“Optimal capital allocation - a generalization of the optimization problem”

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Optimal capital allocation – a generalization of the optimization problem

Abstract

Some recent papers, e.g. Zaks et al.(2006), Frostig et al. (2007) and Dhaene et al. (2012), deal with capital allocation and premium pricing as optimization problems. In these papers, the authors consider some constraints on the convex function and find an explicit solution in each case. In this paper we prove the existence of a unique solution where each class has a different convex function. Moreover, it gives an explicit solution in case the convex functions have first order derivatives.

Keywords: optimization in insurance, premium calculation, convex optimization.

Introduction

A fundamental question in actuarial science is how much money to ask from a policyholder in a heterogeneous portfolio? A closely related question is how to allocate a given amount of capital between the different classes in a portfolio. The first question is called the premium calculation problem; the second is the capital allocation problem. These problems are well known in the literature. Goovaerts et al. (1984) present in their book the main premium principles and their properties. Deprez and Gerber (1985) studied the convexity property of several common premium principles. In addition, they study the problem of optimal reinsurance contract and the problem of splitting the portfolio between several insurers in order to reduce the total premium. Many papers deal with the capital allocation problem, e.g., Cummins (2000) who discussed the advantages and disadvantages of some methods of capital allocation, such as the marginal capital allocation. Denault (2001) and Tsanakas (2009) considered a capital allocation based on methods of game theory. Another approach is to consider some optimization problem as in Zaks et al. (2006) and Frostig et al. (2007) who studied the premium calculation problem, and one of the models of capital allocation that is presented in Dhaene et al. [2012, Section 3.2]. The objective function in these optimization problems is to minimize the expected difference between the premium (or capital) and the loss. The objective function assumes the same convex function for each class. In Dhaene et al. (2012) and Zaks et al. (2006) different weights are allowed, while in Frostig et al. (2007) no weights are allowed. The constraint in these optimization problems is a predetermined level of insolvency or an aggregate capital.

In this paper we generalize the results of these three papers, in the sense of allowing to measure the deference between the allocated capital and the loss by a different convex function for each class. It gives the insurer the flexability to implement its policy in regards of the preferred risks in the portfolio.

Throughout the paper, we refer to two settings of the objective function: the *random setting* and the *deterministic setting*. In the random setting, the random variable (r.v.) S, that describes the claims, appears in the objective function, while in the deterministic setting we consider some real function of the r.v., such as its expectation or its variance. Zaks et al. (2006) and Dhaene et al. (2012) showed the equivalence of the two settings in case of the quadratic function. In these papers, the authors showed that different settings of the weights lead to different common premium principles and capital allocation. Frostig et al. (2007) dealt with the deterministic setting. In this paper, we show the existence of a unique solution considering the random setting. Thereafter, we analyze the case of the deterministic setting in order to obtain an analytical solution.

The rest of the paper is organized as follow. section 1 presents the main notations of this paper. In section 2, we present the main results of the paper follow by some examples in section 3 to illustrate the required calculations. In addition, we show the relations between this paper to the results in Dhaene et al. (2012), Frostig et al. (2007) and Zaks et al. (2006). Brief conclusions are given in the final section.

1. Preliminaries

In this paper, we follow the notations as in Frostig et al. (2007) and Zaks et al. (2006). We consider a heterogeneous portfolio consisting of k classes. Let $X_1,\ldots, X_k$ be k random risks with means $\mu_1,\ldots,\mu_k$ respectively, and let π_i be the premium for a risk in class i, $i = 1,\ldots,k$. Consider a portfolio consisting of n_i random risks $X_{i,1},\ldots,X_{i,n_i}$, distributed as X_i, $i =$

$1,...,k$. Let $n=\sum_{i=1}^{k} n_i$. The aggregate elements of the model are:

$\tilde{\pi}=n_i\pi_i$: the aggregate premium of class i.

$\pi=\sum_{i=1}^{k}\tilde{\pi}_i$: the aggregate premium of the entire portfolio.

$S_i=\sum_{j=1}^{n_i} X_{ij}$: the aggregate risk of class i, with $\tilde{\mu}_i=E[S_i]=n_i\mu_i$.

