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ARTICLE INFO	Wen-Chieh Chang and Han-Hsing Lee (2013). Modelling complex safety covenant of corporate risky bonds under the double exponential jump-diffusion process. *Investment Management and Financial Innovations*, *10*(2-1)
RELEASED ON	Thursday, 01 August 2013
JOURNAL	"Investment Management and Financial Innovations"
FOUNDER	LLC “Consulting Publishing Company “Business Perspectives”

NUMBER OF REFERENCES **0**

NUMBER OF FIGURES **0**

NUMBER OF TABLES **0**

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Modeling complex safety covenant of corporate risky bonds under the double exponential jump-diffusion process

Abstract

This paper employs the structural approach for valuing corporate bonds under the double exponential jump-diffusion process, which offers much more flexibility in describing the empirical asset-return distribution than previous structural models. The authors extend the uniform sampling approach to develop a simple and efficient Monte Carlo simulation method for valuation of corporate risky bond with a complex safety covenant. Unlike the first passage-time model, our model incorporates the caution time setting which allows firm value to stay below the "caution default boundary" for a pre-specified time window. We further extend the model assuming an additional "immediate default boundary" in the safety covenant. These two models are used to illustrate how different assumptions of default boundaries and firm value process affect the possible credit spreads, default probabilities and recovery rates.

Keywords: modified uniform sampling approach, Parisian option, double exponential jump-diffusion process, Monte Carlo simulation, caution time, safety covenant.
JEL Classification: G12, G32, G33.

Introduction

After the worldwide financial tsunami hit in 2008, corporate credit risk has been getting vast attention not only from academics but also from businesses and professionals. Specifically, many firms had good credit ratings but suddenly defaulted during the financial crisis. Hence, accurately modeling the default risk of firms has become a much more important issue now than before. There are two fundamental approaches to modeling default risk of companies. One is the structural approach which models the firm value, pioneered by Black and Scholes, (1973) and Merton (1974), and extended by Black and Cox (1976), Longstaff and Schwartz (1993) Leland (1994) Zhou (2011), Chen and Kou (2009), and others. The other is the reduced-form approach, brought up by Jarrow et al. (1997), Madan and Unal (1998), Duffie and Singleton (1999), and others, using the Poisson process to model the default rate directly instead of modeling firm value. The reduced form approach does not explicitly consider the relationship between a company's capital structure and its default risk.

The primary purpose of this paper is to build a structural model which combines the concept of a more realistic safety covenant and left-skewed, heavier-tailed empirical asset return distribution. In terms of safety covenants of bonds, Fujita and Ishizaka (2002) used the Parisian option framework to propose the concept of "caution time" in the original first-time passage model. In their model, when firm value breaches the default barrier, the firm goes in the caution time condition. The firm defaults if firm value remains below the barrier beyond the pre-specified caution time. In terms of the underlying firm value process, Zhou (2011) demonstrated that incorporating jump risk into the firm value process is crucial since a firm with a good financial standing can suddenly default because of a huge drop in its firm value. Accordingly, a structural model with jump risk can better characterize the credit spread of corporate bonds, as well as various shapes of empirical yield spread curves. Moreover, some empirical studies have documented that the double exponential jump-diffusion process proposed by Kou (2002) performs better than the log-normally distributed jump-diffusion in equities markets (see Ramezani and Zeng, 2007) and structural credit risk models (see Wong and Li, 2006). Therefore, our paper contributes to the literature by providing a new model under the Parisian option framework and the double exponential jump diffusion process. The new model can better fit the empirical evidence of both short-term and long-term default rates as well as yield spreads.

Chesney, Jeanblanc and Yor (1997) defined a new option called the Parisian option, which is an extension of the barrier option. A down-and-out (up-and-out) Parisian option is an option that expires if the underlying asset price goes down (up), hits a specific barrier level and stays below (above) the barrier for a specified time (window) period. The assumption of barrier option is that if firm value drops below the pre-specified level, the firm will shut down immediately. This first-passage time model helps us parsimoniously model safety covenants of bonds, but such safety covenants are often too strict to firms. Therefore, subsequent researchers such as Fujita and Ishizaka (2002) and Francois and Morellec (2004) introduced the Parisian option framework and proposed the much more complex bankruptcy proceedings.

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The Appendix to the article is available at http://web.it.nctu.edu.tw/~hhlee.

Under this notion, Fujita and Ishizaka (2002) incorporated "caution time" in the original first-time passage model. In their model, when the firm value hits the default barrier, the firm goes into caution time condition. The firm defaults if its firm value remains below the barrier beyond the pre-specified caution time. Their model gives more flexibility to the structural form model and safety covenant. On the other hand, in the literature of capital structure models, Francois and Morellec (2004) used the notion of Parisian option to model US bankruptcy codes of Chapter 11 and provided insights into the importance of Chapter 11 modeling[1]. Moreover, Broadie, Chernov and Sundaresan (2007) and Paseka (2003) proposed a capital structure model comprising of two barriers under both barrier and Parisian option framework: one is for modeling under Chapter 11 and the other is for Chapter 7. In this paper, we mainly focus on bond pricing, not capital structure issues. Therefore, we choose to build our model by extending the settings of Fujita and Ishizaka (2002).

Prior research has documented that bond price often drops surprisingly around the time of default (see Duffie and Lando, 2011. Many situations may cause the jump in bond price, such as a natural disaster, lawsuits or sudden financial turmoil. Accordingly, credit spreads styled facts are: (1) credit spreads do not converge to zero even for very short maturity bonds; and (2) term structure of credit spreads has downward, humped, and upward shapes. The double exponential jump diffusion model is thus well-suited for structural modeling. Kou (2002) and Kou and Wang (2004) provided an option pricing approach under the double exponential jump-diffusion process. This process has many distinct features, including the ability to separate the probability and magnitude of upside and downside jumps. Because of the nice features of the double exponential jump-diffusion model, the log-normal return assumption of Merton (1974) model could be improved. Moreover, empirical evidence suggests that return distributions of firm values are skewed to left, and have higher peak and heavier tails than normal distribution. The double exponential jump-diffusion model is more flexible in parameter setting than the log-normal jump-diffusion model of Merton (1976). These features of the double exponential jump diffusion can better fit asset return distribution and credit spreads to empirical data.

All of the above are motivations for us to evaluate risky bonds under the double exponential jump diffusion process and the Parisian option framework. It is well known that deriving an analytical formula for valuation of risky debt under the jump-diffusion process is very difficult, especially under the assumption of a complex safety covenant. Therefore, we develop a Monte Carlo simulation procedure combining the uniform sampling approach of Metwally and Atiya (2002) and the standard Brownian bridge path generation to estimate corporate bonds prices under the Parisian option framework. We build two models for pricing bonds under different safety covenants. Since our main focus is not on capital structure issues but risky bond pricing, we assume the total market value of the firm as the underlying process rather than the asset value (the value of an unleveraged but otherwise identical firm) process of Chen and Kou (2009). Our first model follows the concept of Fujita and Ishizaka (2002) of caution time to estimate bond value. The second model follows the notion of Broadie, Chernov and Sundaresan (2007) in which two distinct default boundaries are assumed: one is default barrier and the other is liquidation barrier. We use these two models to price bonds and for default risk analysis under the double exponential jump-diffusion process.

