{z_i} = \partial_{z_i}\overline{\sigma}\partial_{\overline{\sigma}} + \overline{fQ}'\partial_{\overline{fQ}},$$
$$v_{1i} := g_i\rho_{xz_i}\langle f\rangle,\ \ v_0 = g_i\langle\Psi_i\rangle,$$

$\forall i = \overline{1,n}$ and the norm function

$$\langle\mathcal{X}\rangle_j := \int\mathcal{X}(y_1,\dots y_l)\pi_j(y_j)dy_j,\quad \forall j = \overline{1,l},$$

$$\langle\mathcal{X}\rangle_{1,2} = \int_{R^2}\mathcal{X}(y_1,\dots y_l)\pi_1(y_1)\pi_2(y_2)dy_1dy_2,\dots\ \langle\mathcal{X}\rangle_{l-1,l} =$$
$$= \int_{R^l}\mathcal{X}(y_1,\dots y_l)\pi_1(y_1)\dots\pi_l(y_l)dy_1\dots dy_l,$$

$$\langle\mathcal{X}\rangle_{l-1,l} = \mathcal{X},\ \ \langle fQ\rangle := \overline{fQ},\ \ \langle f^2\rangle = \overline{\sigma}^2.$$

We find solutions to the equations (3)-(5) on the basis of Eigen functions, eigenvalues of the operator $\langle P_2\rangle$, each of which satisfies a corresponding Poisson equation:

$$P_{01}v_1 = f^2 - \langle f^2\rangle_1,$$

$$P_{02}v_2 = \langle f^2\rangle_1 - \langle f^2\rangle_{1,2},\dots,$$
$$\dots P_{0l}v_l = \langle f^2\rangle_{l-2,l-1} - \langle f^2\rangle_{l-1,l}$$

$$P_{01}\eta_1 = fQ - \langle fQ\rangle_1,\dots,P_{0j}\eta_j = \langle fQ\rangle_{j-2,j-1} -$$
$$-\langle fQ\rangle_{j-1,j},\dots,P_{0l}\eta_l = \langle fQ\rangle_{l-2,l-1} - \langle fQ\rangle_{l-1,l}.$$

Theorem 1: Let the equation be:

$$-\langle P_2\rangle v_n = \mu_n v_n,\ \ v_n \in dom(\langle P_2\rangle), \tag{6}$$

and $h \in H$. Then, the solution $w_{\bar{0},\bar{0}'}$ has the form:

$$w_{\bar{0},\bar{0}'} = \sum_{n=1}^{\infty} k_n v_n T_n,\ \ k_n = (v_n, h),\ \ T_n = e^{-t\mu_n}. \tag{7}$$

Theorem 2: Let k_n, v_n, T_n be equations (6)-(7):

$$\mathcal{W}_{jk,n} := (v_k, \mathcal{W}_j v_n),\ \ M_{k,n} := \frac{T_k - T_n}{\mu_k - \mu_n}. \tag{8}$$

Then, the solution $w_{\bar{1}_j,\bar{0}'}$ of equation (4) has the form:

$$w_{\bar{1}_j,\bar{0}'} = \sum_n \sum_{k\neq n} k_n \mathcal{W}_{jk,n} v_k M_{k,n} - \sum_n k_n \mathcal{W}_{jn,n} v_n t T_n. \tag{9}$$

Note that $w_{\bar{1}_j,\bar{0}'}$ is linear in the parameter group $(\vartheta_{3j}, \vartheta_{2j}, u_{2j}, u_{1j})$.