$S=\sum_{i=1}^{k_i} S_i$: the aggregate risk of the entire portfolio,

$\mu=E[S]=\sum_{i=1}^{k} n_i\mu_i$.

In addition, let $q_{1-\alpha}$ represent the $1-\alpha$ percentile of $S-\mu$ that is,

$P(S-\mu\leq q_{1-\alpha})=1-\alpha$.

When pricing insurance policies, it is common to ask that the premium will fulfill the following principles:

1. Each policyholder pays at least a premium equal to his risk's expectation, i.e. $\pi_i\geq\mu_i$
2. Each policyholder pays no more than the maximum loss.
3. The probability that the total claims (S) exceed the total premiums paid (π) is less than or equal to a predetermined α, where $0<\alpha<1$, i.e. $P(S>\pi)\leq\alpha$
4. The premium (π_i) is a non-decreasing function of the expected claim (μ_i).

In Frostig et al. (2007), the authors found an explicit solution for the *primal optimization problem.* In the primal optimization problem, the authors considered a single convex function in order to measure the difference between the premium and the expected claims in each class. This difference is also known as the residual risk. They found premiums that minimize the aggregate residual risk of the portfolio and they verified that principles 1, 3 and 4 hold. In mathematical notations, in Frostig et al. (2007), the following optimization problem is studied:

$$\begin{cases}\min\limits_{\pi}\left\{\sum_{i=1}^{k} f(\tilde{\pi}_i-\tilde{\mu}_i)\right\}\\ s.t.\quad \sum_{i=1}^{k}\tilde{\pi}_i=\mu+q_{1-\alpha}\\ \pi_i\geq\mu_i\quad \forall_i=1,...,k\end{cases}\tag{1}$$

where the function f: $(-\infty,\infty)\to[0,\infty)$ is strictly convex.

In (1), there is a consideration of the expected claims without any consideration of the r.v. itself. Therefore, this is a deterministic setting of the objective function, which is common practice. We can rewrite the objective function in (1) as $f\left(E\left[\tilde{\pi}_i-S_i\right]\right)$. On the other hand, the real life situation is the random setting, i.e., $E\left[f\left(\tilde{\pi}_i-S_i\right)\right]$. It appears that in some cases (see Dhaene et al. (2012), Zaks et al. (2006)), these two settings lead to the same optimal solution. Note that in case of expectations, the deterministic setting is a special case of the random setting. The two settings coincide in case we set S_i to be a constant equal to $\tilde{\mu}_i$ since $E\left[f\left(\tilde{\pi}_i-\tilde{\mu}_i\right)\right]=f\left(\tilde{\pi}_i-\tilde{\mu}_i\right)$.

2. The optimal solution

In this section we bring the main results of the paper. In section 2.1, we show the existence of a unique solution for the random setting. Section 2.2 considers the deterministic setting, for which we find an analytic solution.

2.1. The existence of a unique solution. A common way to solve this type of optimization problem is by Lagrange multipliers. Sometimes it is possible to solve the first order condition and to obtain an explicit solution. Frostig et al. (2007) found an explicit solution following Majorization's considerations for the case of the same convex function for each class without any weights. Here we allow a different convex function for each class. We prove the existence of a unique solution without the need for derivatives. In proposition 2.1., we consider the case of the random setting.

Proposition 2.1. *Let* g: $(-\infty,\infty)\to[0,\infty)$ *be a strictly convex function. Then the following minimization problem has a unique solution*

$$\begin{cases}\min\limits_{\pi}\left\{\sum_{i=1}^{k} E\left[g_i\left(S_i-\tilde{\pi}_i\right)\right]\right\}\\ s.t.\ \sum_{i=1}^{k}\tilde{\pi}_i=\mu+C\\ \pi_i\geq\mu_i\quad\forall i=1,...,k\end{cases}\tag{2}$$

for any constant $C>0$.

Proof. First we show that $h(x)=E[g(S-x)]$ is a strictly convex function. Let $x,y\in R$ and let $0<\alpha<1$. Hence,

$h(\alpha x + (1-\alpha)y) = E[g(S - \alpha x - (1-\alpha)y)]$

$= E[g(\alpha(S-x) + (1-\alpha)(S-y)] <$

$< E[\alpha g((S-x) + (1-\alpha) g(S-y)] =$

$= \alpha h(x) + (1-\alpha) h(y)$,

where the inequality holds by the convexity of g Therefore, we obtain that $\sum_{i=1}^{k} E\left[g_i\left(S_i - \tilde{\pi}_i\right)\right]$ is a strictly convex function. The constraints create a closed set, thus there is a unique minimum.