Our contribution is twofold: first, we develop a simple and efficient Monte Carlo simulation method to value corporate risky bonds under the jump-diffusion process and complex safety covenant under the Parisian option framework. This approach can significantly reduce computation time and estimation bias compared to a standard approach. This is a substantial advantage over the standard short-step Monte Carlo simulation if one needs a more accurate and faster calculation of bond prices. Secondly, to make our model more realistic, we build two models which adopt different default boundary assumptions for valuing corporate bonds under the double exponential jump-diffusion process proposed by Kou (2002). We use these two models to evaluate corporate risky bonds and illustrate how different assumptions of default boundaries affect the possible credit spreads, default probabilities and recovery rates under the double exponential jump-diffusion process.

The remaining sections of this paper are organized as follows. Section 1 presents the structural model under the double exponential jump-diffusion process and the Parisian option framework. Section 2 proposes a fast numerical method to simulate corporate bond values and reports the numerical performance of our simulation procedure. Section 3 presents the default analysis of our new models and compares it with prior structural models. The final section concludes the paper.

[1] Francois and Morellec (2004) used the down-and-out Parisian option for modeling risky bonds under Chapter 11 of the US Bankruptcy Code. They pointed out that Parisian option's special feature of window period could fundamentally represent that a corporation in financial distress renegotiate debt under Chapter 11 of the US Bankruptcy Code. This model lets bondholders and shareholders have an unambiguous effect on default incentives and credit spread.

1. Model

1.1. Firm value model. Under the double exponential jump diffusion process of Kou (2002), the dynamics of a firm's value comprises of two parts: one is a pure diffusion process, as a geometric Brownian motion; the other is the jump part. Jump sizes follow the double exponential distribution and jump times are driven by event times of a Poisson process.

We use the following equation for modeling market value $V(t)$, which follows a double exponential jump diffusion process under risk-neutral measure Q:

$$\frac{dV(t)}{V(t^-)} = \left(r - \lambda\xi\right)dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(Z_i - 1)\right) \quad (1)$$

The solution of the equation is given by

$$V(t) = V(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2 - \lambda\xi\right)t + \sigma W(t)\right\}\prod_{i=1}^{N(t)} Z_i, \quad (2)$$

where r is the constant risk-free interest rate, σ is the volatility of the firm value, and ξ is the mean of percentage jump size:

$$\xi \equiv E\left[Z - 1\right] = E\left[e^Y - 1\right] = \frac{p\eta_u}{\eta_u - 1} + \frac{q\eta_d}{\eta_d + 1} - 1. \quad (3)$$

$W(t)$ is a standard Brownian motion under risk-neutral measure Q, $N(t)$ is a homogenous Poisson process with mean λ, and Z_i is a series of independent identically distributed nonnegative random variables such that $Y = \ln(Z)$ has a density of the double exponential distribution:

$$f_y(y) = p \cdot \eta_u e^{-\eta_u y} \cdot I_{\{y \geq 0\}} + q \cdot \eta_d e^{\eta_d y} \cdot I_{\{y<0\}}, \quad (4)$$
$$\eta_u > 1,$$

where p, $q \geq 0$, $p + q = 1$, $I_{\{y\geq 0\}}$, $I_{\{y<0\}}$ are indicator functions. The condition $\eta_u > 1$ is to ensure that expectation of $V(t)$ is finite. p and q are probabilities of upside jump and downside jump, respectively. Mean values of the two exponential distributions are $1/\eta_u$ and $1/\eta_d$. Mean of Y is $p/\eta_u - q/\eta_d$. In this model, $W(t)$, $N(t)$, and Y are assumed to be independent. The return process $X(t) \equiv \ln(V(t)/V(0))$ is as in the following equation:

$$X(t) = \left(r - \frac{1}{2}\sigma^2 - \lambda\xi\right)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad (5)$$

where $X(0) = 0$, and the equation is also under risk-neutral probability measure Q. If Y is normally distributed, the model is the same as the Merton jump-diffusion model (1976).

1.2. Pricing corporate debt. The next step is to develop the corporate bond pricing models. For simplicity, we focus on zero-coupon bond for our research. We follow assumptions of the total market firm value process described in the previous section and develop two models for valuing corporate risky debt in this paper. First, we build a model with a realistic safety covenant assumption under the Parisian option framework. Unlike the first passage time model, the caution time setting allows firm value to stay below the "caution default boundary" for a time window of a pre-specified length of time. The bond defaults when the consecutive time of the firm value remains below this barrier for longer than the pre-specified time (window) period. Secondly, we further extend the first model to let it have two default boundaries at the same time: one is the "caution default boundary" and the other is the "immediate default boundary" corresponding to the original first passage time model.

Model 1: Parisian framework (caution default boundary).

In the spirit of Black and Cox (1976), we assume the exponential default boundary $H(t) = e^{-\phi(T-t)}F$, where ϕ is boundary discount rate and F is face value of the bond. Under the Parisian option framework, a bond defaults if a firm's market value is below the boundary, $H(t)$, and the consecutive time, $\hat{t}$, of firm value below this boundary is over a pre-specified time period (caution time) w[1]. The first time the bond defaults is defined as time τ. Mathematically,

$$H(t) = e^{-\phi(T-t)}F, \; T > 0, \; 0 \leq t \leq T \quad (6)$$

$$\tau \equiv \inf\left\{t \geq 0 \middle| \hat{t} \geq w\right\} \quad (7)$$

where $\hat{t}(V(t),t) = \begin{cases} 0 \text{ if } V(t) > H(t) \\ t - g_t \text{ if } V(t) \leq H(t) \end{cases}$

and $g_t = \sup\left\{s \leq t \,|\, V(s) = H(s)\right\}$,

where g_t is the last time before t when firm value breaches the boundary. Figure 1 shows the "caution time" framework.

Model 2: Caution and immediate default boundaries.

Model 2 is an extension of Model 1. Several structural models in the literature have introduced two different barriers: the default barrier and the liquidation barrier (for example, see Broadie, Chernov and Sundaresan, 2007). Model 2 is under the Parisian option framework and the barrier option framework at the same time. Hence, there are two

[1] Barrier option is a special case of Parisian option, when the caution time is equal to zero.

situations in which a bond may default. First, if a firm's market value is below a boundary $H(t)$ (an exponential barrier), and the time $\hat{t}$ for which the value remains below this barrier $H(t)$ exceeds the length of the time window (caution time) w. Second, if a firm's market value breaches the lower barrier $L(t)$, the bond defaults immediately. We assume that immediate liquidation boundary is a constant fraction of caution default boundary, i.e. $L(t) = \rho H(t)$, $0 \leq \rho \leq 1$. Figure 2 illustrates these two default conditions of Model 2. The first time the bond defaults is defined as time τ.