Theorem 3: Let k_n, v_n and T_n are set by (6)-(7) and $M_{k,n}$ from (8)-(9), then:

$$\tilde{\mathcal{V}}_{ik,n} := (v_k, \mathcal{V}_i \partial_{Z_i} v_n),\ \ \mathcal{V}_{ik,n} := (v_k, \mathcal{V}_i v_n),$$

$$M_{ik,n} := \frac{T_k - T_n}{(\mu_k - \mu_n)^2} + \frac{tT_n}{\mu_k - \mu_n}.$$

Then, the solution $w_{\bar{0},\bar{1}_i'}$ has the form:

$$\begin{aligned}w_{\bar{0},\bar{1}_i'} &= \sum_n \sum_{k\neq n} k_n \tilde{\mathcal{V}}_{ik,n} v_k M_{ik,n} - \sum_n k_n \tilde{\mathcal{V}}_{in,n} v_n t T_n + \\ &+ \sum_n \sum_{k\neq n} (\partial_{Z_i} k_n) \mathcal{V}_{ik,n} v_k M_{ik,n} - \sum_n (\partial_{Z_i} k_n) \mathcal{V}_{in,n} v_n t T_n + \\ &+ \sum_n \sum_{k\neq n} k_n \mathcal{V}_{ik,n} v_k (\partial_{Z_i} \mu_n) N_{ik,n} - \\ &- \sum_n k_n \mathcal{V}_{in,n} v_n (\partial_{Z_i} \mu_n) \frac{1}{2} t^2 T_n.\end{aligned}$$

We proved that $u_{\bar{0},\bar{1}_i'}$ are linear in

$$\left(v_{1i}\bar{\sigma}',\ b_{1i}\overline{fQ'},\ b_{0i}\bar{\sigma}',\ b_{0i}\overline{fQ'}\right).$$

We have obtained the approximate solution:

$$w^{\bar{\eta},\bar{\gamma}'} \approx w_{\bar{0},\bar{0}'} + \sum_{j=1}^{l} \sqrt{\eta_j}\, w_{\bar{1}_j,\bar{0}'} + \sum_{i=1}^{n} \sqrt{\gamma_i}\, w_{\bar{0},\bar{1}_i'}$$

for the valuation of derivative assets. For a more accurate result, we assume that the payoff function $H(x)$ and its derivative are smooth and limited functions. Thus, we restrict our analysis of derivatives to a smooth and limited payoff; in this case, the closeness estimates is based on the following theorem.

Theorem 4: If there is fixed $(t, x, y_1, \dots, y_l, z_1, \dots, z_r)$, then there exists a C constant such that for all $\eta_j \le 1$, $\gamma_i \le 1$, the following inequality takes place:

$$\left| w^{\bar{\eta},\bar{\gamma}'} - \left(w_{\bar{0},\bar{0}'} + \sum_{j=1}^{l} \sqrt{\eta_j}\, w_{\bar{1}_j,\bar{0}'} + \sum_{i=1}^{n} \sqrt{\gamma_i}\, w_{\bar{0},\bar{1}_i'} \right) \right| \le$$

$$\le C\left(\sum_{j=1}^{l} \varepsilon_j + \sum_{i=1}^{r} \delta_i \right).$$

Theorem 4 provides us information on how the approximate price behaves when $\eta_j \to 0$ and $\gamma_i \to 0$.

2. APPLICATION OF THE DESCRIBED METHODOLOGY

The models developed by these scholars have their advantages and disadvantages, but each of them is used to increase the liquidity of financial markets. The findings are credit spread of credit market instruments, calculating option prices for interest rates, determining the risk and derivatives' rate of return of the stock market financial instruments.

Using the method of Eigen function expansions, we derive analytical solutions for zero-coupon bonds and bond options under CEV processes for the shadow rate. This class of models can be used to model low interest rate regimes.

Let's assume that the asset is defined by $S_t = I_{\{\theta > t\}} X_t$. Since S must be positive, the space of states X will be $(a_1, a_2) = (0, \infty)$. A multiscale diffusion is formed on the default leap using a method of continuous variations (Carr & Linetsky, 2006). In particular, $\tilde{P}$ is the dynamics of X before default is set:

$$dX_t = (v + bX_t^{2\kappa}) X_t dt +$$
$$+ \left(f(Y_1, \dots, Y_l, Z_1, \dots, Z_n) X_t^{\kappa} \right) X_t d\tilde{B}_t^x,$$
$$h(X_t) = v + bX_t^{2k}.$$

For ease of calculation, the risk-free interest rate is zero: $r = 0$. Let's calculate the approximate price of a European option, which is described by S.