2.2. An analytical solution. In the previous section, we proved the existence and the uniqueness of a solution in a framework of the random setting. This section is devoted to finding an expression for the optimal solution in case of the deterministic setting.

In Proposition 2.2., we consider the deterministic setting and obtain the optimal premium $\tilde{\pi}_1$ as a fixed point. Based on this optimal $\tilde{\pi}_1$, we express the optimal premiums for the other classes. First, we analyze the optimization problem without the set of constraints $\pi_i \geq \mu_i$ $i = 1,\ldots,k$. Thereafter, we conclude in Corollary 2.3 that the obtained optimal solution satisfies these constraints as well.

Proposition 2.2. *Let $g_1,\ldots g_k$: $(-\infty, \infty) \to [0, \infty)$ be continuous and strictly convex functions such that g_i has a first order derivative, denoted by ρ_i, for $i = 1,\ldots,k$, and let $C > 0$. Hence, the solution to the optimization problem*

$$\begin{cases} \min\limits_{\pi}\left\{\sum_{i=1}^{k} g_i\left(\tilde{\pi}_i - \tilde{\mu}_i\right)\right\} \\ s.t. \sum_{i=1}^{k} \tilde{\pi}_i = C + \mu \end{cases} \tag{3}$$

satisfies the following conditions

$$\tilde{\pi}_1 - \tilde{\mu}_1 = \rho_1^{-1}\left(\rho_k\left(C - \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1\left(\tilde{\pi}_1 - \tilde{\mu}_1\right)\right)\right)\right) \tag{4}$$

$$\tilde{\pi}_i - \tilde{\mu}_i = \rho_i^{-1}\left(\rho_k\left(C - \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1\left(\tilde{\pi}_1 - \tilde{\mu}_1\right)\right)\right)\right) \tag{5}$$

Proof. Note that since g_i is strictly convex, then its first derivative ρ_i is a strictly increasing function and its inverse function ρ_i^{-1} exists for every $i = 1,\ldots,k$.

Denote $x_i = \tilde{\pi}_i - \tilde{\mu}_i$, and consider the following equivalent problem:

$$\begin{cases} \min\limits_{\pi}\left\{\sum_{i=1}^{k} g_i\left(x_i\right)\right\} \\ s.t. \sum_{i=1}^{k} x_i = C \end{cases} \tag{6}$$

Substitute $x_k = C - \sum_{i=1}^{k-1} x_i$ in the objective function to obtain

$$G(\mathbf{x}) = \sum_{i=1}^{k-1} g_i\left(x_i\right) + g_k\left(C - \sum_{i=1}^{k-1} x_i\right). \tag{7}$$

Thus, we analyze $G(\cdot)$ in order to find its minimum. The $i-th$ element of the gradient of G is thus $\rho_i\left(x_i\right) - \rho_k\left(C - \sum_{j=1}^{k-1} x_j\right)$ for every $i = 1,\ldots,k-1$.

In order to find the critical points we compare $\nabla G(\mathbf{x}) = 0$, and we obtain the relation

$$\rho_i\left(x_i\right) = \rho_k\left(C - \sum_{j=1}^{k-1} x_j\right) \quad \forall i = 1,\ldots,k-1. \tag{8}$$

In particular, we obtain $\rho_i(x_i) = \rho_1(x_1)$ for every $i = 1,\ldots,k-1$. Hence the following holds:

$$x_i = \rho_i^{-1}\left(\rho_1\left(x_1\right)\right). \tag{9}$$

Substituting the last equality in (8), we attain the following condition on x_1:

$$x_1 = \rho_1^{-1}\left(\rho_k\left(C - \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1\left(x_1\right)\right)\right)\right). \tag{10}$$

We denote by x_1^* the solution of (10). Following (9), (10) and by the relation $x_k = C - \sum_{i=1}^{k-1} x_i$, we obtain

$$x_i^* = \rho_i^{-1}\left(\rho_k\left(C - \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1\left(x_1^*\right)\right)\right)\right) \quad \forall i = 1,\ldots,k. \tag{11}$$

$\mathbf{x}^* = \left(x_1^*,\ldots,x_k^*\right)$ is the only point that satisfies the Lagrange first order condition. By Proposition 2.1 we know that there is a unique minimum, hence $\mathbf{x}^*$ is the optimal solution.