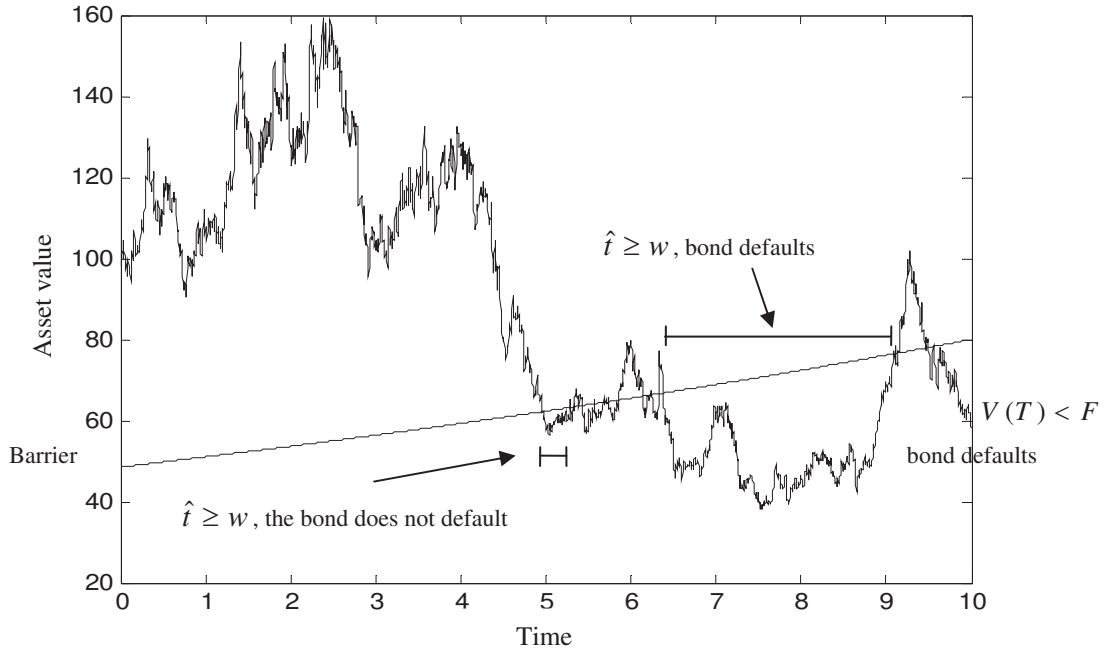


Fig. 1. Situations when corporate bonds default in Model 1 – Parisian framework (caution default boundary)

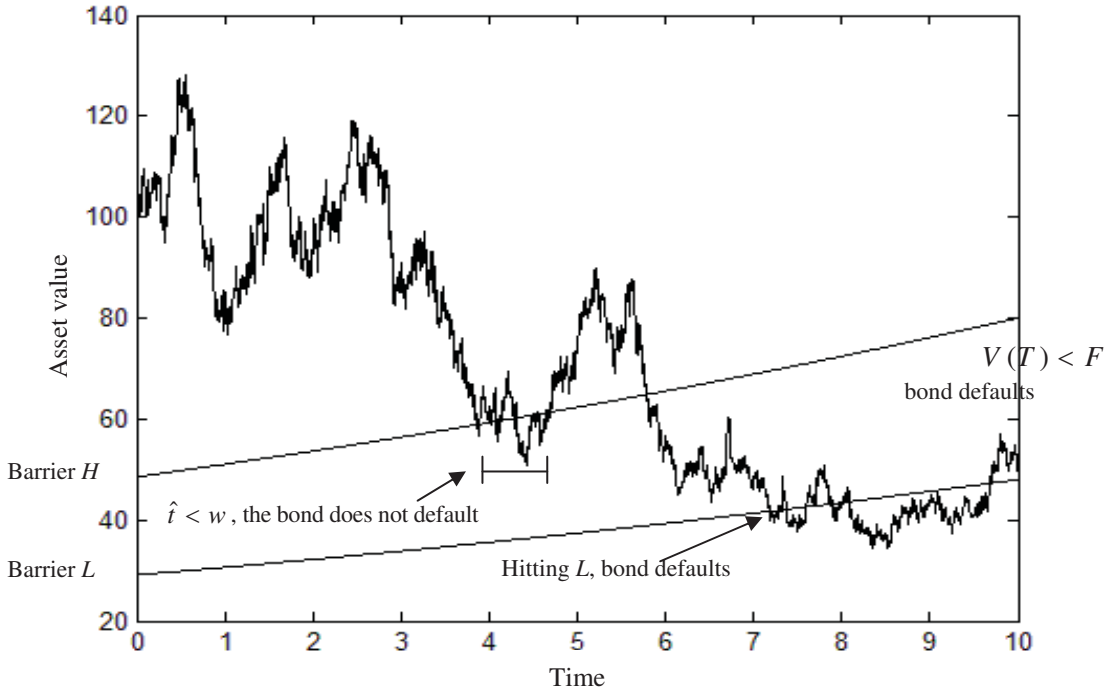


Fig. 2. Situations when corporate bonds default in Model 2 – caution and immediate default boundaries

Mathematically,

$$H(t) = e^{-\phi(T-t)}F,\ T > 0,\ 0 \le t \le T, \tag{8}$$

$$L(t) = \rho H(t),\ 0 \le \rho \le 1, \tag{9}$$

$$\tau \equiv \inf\left\{t \ge 0 \middle| \hat{t} \ge w\right\} \cup (V \le L), \tag{10}$$

where $\hat{t}(V(t),t) = \begin{cases} 0 \text{ if } V(t) > H(t) \\ t - g_t \text{ if } V(t) \le H(t) \end{cases}$

and $g_t = \sup\left\{s \le t \,|\, V(s) = H(s)\right\}$.

Therefore, Model 1 can be regarded as a special case of Model 2, where $\rho = 0$.

1.3. Bond pricing equation. If $\tau \ge T$, the bond does not default before maturity. In this case, the bond defaults when the firm value falls below the face value of bond at maturity. If $\tau < T$, it means that the bond defaults before maturity. While bond defaults, bondholders only receive the value of the firm minus the write-down of value at default time. In general, write-down value is a non-increasing function of firm value. We assume the equation of write-down value is of a linear form:

$$\alpha(V) = \alpha_1 \cdot V, \tag{11}$$

where α_1 is a non-negative constant and $0 \le \alpha_1 < 1$. We can evaluate the price of a risky bond by using a fundamental bond pricing approach, that is, discounted cash flows. Bond price $B(V, T)$ is as follows:

$$B(V,T) = E^Q\left[\left\{\exp(-rT)\cdot\left(F \cdot I_{\{V(T)\ge H(T)\}} + \left(V(T) - \alpha(V(T))\right)\cdot I_{\{V(T)<H(T)\}}\right)\right\}I_{\{\tau\ge T\}} + \right.$$
$$\left. + \exp(-rT)\left(V(\tau) - \alpha\left(V(\tau)\right)\right)I_{\{\tau<T\}}\right], \tag{12}$$

where Q is risk-neutral probability measure, I is indicator function, T is bond maturity, and τ is the time of default. This equation comprises two parts: the first part of equation is present value of the cash flow which the bondholder could receive at maturity, and the second part is present value of the cash flow which the bondholder could receive if the bond defaults before maturity. In the Monte Carlo method, $B(V, T)$ is obtained as in the following equation:

$$B(V,T) = \lim_{N\to\infty}\sum_{i=1}^{N}\frac{1}{N}\left[\left\{\exp(-rT)\cdot\left(F \cdot I_{\{V_i(T)\ge H(T)\}} + \left(V_i(T) - \alpha(V_i(T))\right)\cdot I_{\{V_i(T)<H(T)\}}\right)\right\}I_{\{\tau\ge T\}} + \right.$$
$$\left. + \exp(-r\tau)\left(V_i(\tau) - \alpha\left(V_i(\tau)\right)\right)I_{\{\tau<T\}}\right]. \tag{13}$$

This equation for evaluating corporate bond price could be easily computed by Monte Carlo simulations with standard software programs.