We write down the operator $\langle P_2 \rangle$ and associated with it densities at a rate $p(x)$:

$$\langle P_2 \rangle = \frac{1}{2} \cdot \bar{\sigma}^2 x^{2\kappa+2} \partial_{xx}^2 + \left(\nu + bx^{2\kappa}\right) x \partial_x - \left(\nu + bx^{2\kappa}\right), \quad (10)$$

$$p(x) = \frac{2}{\bar{\sigma}^2} \cdot x^{\frac{2b}{\bar{\sigma}^2 - 2 - 2\kappa}} exp\left(Wx^{-2\kappa}\right), \ W = \frac{\nu}{\bar{\sigma}^2 |\kappa|}.$$

We consider the diffusion operator (10), the end of the interval $a_2 = \infty$ is a natural boundary. However, the classification of end point $a_1 = 0$ depends on the value of k and $\frac{b}{\bar{\sigma}^2}$, i.e.:

1) $\frac{b}{\bar{\sigma}^2} \geq \frac{1}{2}$, $k < 0$, $a_1 = 0$ is a trivial case;

2) $\frac{b}{\bar{\sigma}^2} \in \left(0, \frac{1}{2}\right)$, $k \in \left[\frac{b}{\bar{\sigma}^2} - \frac{1}{2}, 0\right)$, $a_1 = 0$ this number plays a role of a starting point;

3) $\frac{b}{\bar{\sigma}^2} \in \left(0, \frac{1}{2}\right)$, $k < \frac{b}{\bar{\sigma}^2} - \frac{1}{2}$, $a_1 = 0$ at such a condition, the start of the interval is constant.

If $(b, \bar{\sigma}, k)$ satisfy $\frac{b}{\bar{\sigma}^2} \in \left(0, \frac{1}{2}\right)$, and $k \in \left[\frac{b}{\bar{\sigma}^2} - \frac{1}{2}, 0\right)$, $a_1 = 0$ then a_1 is considered as a keeling boundary. In this case, we find Eigen functions and eigenvalues (7) boundary conditions:

$$\lim_{x \to 0} v_n = 0, \text{ if } \frac{b}{\bar{\sigma}^2} \in \left(0, \frac{1}{2}\right).$$

The solution has the following form (Mendoza-Arriaga et al., 2010):

$$v_n = W^{\frac{\upsilon}{2}} \sqrt{\frac{(n-1)!\,\nu}{\Gamma(\upsilon + n)}} x \, exp\left(-Wx^{-2\kappa}\right) L_{n-1}^{(\upsilon)}\left(Wx^{-2\kappa}\right),$$

$$n = 1, 2, 3, \dots ,$$

$$\mu_n = 2\nu|\kappa|(n + \upsilon), \ \upsilon = \frac{1 + 2\left(\frac{2}{\bar{\sigma}^2}\right)}{2|\kappa|},$$

where $L_n^{(\upsilon)}$ are generalized Laguerre polynomials. For the system (10), operators $\mathcal{W}_j$ and $\mathcal{V}_j$ have the form:

$$\mathcal{W}_j = -\nu_{j3} x^{\kappa+1} \partial_x x^{2\kappa+2} \partial_{xx}^2 - \nu_{j2} x^{2\kappa+2} \partial_{xx}^2,$$

$$\mathcal{V}_j = -\nu_{j1} x^{\kappa+1} \partial_x - \nu_{j0}.$$

Payoff for a European call option with the execution price $K > 0$ can be expanded as follows:

$$\left(K - S_t\right)^+ = \left(K - X_t\right)^+ I_{\{\theta > t\}} + K\left(1 - I_{\{\theta > t\}}\right). \quad (11)$$