In Corollary 2.3 we state a condition for which $\mathbf{x}^*$ is strictly positive, i.e. the optimal solution we have found for (3) satisfies $\pi_i \geq \mu_i$ for every $i = 1,\ldots,k.$, as required in (2).

Corollary 2.3. *Let $g_1,...,g_k$ be as in Proposition 2.2. In addition, Let g_i be an increasing function for non-negative values for every i = 1,…,k. If* $C > max\left(0, \sum_{j=1}^{k-1} \rho_j^{-1}(\rho_1(0))\right)$ *then* x_j^* *in (11) satisfies* $x_j^* > 0$ *for every i = 1,…,k.*

Proof. By assumption, g_i is convex and it is increasing for non-negative values, thus ρ_i and ρ_i^{-1} are positive and increasing functions for non-negative values.

Let x_1^* be the solution to equation (10). If x_1^*=0 then by the relation $C - \sum_{j=1}^{k-1} \rho_j^{-1}(\rho_1(0)) > 0$ it follows that

$$\rho_k\left(C - \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1(x_1^*)\right)\right) > 0 .$$

Therefore, according to (10), x_1^* is strictly positive. This is a contradiction to $x_1^* = 0$.

For any $x_1^* < 0$ by the increasing property of the derivatives we conclude that

$$\sum_{j=1}^{k-1} \rho_j^{-1}(\rho_1(0)) > \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1\left(x_1^*\right)\right) .$$

Hence, we obtain that $C - \sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1(x_1^*)\right) > 0.$

Again, we obtain from (10) that $x_1^* > 0$, a contradiction. Therefore $x_1^* > 0$ for every i = 1,…,k

Remark 1.4. *If every g_i is a decreasing function for negative values and an increasing function for positive values, i.e. $\rho_i(0) = 0$, then we obtain* $\sum_{j=1}^{k-1} \rho_j^{-1}\left(\rho_1(x)\right) \le 0$ *for every* $x \le 0$. *Hence, for every C > 0, the assumptions of Corollary 2.3 hold and therefore* $x_i^* > 0$ *for every i = 1,…,k.*

Proposition 2.2 gives us a numerical method to find the optimal solution. Sometimes it is possible to solve equations (10) and (11) directly, as we show in the next section.

3. Examples

Example 3.1. *Equation (10) can be rewritten as*

$$\sum_{j=1}^{k} \rho_j^{-1}\left(\rho_1(x_1)\right) = C \tag{12}$$

It is straightforward that if $g_i(x) = g_j(x)$ for every j, i = 1,…,k, then $x_i^* = \frac{C}{k}$ for every i = 1,…,k. This is the case in Frostig et al. (2007).

Example 3.2. *Let $g(x)$ be a strictly convex function and define* $g_i(x) = \frac{1}{r_i} g(x)$ *for every i = 1,…,k.*

Denote $\rho(x) = g'(x)$. It follows that $\rho_i(x) = \frac{1}{r_i}\rho(x)$ *and* $\rho_i^{-1}(x) = \rho^{-1}(r_i x)$, *hence*

$$\rho_i^{-1}\left(\rho_1(x_1)\right) = \rho^{-1}\left(\frac{r_i}{r_1}\rho(x_1)\right)$$

In case ρ^{-1} is homogeneous of order n, then $\rho_i^{-1}\left(\rho_1(x_1)\right) = \left(\frac{r_i}{r_1}\right)^n x_1$. Consequently, we obtain

$$x_1 = \frac{r_1^n C}{\sum_{i=1}^{k} r_i^n}. \tag{13}$$

Therefore,

$$x_j^* = \frac{r_j^n C}{\sum_{i=1}^{k} r_i^n} \quad \forall i = 1,...,k. \tag{14}$$

In Dhaene et al. (2012) (section 3.2, eq. (35)) and Zaks et al. (2006), the authors consider $g(x)$ to be the quadratic function, hence its first order derivative is homogeneous of order 1.