2. Monte Carlo method

The Parisian option is a path-dependent option which takes considerable time to simulate. In order to reduce the computing time of the Monte Carlo method, Metwally and Atiya (2002) provided an approach called uniform sampling for accelerating simulation time of the calculation. This approach is based on the Brownian bridge concept developed by Karatzas and Shreve (1991) and Revuz and Yor (1994). The Brownian bridge concept is that if one has a Wiener process defined by a series of time-indexed random variables $\{W(t_1), W(t_2),\ldots\}$, one could use the Brownian bridge method to insert a random variable $W(t_k)$, where $t_i < t_k < t_{i+1}$, into the series in such a manner that the result of the original series remains unchanged. Given $W(t)$ and $W(t + \Delta t_1 + \Delta t_2)$, one needs to obtain $W(t + \Delta t_1)$. Standard Brownian bridge method for computing $W(t + \Delta t_1)$ is to take a weighted average of $W(t)$ and $W(t + \Delta t_1 + \Delta t_2)$ plus an independent normal random variable:

$$W(t + \Delta t_1) = \alpha^* W(t) + \beta^* W(t + \Delta t_1 + \Delta t_2) + \gamma^* Z, \tag{14}$$

$$\alpha^* = \frac{\Delta t_2}{\Delta t_1 + \Delta t_2},$$

$$\beta^* = 1 - \alpha^*,$$

$$\gamma = \sqrt{\Delta t_1 \alpha^*},$$

where α^*, β^* and γ^* are constants to be determined, and Z is a standard normal random variable.

Metwally and Atiya (2002) follow the Brownian bridge concept to calculate the probability of firm value not crossing the barrier. Given two endpoints values, their simulation procedure is as the following. Let the jump times be T_1, T_2, …, T_M, the first variables to be generated. Assuming $x\left(T_i^-\right)$ is the instant process value before the *i-th* jump and $x\left(T_i^+\right)$ is the instant process value after the *i-th* jump. Hence, between the two jumps, $x\left(T_i^+\right)$ and $x\left(T_{i+1}^-\right)$, the process is a pure Brownian motion. Let B_S be a Brownian bridge between intervals $\left[T_i, T_{i+1}\right]$. The probability of no barrier crossing in the interval $\left[T_i, T_{i+1}\right]$ is:

$$P_i = P\left(\inf_{T_i \le s \le T_{i+1}} B_s > \ln H \,\middle|\, B_{T_i^+} = x(T_i^+), B_{T_{i+1}^-} = x(T_{i+1}^-)\right) =$$

$$= \begin{cases} 1 - \exp\left(-\dfrac{2\left[\ln H - x(T_i^+)\right]\left[\ln H - x(T_{i+1}^-)\right]}{\tau\sigma^2}\right) & \text{if } x(T_{i+1}^-) > \ln H. \\ 0 \quad \text{otherwise} \end{cases} \tag{15}$$

This method is very efficient and unbiased in pricing vanilla barrier option[1]. Metwally and Atiya (2002) assume barrier H is flat. In our model, we assume that default boundaries are exponential functions of time. Therefore, we have to modify the drift-term of the double exponential jump-diffusion process to let the barrier be constant. In this case, our new process of firm value return and barrier become:

$$X'(t) = ct + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \tag{16}$$

$$c = r - \phi - \frac{1}{2}\sigma^2 - \lambda\xi,$$

$$H' = \mathrm{e}^{-\phi T} F,$$

$$L' = \rho H'.$$

2.1. Algorithms of the modified uniform sampling approach. This section describes our Monte Carlo method – the modified uniform sampling approach for bond pricing under the Parisian option framework and the double exponential jump diffusion process. To incorporate caution time setting to the uniform sampling method by Metwally and Atiya (2002), once the boundary crossing is detected in any interval between jumps, we switch the simulation procedure to standard Monte Carlo method before the next jump instant. We present the algorithm of Model 1 in detail and refer readers to online Appendix (http://web.it.nctu.edu.tw/~hhlee) for the algorithm of Model 2 to conserve space.

Model 1: Caution default boundary

Step 1. For n = 1 to N, conduct the Monte Carlo experiment as *Steps 2* to *5*.

Step 2. Generate jump-instants T_i, by generating inter-jump times (t_i) from a given density function (in this paper, we use exponential distribution) and set $T_i = T_{i-1} + t_i$. Repeat *Step 2* until $\sum_{i=1}^{M+1} t_i > T$, where M is the number of jumps that happen during the entire life of contract.

Step 3. For i = 1 to M + 1, generate the return of firm value for all jump points.

(I) Let $x(t) = \ln V(t)$, initial value $x(T_0) = x(0) = \ln V(0)$ and generate the return of firm value before jump $x\left(T_i^-\right)$ from Gaussian distribution under mean $x(T_{i-1})$ + + $c(T_i - T_{i-1})$ and standard deviation $\sigma\sqrt{T_i - T_{i-1}}$.

(II) Generate a random variable from uniform distribution U[0, 1]. Use the random variable and cumulative distribution function (CDF) of the double exponential distribution to generate jump size J_i[2].

(III) Compute the return of firm value after jump: $x\left(T_i^+\right) = x\left(T_i^-\right) + J_i$.

Step 4. For intervals i = 0 to M, set *default* = 0, *check-time* = 0, i = 0 at first, let $x\left(T_0^+\right) = x(0)$, while (*default* = 0) or ($i < M + 1$), and we continue the loop. *Check-time* denotes the consecutive time that firm value remains below the default boundary.

(I) If $x(T_i^+) > \ln(H')$, set *check-time* = 0

(1) Compute the probability of no barrier crossing P_i based on equation (15).

(2) Following Metwally and Atiya, 2002, let $b = (T_{i+1} - T_i)/(1 - P_i)$. b is used to setup a time interval for uniform sampling in the following steps to simultaneously decide if the firm value crosses the barrier and the first passage time if the firm value does cross the barrier.

(3) Generate s from a uniform distribution in the interval $[T_i, T_i + b]$.

(4) If $s \notin \left[T_i, T_{i+1}\right]$, firm value does not cross the barrier, skip (5), and set $i = i + 1$. Repeat *Step 4*.

(5) If $s \in \left[T_i, T_{i+1}\right]$, then the firm value crosses the barrier for the first time at time s in interval $[T_i, T_{i+1}]$.

Since we know the firm values at time s and T_{i+1} are $\ln H'$ and $x\left(T_{i+1}^-\right)$, we assume that one year could be divided by K days and use the standard Brownian

[1] The standard Monte Carlo simulation for pure Brownian motion simply samples the firm value path at each discrete time interval: $V(t) = V(0)\exp\left[(r - \frac{1}{2}\sigma^2)t + \sigma\varepsilon\sqrt{t}\right]$, where ε is a random sample from a standard normal distribution. This method can cause estimation bias because the firm value below the barrier may occur but be ignored during the time between the end points (Metwally and Atiya, 2002). Shortening the length of time intervals can reduce bias, but computation becomes very slow and bias still cannot be eliminated entirely.

[2] In an online Appendix, we provide the detailed derivations of the double exponential jump diffusion CDF and we also present the method to generate jump size under double exponential jump diffusion.

bridge method (equation (14)) to simulate the firm value process from s to T_{i+1}. We check the process of each point to see whether the firm value crosses the barrier before time T_{i+1}.

For intervals $j = 1$ to $[T_{i+1} - s]\cdot K$, repeat (a) to (c).