The first item on the right-hand side (11) is the option payoff before the default at time t. The second item represents option pay off after the default, which may occur at time t. Thus, the option value with execution price K is designated by $w^{\bar{\eta},\bar{\gamma}'}(t, x; K)$ can be expressed as a sum of two parts:

$$w^{\bar{\eta},\bar{\gamma}'}(t, x; K) = w_0^{\bar{\eta},\bar{\gamma}'}(t, x; K) + w_D^{\bar{\eta},\bar{\gamma}'}(t, x; K),$$

where

$$w_0^{\bar{\eta},\bar{\gamma}'}(t, x; K) = \widetilde{E}_{x, y_1, \dots, y_1, z_1, \dots, z_r}\left[\left(K - X_t\right)^+ I_{\{\theta > t\}}\right],$$

$$w_D^{\bar{\eta},\bar{\gamma}'}(t, x; K) = K - K\widetilde{E}_{x, y_1, \dots, y_1, z_1, \dots, z_r}\left[I_{\{\theta > t\}}\right] =$$

$$K - K\int_0^\infty \widetilde{E}_{x, y_1, \dots, y_1, z_1, \dots, z_r}\left[\delta_{x'}\left(X_t\right) I_{\{\theta > t\}}\right] dx' =$$

$$= K - K\int_0^\infty w_1^{\bar{\eta},\bar{\gamma}'}(t, x; x') dx',$$

$$w_1^{\bar{\eta},\bar{\gamma}'}(t, x; x') = \widetilde{E}_{x, y_1, \dots, y_1, z_1, \dots, z_r}\left[\delta_{x'}\left(X_t\right) I_{\{\theta > t\}}\right].$$

Note that $1 \notin L^2\left(R^+, p\right)$, we used the fact that the integral of Dirac's function is equation. As payoff functions:

$H_0(x) = (K - x)^+$ and
$H_1(x) = \delta_{x'}(x) 4_{x'} \in L^2\left(R^+, p\right)$, we can calculate:

$$k_{0,n} = \left(v_n(\cdot), (K - \cdot)^+\right), \ k_{1,n} = \left(v_n, \delta_{x'}\right).$$

The coefficients can be found in Mendoza-Arriaga et al. (2010).

The approximate European option price which are described by system (10) can now be calculated using theorems 1-4.

Note that volatility $\sigma^{\bar{\varsigma},\bar{\gamma}'}$ of the option with price $w^{\bar{\eta},\bar{\gamma}'}(t, x; K)$ is solved by the formula:

$$w^{\bar{\eta},\bar{\gamma}'}(t, x; K) = w^{BS}\left(t, x, \sigma^{\bar{\eta},\bar{\gamma}'}; K\right),$$

where $w^{BS}\left(t, x, \sigma^{\bar{\eta},\bar{\gamma}'}; K\right)$ is Black-Scholes price with volatility $\sigma^{\bar{\eta},\bar{\gamma}'}$.

Note that figures are constructed component-wise in each corresponding time scale, similarly to both components in works of Lorig (2014), Burtnyak and Malytska (2016).

CONCLUSION

This paper expands the methodology of approximate pricing for a wide range of derivative assets. Using a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular disturbances, an analytic formula for the approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on l-fast variables and r-slowly variables, $l \geq 1$, $r \geq 1$, $l \in N$, $r \in N$ and a local variable. Applying the analysis of singular and regular perturbations at different time scales, we obtained explicit formulas for derivatives price approximations. The theorem of closeness estimates of financial instruments approximate prices is proved.

The main advantage of our pricing methodology is that by combining methods from spectral theory, regular perturbation theory, and the theory of singular perturbations, we reduce everything to the solution of the equations to find their Eigen functions and eigenvalues.