The last two examples show the relationships between the optimal solutions of previous works and the optimal solution presented in this paper. In the following example, we demonstrate the advantage of Proposition 2.2 by considering the exponential function $\alpha e^{\beta x}$ in the objective function, where α, β > 0. Moreover, we allow different parameters α, β for different classes.

Example 3.3. *Let* $g_i(x) = \alpha_i e^{\beta_i x}$, *i = 1,…,k, where α_i, β_i > 0 Following the notations in Proposition 2.2, we obtain*

$$\rho_i(x) = \alpha_i \beta_i e^{\beta_i x}$$

$$\rho_i^{-1}(x) = \frac{1}{\beta_i} ln\frac{x}{\alpha_i \beta_i} = \frac{1}{\beta_i} lnx - \frac{1}{\beta_i} ln(\alpha_i \beta_i) .$$

In particular, $\rho_1(x_1) = \alpha_1 \beta_1 e^{\beta_1 x_1}$. We will calculate the solution of (10) in steps. First, for every i = 1,…,k – 1, we obtain

$$\rho_i^{-1}\left(\rho_1\left(x_1^*\right)\right)=\frac{1}{\beta_i}ln\frac{\alpha_1\beta_1 e^{\beta_1 x_1^*}}{\alpha_i\beta_i}=$$

$$=\frac{1}{\beta_i}\left(\beta_1 x_1^*+ln\left(\alpha_1\beta_1\right)-ln\left(\alpha_i\beta_i\right)\right).$$

It follows that

$$\sum_{j=1}^{k-1}\rho_j^{-1}\left(\rho_1\left(x_1^*\right)\right)=$$

$$=\sum_{j=1}^{k-1}\frac{1}{\beta_j}\left(\beta_1 x_1^*+ln\left(\alpha_1\beta_1\right)-ln\left(\alpha_j\beta_j\right)\right)=$$

$$=\beta_1 x_1^*\sum_{j=1}^{k-1}\frac{1}{\beta_j}+ln\left(\alpha_1\beta_1\right)\sum_{j=1}^{k-1}\frac{1}{\beta_j}-\sum_{j=1}^{k-1}\frac{ln\left(\alpha_j\beta_j\right)}{\beta_j}\ .$$

Denote by $A_{k-1}=\sum_{j=1}^{k-1}\frac{1}{\beta_j}$ and $B_{k-1}=\sum_{j=1}^{k-1}\frac{ln\left(\alpha_j\beta_j\right)}{\beta_j}$.

Hence,

$$\sum_{j=1}^{k-1}\rho_j^{-1}\left(\rho_1\left(x_1^*\right)\right)=\beta_1 x_1^* A_{k-1}+A_{k-1}ln\left(\alpha_1\beta_1\right)-B_{k-1}.$$

Substituting x_1^* by 0 in the last equality leads to the restriction on C according to Corollary 2.3. That is,

$$C>A_{k-1}ln\left(\alpha_1\beta_1\right)-B_{k-1}\ .$$

We continue with the calculation of (10)

$$\rho_k\left(C-\sum_{j=1}^{k-1}\rho_j^{-1}\left(\rho_1\left(x_1\right)\right)\right)=$$

$$=\rho_k\left(C-\left(\beta_1 x_1^* A_{k-1}+A_{k-1}ln\left(\alpha_1\beta_1\right)-B_{k-1}\right)\right)=$$

$$=\alpha_k\beta_k e^{\beta_k\left(C-\beta_1 x_1^* A_{k-1}-A_{k-1}ln\left(\alpha_1\beta_1\right)+B_{k-1}\right)}\ .$$