(a) If $x(s + j/K) < \ln H'$), then *check-time* = *check-time* + $1/K$

(b) If $x(s + j/K) \geq \ln H'$), then we reset *check-time* = 0

(i) If *check-time* $\geq w$, then *default* = 1.

$$DiscBond_n = \exp[(\phi - r)\tau]\left[(1-\alpha_1)(\exp(x(\tau)))\right]$$

where $\tau(= s + {}^{j}/_{K})$ is the default time.

Exit loop, compute another Monte Carlo iteration (Step 2-5).

(ii) Else, $j = j + 1$

(c) when $j = [T_{i+1} - s]\cdot K$, and *check-time* < w, let $i = i + 1$, repeat Step 4.

If $x(T_i^+) \leq \ln(H')$, since we know the firm value at time T_i and T_{i+1} is $x(t_i+)$ and $x(t_{i+1}-)$, respectively, we directly use the standard Brownian Bridge method (equation (14)) to simulate the firm value process in interval $[T_i,T_{i+1}]$ as (Step 4). (I) (5). We check each path to see whether it crosses the barrier before time T_{i+1}, for intervals $j = 1$ to $[T_{i+1} - T_i]\cdot K$:

(1) If $x(T_i + j/K) < \ln H'$, then *check-time* = *check-time* + $1/K$.

(2) If $x(T_i + j/K) \geq \ln H'$, then we reset *check-time* = 0.

(a) If *check-time* $\geq w$, then *default* = 1

$$DiscBond_n = \exp[(\phi - r)\tau]\left[(1-\alpha_1)(\exp(x(\tau)))\right]$$

Exit loop, compute another Monte Carlo iteration (Step 2-5).

(b) Else, $j = j + 1$

(3) When $j = [T_{i+1} - T_i]\cdot K$, and *check-time* < w, let $i = i + 1$, repeat Step 4.

(c) When $i = M + 1$, check $x(T)$:

(1) If $x(T) \geq \ln(H')$ and *default* = 0

$$DiscBond_n = \exp(-rT)\cdot F$$

(2) Else, let *default* = 1

$$DiscBond_n = \exp[(\phi - r)T]\left[(1-\alpha_1)(\exp(x(T)))\right]$$

Exit loop, compute another Monte Carlo iteration (Step 2-5).

Step 5. If $n = N$, we finish the Monte Carlo simulation. We can calculate the estimated risky bond price as:

$$DiscBond = \frac{1}{N}\sum_{n=1}^{N} DiscBond_n.$$

2.2. Performance of the modified uniform sampling approach. For each method, we run the MATLAB program on an Intel T4400 2.20 GHz CPU for one million Monte Carlo iterations to compute the bond price. We compare the results of modified uniform sampling method with standard Monte Carlo method in Model 1 and Model 2 under different bond face values. In Table 1, we use parameter settings as follows: $V(0) = 100$, $F = 80$, $r = 0.05$, $\phi = 0.05$, $\lambda = 0.2$, $\alpha_1 = 0.4$, $p = 0.5$, $q = 0.5$, $\eta_1 = 2.79667154579233$, $\eta_2 = 2.12168612641381$, $T = 1$, $w = 1/12$, $\sigma^2 = 0.02$. This choice of base-case parameters is described in more detail in section 3.1.

Table 1. Comparisons of different simulation methods: Model 1

Panel A: $F = 80$			
Method	Std. error	CPU time (seconds per million iterations)	Std. error × CPU time
Standard Monte Carlo Δ = 1/12	0.0227	1096	24.8792
Standard Monte Carlo Δ = 1/52	0.0217	4695	101.8815
Standard Monte Carlo Δ = 1/252	0.0201	21007	422.2407
Uniform sampling $K = 252$	0.0123	66	0.8118
Panel B: $F = 90$			
Method	Std. error	CPU time (seconds per million iterations)	Std. error × CPU time
Standard Monte Carlo Δ = 1/12	0.0342	1123	38.4066
Standard Monte Carlo Δ = 1/52	0.0334	5236	174.8824
Standard Monte Carlo Δ = 1/252	0.0325	20629	670.4425
Uniform sampling $K = 252$	0.0184	83	1.5272
Panel C: $F = 95$			
Method	Std. error	CPU time (seconds per million iterations)	Std. error × CPU time
Standard Monte Carlo Δ = 1/12	0.0382	905	34.5710
Standard Monte Carlo Δ = 1/52	0.0381	4148	158.0388
Standard Monte Carlo Δ = 1/252	0.0379	19437	736.6623
Uniform sampling $K = 252$	0.0184	93	1.7112

In Panel A of Table 1, the uniform sampling method greatly reduces simulation time. From the result of Std. error × CPU time, one can find that the uniform sampling method is more efficient than the standard Monte Carlo method. It is apparent that the standard error of the uniform sampling method is also smaller than standard short-step Monte Carlo simulation. This implies that uniform sampling offers a more accurate Monte Carlo simulation result with the same time discretization Δt. In addition to its efficiency, the uniform sampling method has lower bias than the standard Monte Carlo method. The reason is that the uniform sampling method uses uniform distribution to generate the time of hitting the barrier. Hence, under the barrier option framework of Metwally and Atiya (2002), the uniform sampling approach produces unbiased estimates of barrier options. Our method is, however, not completely bias-free because time discretization of diffusion paths of the Brownian bridge generated from equation (14) introduces bias in the estimate. Nonetheless, the bias shall be very small compared with standard Monte Carlo method because generation of diffusion paths of equation (14) is needed only when firm value hits the barrier while the time period is chosen to be quite small under this condition.

This setting of $F = 80$ makes the modified uniform sampling method very efficient since the initial firm value $V(0)$ is far away from the default boundary and, therefore, it has fewer diffusion paths of standard Brownian bridge that need to be simulated (equation (14)). In Panels B and C of Table 1, we increase the face value of bond to reduce the advantage of our modified uniform sampling method while keeping other parameters the same. We increase the face value from 80 to 90 and 95 in Panel B and Panel C, respectively. Standard error increases significantly when the face value is higher, in standard Monte Carlo methods.

The probability of using diffusion path generation of the standard Brownian bridge method increases with increase of face value in the modified uniform sampling method, while the CPU time increases only slightly. Computation time is only 6% ($F = 80$) to 10% ($F = 95$) of the standard Monte Carlo simulation when $\Delta = 1/12$. Moreover, our method is much more accurate and efficient than the standard Monte Carlo simulation even when $\Delta = 1/252$. We conduct a performance analysis for Model 2 similar to Table 1 by adding the immediate default boundary and letting $\rho = 0.8$. From Table 2, one find that after adding the second default boundary, computational efficiency of Model 2 is only slightly higher than Model 1. The modified uniform sampling method still has lower standard error and is much more efficient than the standard Monte Carlo method.

3. Numerical results and empirical implications

This section presents the default analysis of impact from two of the distinctive features of our model: the complex bond safety covenant and the double exponential jump diffusion process of firm value. To compare with prior structural models, we illustrate the effect of the caution time setting on default probabilities, credit spreads, and recovery rates, as well as the differences between our model and the Merton's jump-diffusion model [18]. Finally, we compare the difference between Model 1 and Model 2.