REFERENCES

1. Aboulaich, R., Baghery, F., & Jrai, A. (2013). Option Pricing for a StochasticVolatility Jump-Diffusion Model. *International Journal of Mathematics and Statistics, 13*(1), 1-19.
2. Andersen, L., & Piterbarg, V. (2007). Moment Explosions in Stochastic Volatility Models. *Finance Stoch, 11,* 29-50.
3. Black, F., & Scholes, M. (1973). The Pricing of Options and Other Corporate Liabilities. *Journal of Political Economy, 81*(3), 637-654. Retrieved from https://www.cs.princeton.edu/courses/archive/fall09/cos323/papers/black_scholes73.pdf
4. Borodin, A., & Salminen, P. (2002). Handbook of Brownian Motion: Facts and Formulae. Birkhauser.
5. Burtnyak, I. V., & Malytska, A. (2016). System Approach to Calculating Options on the Basis of CEV Model. *Business Inform, 10,* 155-159. Retrieved from http://www.business-inform.net/annotated-catalogue/?year=2016&abstract=2016_10_0&lang=ua&stqa=20
6. Carr, P., & Linetsky, V. (2006) A Jump to Default Extended CEV Model: An Application of Bessel Processes. *Finance and Stochastics, 10,* 303-330. Retrieved from http://users.iems.northwestern.edu/~linetsky/JDCEV.pdf
7. Cox, J., & Ross, S. (1976). The Valuation of Options for Alternative Stochastic Process. *Journal of Financial Economics, 3,* 145-166.
8. Davydov, D., & Linetsky, V. (2001). Pricing and Hedging Path-Dependent Options Under the CEV Process. *Management Science, 47,* 949-965. Retrieved from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.648.7253&rep=rep1&type=pdf
9. Davydov, D., & Linetsky, V. (2003). Pricing Options on Scalar Diffusions: An Eigenfunction Expansion Approach. *Operations Research, 51,* 185-209. Retrieved from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.525.4805&rep=rep1&type=pdf
10. Emanuel, D. C., & MacBeth, J. D. (1982). Further Results on the Constant Elasticity of Variance Call Option Pricing Model. *The Journal of Financial and Quantitative Analysis, 17,* 533-554.
11. Fouque, J. P., Papanicolaou, G., & Sircar, K. R. (2000). *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press.
12. Heston, S. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies, 6*(2), 327-343. Retrieved from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.139.3204&rep=rep1&type=pdf
13. Hull, J., & White, A. (1989). The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance, 42,* 281-300. Retrieved from http://faculty.baruch.cuny.edu/lwu/890/hullwhite87.pdf
14. Lindsay, A. E., & Brecher, D. R. (2012). Simulation of the CEV Process and the Local Martingale Property. *Mathematics and Computers in Simulation, 82,* 868-878.
15. Linetsky, V. (2007). Spectral Methods in Derivatives Pricing. In J. R. Birge & V. Linetsky (Eds.), *Financial Engineering of Handbooks in Operations Research and Management Science* (pp. 223-299). Elsevier.
16. Lorig, M. J. (2014). Pricing Derivatives on Multiscale Diffusions: an Eigenfunction Expansion Approach. *Mathematical Finance, 24*(2), 331-363.
17. Mendoza-Arriaga, R., Carr, P., & Linetsky, V. (2010). Time-Changed Markov Processes in unified Credit equity Modeling. *Mathematical Finance, 20,* 527-569.
18. Schroder, M. (1989). Computing the Constant Elasticity of Variance Option Pricing Formula. *The Journal of Finance, 44,* 211-219. Retrieved from http://www.ase.ro/upcpr/profesori/167/schroder.pdf
19. Stein, E., & Stein, J. (1991). Stock Price Distributions with Stochastic Volatility: an Analytic Approach. *Review of financial studies, 4*(4), 727. Retrieved from http://finance.martinsewell.com/stylized-facts/volatility/SteinStein1991.pdf