To simplify the notations, let

$$D=C-\beta_1 x_1^* A_{k-1}-A_{k-1}ln\left(\alpha_1\beta_1\right)+B_{k-1}.$$

By substituting it in ρ_1^{-1}, we obtain

$$\rho_1^{-1}\left(\rho_k\left(C-\sum_{j=1}^{k-1}\rho_j^{-1}\left(\rho_1\left(x_1\right)\right)\right)\right)=$$

$$=\rho_1^{-1}\left(\alpha_k\beta_k e^{\beta_k D}\right)=$$

$$=\frac{1}{\beta_1}ln\left(\alpha_k\beta_k e^{\beta_k D}\right)-\frac{1}{\beta_1}ln\left(\alpha_1\beta_1\right)\ .$$

The last expression is equal to

$$\frac{\beta_k}{\beta_1}\left(\frac{ln\left(\alpha_k\beta_k\right)}{\beta_k}+D-\frac{1}{\beta_k}ln\left(\alpha_1\beta_1\right)\right)$$

and by replacing D we rearrange it as follows:

$$\rho_1^{-1}\left(\rho_k\left(C-\sum_{j=1}^{k-1}\rho_j^{-1}\left(\rho_1\left(x_1\right)\right)\right)\right)=$$

$$=\frac{\beta_k}{\beta_1}\left(C-\beta_1 x_1^* A_{k-1}-A_k ln\left(\alpha_1\beta_1\right)+B_k\right)\ .$$

Note the changes of the indexes from $k-1$ to k.

Finally, we find x_1^* from (10):

$$x_1^*=\frac{\beta_k}{\beta_1}\left(C-\beta_1 x_1^* A_{k-1}-A_k ln\left(\alpha_1\beta_1\right)+B_k\right),$$

Hence,

$$x_1^*=\frac{\beta_k}{\beta_1\left(1+\beta_k A_{k-1}\right)}\left(C-A_k ln\left(\alpha_1\beta_1\right)+B_k\right)$$

and by (11) we obtain the optimal solution for every class $i=2,\ldots,k-1$:

$$x_i^*=\frac{\beta_k}{\beta_i}\left(\frac{ln\left(\alpha_k\beta_k\right)}{\beta_k}+D-\frac{1}{\beta_k}ln\left(\alpha_i\beta_i\right)\right)$$

and

$$x_k^*=C-\beta_1 x_1^* A_{k-1}-A_{k-1}ln\left(\alpha_1\beta_1\right)+B_{k-1}\ .$$

Conclusions

In this paper, we generalized the results of optimal capital allocation as in Dhaene et al. (2012), Frostig et al. (2007) and Zaks et al. (2006) by allowing to measure the difference between the allocated capital and the loss with an arbitrary convex function for each class. It gives the insurer the flexability to implement its policy in regard to the preferred risks in the portfolio. We studied the random setting and showed the existence of a solution to the optimization problem. Moreover, we proved that there is a unique solution. To prove the existence of a unique solution, there were no requirement for the convex functions to have any derivatives. An explicit analytic solution was obtained for the deterministic setting under the assumption that any convex function has a first derivative. Finally, we demonstrated the advantage of our results by considering the exponential function with different parameters for each class.

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References

1. Cummins J.D. (2000). Allocation of capital in the insurance industry, *Risk Management and Insurance Review*, 3 (1), pp. 7-27.
2. Denault M. (2001). Coherent allocation of risk capital, *Journal of Risk*, 4 (1), pp. 1-34.
3. Deprez O. and Gerber H.U. (1985). On convex principles of premium calculation, *Insurance: Mathematics and Economics*, 4 (3), pp. 179-189.
4. Dhaene J., Tsanakas A., Valdez E.A. and Vanduffel S. (2012). Optimal capital allocation principles, *Journal of Risk and Insurance*, 79 (1), pp. 1-28.
5. Frostig, E., Zaks, Y., and Levikson B. (2007). Optimal pricing for a heterogeneous portfolio for a given risk factor and convex distance measure, *Insurance: Mathematics and Economics*, 40 (3), pp. 459-467.
6. Goovaerts M.J., de Vylder F. and Haezendonck J. (1984). *Insurance premiums*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands.
7. Tsanakas A. (2009). To split or not to split: capital allocation with convex risk measures, *Insurance: Mathematics and Economics*, 44 (2), pp. 268-277.
8. Zaks Y., Frostig E., and Levikson B. (2006). Optimal pricing of a heterogeneous portfolio for a given risk level, *ASTIN Bulletin*, 36 (1), pp. 161-185.