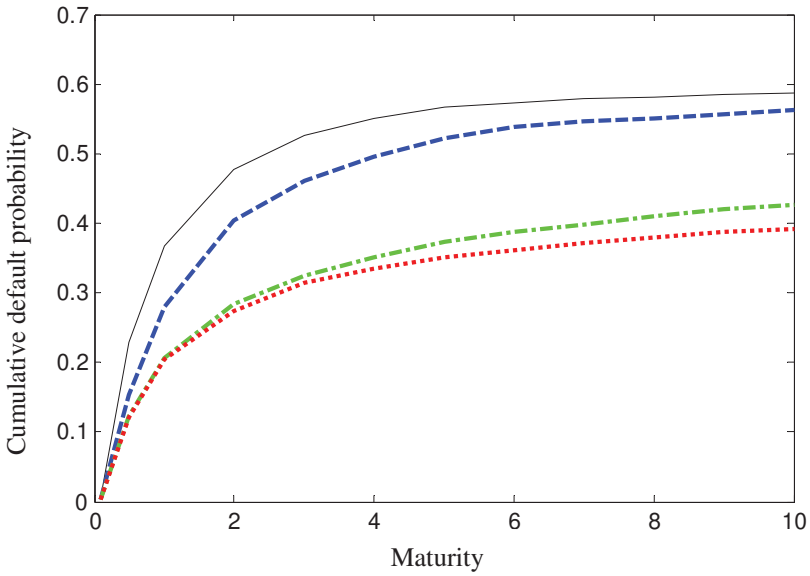
Table 2. Comparisons of different simulation methods: Model 2

Panel A: $F = 80$			
Method	Std. error	CPU time (seconds per million iterations)	std. error × cpu time
Standard Monte Carlo Δ = 1/12	0.0223	1217	27.1391
Standard Monte Carlo Δ = 1/52	0.0209	4827	100.8843
Standard Monte Carlo Δ = 1/252	0.0196	23052	451.8192
Uniform sampling $K = 252$	0.0124	79	0.9796
Panel B: $F = 90$			
Method	Std. error	CPU time (seconds per million iterations)	Std. error × CPU time
Standard Monte Carlo Δ = 1/12	0.0334	1017	33.9678
Standard Monte Carlo Δ = 1/52	0.0326	4472	145.78724
Standard Monte Carlo Δ = 1/252	0.0324	21071	682.7004
Uniform sampling $K = 252$	0.0184	88	1.6192
Panel C: $F = 95$			
Method	Std. error	CPU time (seconds per million iterations)	Std. error × CPU time
Standard Monte Carlo Δ = 1/12	0.0379	963	36.4977
Standard Monte Carlo Δ = 1/52	0.0377	4208	158.6416
Standard Monte Carlo Δ = 1/252	0.0375	18493	693.4875
Uniform sampling $K = 252$	0.0214	97	2.0758

3.1. Empirical implication of caution time settings (Model 1). Following Zhou's approach (2001), we control for the overall mean and volatility of firm value returns when changing parameter values that affect random components of firm value. Accordingly, we know that the variations of bond values are truly

caused by different combinations of model parameters rather than by changes in overall mean and volatility of firm value returns. We denote X as the return of firm value, and control EX and $Var(X)$ by the moments of return X in different models to observe the effect caused by changing the parameter. In order to perform the analysis, we have to obtain, under the risk-neutral measure Q, mean and volatility of firm return of these models. From Ramezani and Zeng (2007), we know the moments of firm value returns of these models under physical measure. Therefore, we can easily obtain the moments of firm value returns under risk-neutral measure Q[1]. In the following figures, each estimate is obtained through one million Monte Carlo simulations for acceptable precision.

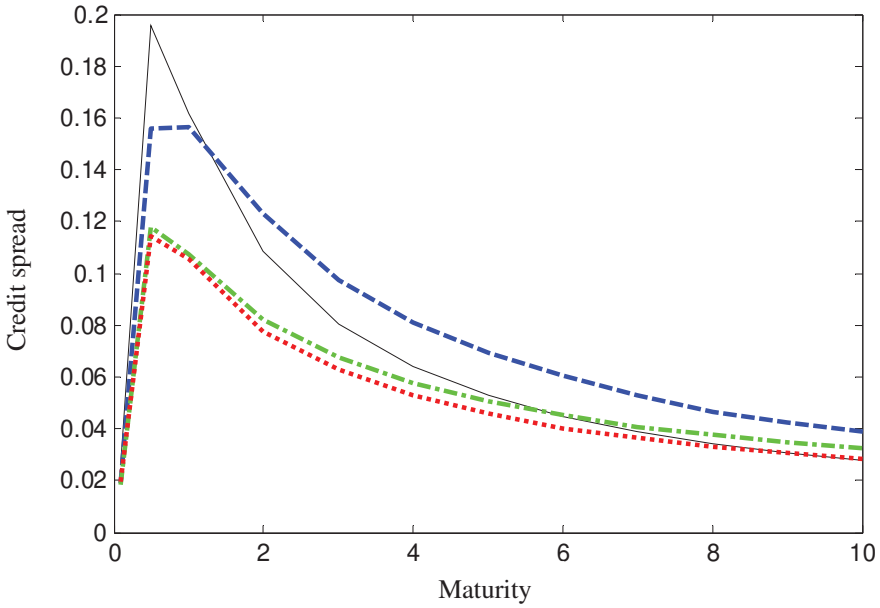
First, we present the difference between the structural models under a barrier option and a Parisian option framework. In this case, we perform the analysis of the double exponential jump-diffusion model. We control parameter settings such that total variance = 0.09, total mean = 0.005, $V(0)$ = 100, F = 80, r = 0.05, ϕ = 0.05, λ = 0.05, α_1 = 0.4, p = 0.5, q = 0.5 and jump variance = 0.35. The choices of parameters such as total variance, r, λ, F and α_1 are similar to those in previous literature and it is plausible to set $\phi = r$ and $p = q = 0.5$. One set of parameters satisfy the above condition is that η_u = 2.79667154579233, η_d = 2.12168612641381, and variance of pure diffusion σ^2 = 0.0725. We change the caution time from zero to 5 days, 10 days, 15 days, 1 month, 6 months, and 1 year to observe the effects of these changes. Note that the model with zero caution time goes back to the model under the barrier option framework. Because there are no apparent differences among caution time beyond 15 days, Figures 3 and 4 only present the results of no caution time up to 15 days. Figure 3 presents the relationship between cumulative default probability and maturity under various settings of caution time. It shows that longer caution time results in lower cumulative default probability. Figure 4 shows that credit spreads decrease as caution time increases for all caution time settings for bonds over 1 year to maturity. Interestingly, credit spreads for bonds that are over 2 years to maturity without caution time are lower than those with 5 days caution time. To further explore this result, we examine average recovery values of models with no caution time and 5 days caution time. Recovery value is the firm value at default minus write-down value in equation (11). We simultaneously perform the simulation and record the recovery values of no caution time and 5 days caution time from each iteration. One can find that the average recovery value for bonds over 2 years to maturity for the model with no caution time is higher than that of 5 days caution time. This explains the phenomenon of credit spreads being lower under no caution time than with 5 days caution time. Average recovery values in Figure 5 explain the results of credit spreads in Figure 3 since the model with 5 days caution time has lower average recovery values than zero caution time with over 2 years to maturity. In other words, the financial standing of a firm can be even worse as the firm value keeps dropping, given the 5 days caution time. The high credit spreads under 5 days caution setting is due to the low recovery values.



Note: (–) denotes no aution time; (--) denotes 5 days; (-·) denotes 10 days; (...) denotes 15 days.

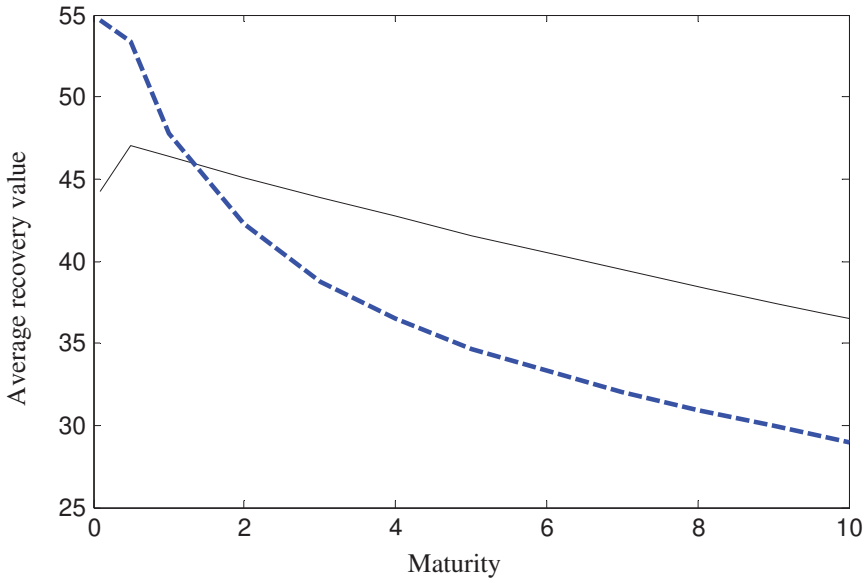
Fig. 3. Relationships between cumulative default probability and maturity under various caution time settings

[1] In an online appendix, we present the first three moments of the double exponential jump diffusion model.



Note: (−) denotes no caution time; (--) denotes 5 days; (-·) denotes 10 days; (...) denotes 15 days.

Fig. 4. Relationships between credit spread and maturity under various caution time settings



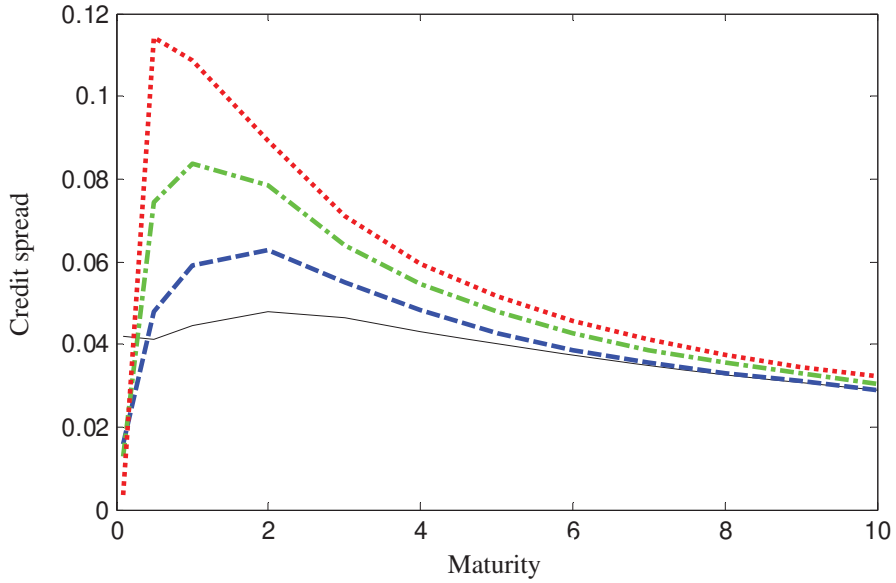
Note: (−) denotes no caution time; (--) denotes 5 days.

Fig. 5. Relationship between average recovery value and maturity under various caution time settings

3.2. Empirical implications of double exponential jump diffusion process. Next, we report the advantage and flexibility of modeling asset value under the double exponential jump-diffusion process. We compare the double exponential jump-diffusion model with the log-normal jump-diffusion model by Merton (1976), given the same EX, $Var(X)$ and jump intensity λ. Given no caution time, we control the total mean $EX = 0$, total variance $Var(X) = 0.09$, $r = 0.05$ and $\lambda = 0.2$, and adjust the remaining parameters to generate various levels of skewness. In Merton's jump-diffusion model, flexibility of skewness is limited. In this case, we set *skewness* = 0 and variance of pure diffusion $\sigma^2 = 0.01$ and find a possible combination of parameters (mean of jump size $\alpha = 0.016250257$ and variance of jump size $\beta = 0.399738782$) which can satisfy these constants under the Merton jump-diffusion model. Next, we generate three different sets of combinations each with different levels of skewness under the double exponential jump diffusion process. Since log-normal jump-diffusion has no way to adjust the probabilities of upside and downside jumps, to facilitate comparison between these two models, we keep the probabilities of both upward jump p and downward jump q in our model as 50%. In the first case, we set *skewness* = -0.5, $p = 0.5$ and $q = 0.5$, and solve for the remaining parameters ($\eta_u = 2.470527501$, $\eta_d = 2.241299129$ and $\sigma^2 = 0.017418446$). In the second case, we control *skewness* = 0, $p = 0.5$ and

q = 0.5, and solve for the remaining parameters (η_u = 2.616159165, η_d = 2.616159142 and σ^2 = 0.03155712). In the third case, we set *skewness* = 0.5, p = 0.05 and q = 0.5, and solve for the remaining parameters (η_u = 2.85627674, η_d = 3.658956675 and σ^2 = 0.050546344). Figure 6 demonstrates the relationship between credit spreads and time to maturity for different levels of skewness under the Merton's log-normal jump-diffusion model and our double exponential jump-diffusion model.

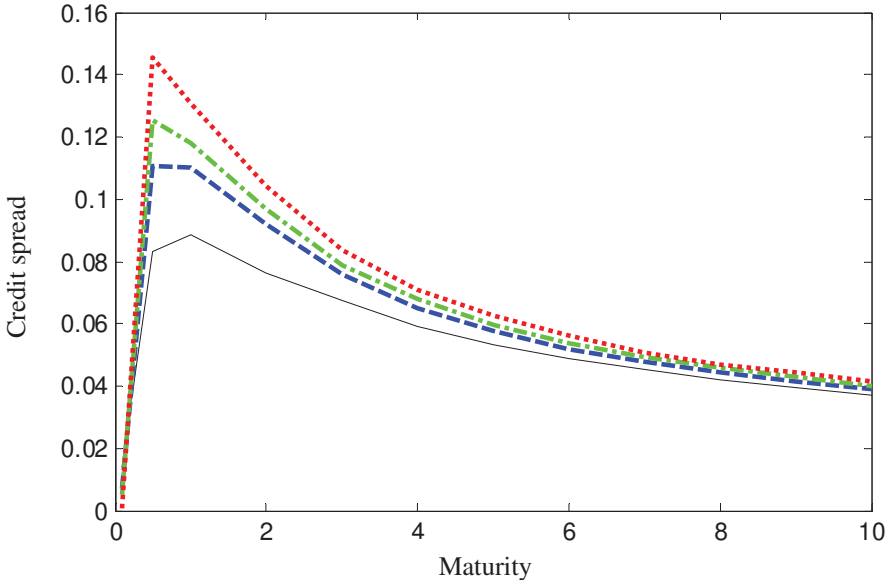
As shown in Figure 6, under the setting of negative skewness, bond credit spread is lower than that under positive skewness most of the time, which is not intuitive since a firm with negatively skewed firm return distribution has larger probability of losing a greater amount of value in the short term. Accordingly, it should have a higher credit spread due to the higher probability of making its corporate bond default. In Figure 6, we do observe that the negatively skewed return distribution leads to a higher credit spread than the other settings in a very short run as bond maturity approaches zero. Therefore, we conjecture that weights of variances between pure diffusion and jump size, instead of the skewness of return distribution, cause this phenomenon.



Note: (−) denotes Merton jump-diffusion model with *skewness* = 0, λ = 0.2, σ^2 = 0.01, α = 0.016250257, β = 0.399738782; the double exponential jump-diffusion model, λ = 0.2, p = 0.5 and q = 0.5; (--) denotes *skewness* = -0.5; (-.) denotes *skewness* = 0; (...) denotes *skewness* = 0.5.

Fig. 6. Relationships between credit spread and maturity in different models and skewness

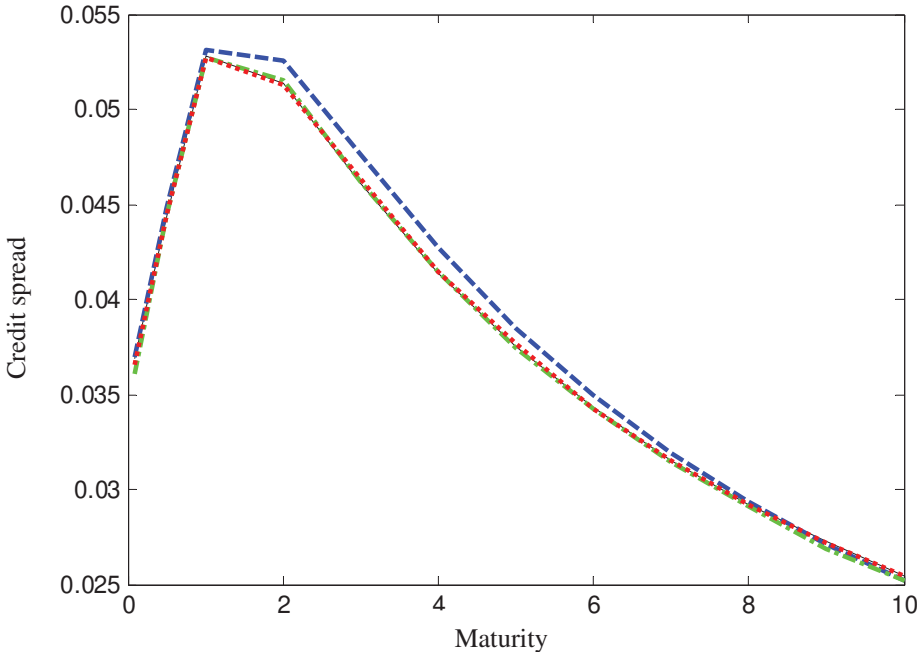
To confirm our conjecture, we change jump intensity λ from 0.2 to 0.05 so that we can adjust the weights of variances between pure diffusion and jump size under the same skewness, as well as probabilities of upside and downside jumps. We generate a parameter setting in which α = 0.03403074, β = 0.825697084 and σ^2 = 0.48714694 under Merton's jump-diffusion model. In addition, we also solve the three parameter settings under the double exponential jump diffusion process: (1) η_u = 1.83694595044413, η_d = 1.58459906319395 and σ^2 = 0.0552696595816093 given *skewness* = -0.5; (2) η_u = 1.92390672760078, η_d = 1.92390672759791 and σ^2 = 0.0629838660328502 given *skewness* = 0; and (3) η_u = 2.07131120647897, η_d = 3.54067693331008 and σ^2 = 0.0743575057803415 given *skewness* = 0.5. We increase the weights of variance in the pure diffusion part in each setting. Figure 7 demonstrates that for all settings, compared with those in Figure 6, short-term credit spreads increase after 0.5 year to maturity and decrease in very short maturities of less than half-a-year. In the long term, credit spreads are very close to each other since they are under the same EX and $Var(X)$. This result supports our conjecture that it is the weights of variances between pure diffusion and jump size, instead of the skewness of return distribution, that determine short-term credit spreads. More precisely, the variance of jump size crucially affects very short-term credit spreads. It appears that the skewness of return distribution is not a main factor determining the shape of credit spreads; skewness still constraints the variability of parameter combinations. Our model under the assumption of double exponential jump diffusion process has more flexibility in parameter setting if one controls the moments of firm value returns. One can use this model to generate more different types of credit spread curves than the Merton jump-diffusion model. It is worth noting that one can still adjust the probabilities of upside jump p and downside jump q in our model, though we fix them at 50% in the discussion above.



Note: (−) denotes Merton jump-diffusion model with *skewness* = 0, λ = 0.05, α = 0.03403074, β = 0.825697084 and σ^2 = = 0.48714694; the double exponential jump diffusion model λ = 0.05, p = 0.5 and q = 0.5: (--) denotes *skewness* = -0.5; (-·) denotes *skewness* = 0; (···) denotes *skewness* = 0.5.

Fig. 7. Relationships between credit spread and maturity in different models and skewness

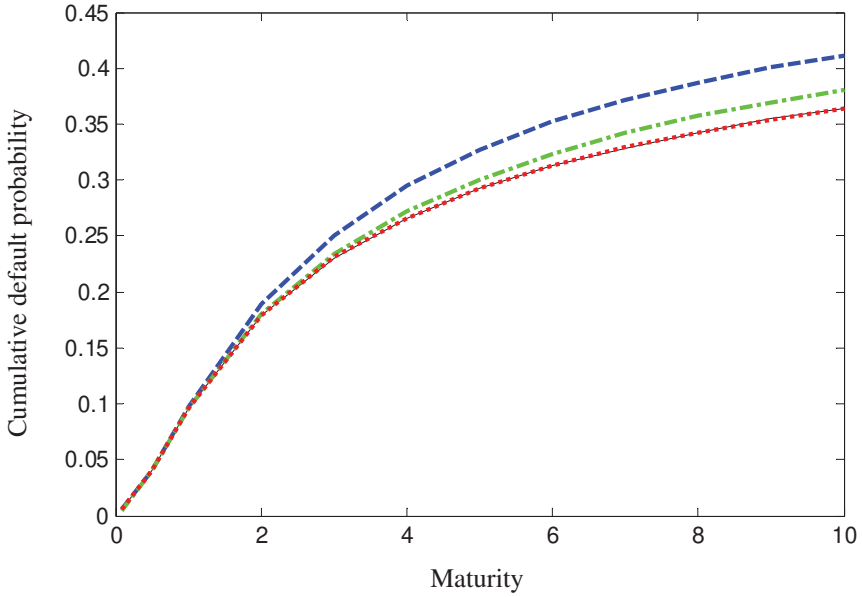
3.3. Empirical implications of the complex bond safety covenant after adding immediate default boundary (Model 2). In this part, we compare the difference between Model 1 and Model 2. We use the same parameters setting as in 4.1 by letting $V(0)$ = 100, F = 80, r = 0.05, ϕ = 0.05, λ = 0.05, α_1 = 0.4, p = 0.5, q = 0.5, η_u = 2.79667154579233, η_d = = 2.12168612641381 and caution time window period w = 15 days. Recall that the immediate default barrier is a fraction of the caution default boundary, i.e., $L(t) = \rho H(t)$. When ρ = 0, the imme-diate default boundary does not exist. We present the changes of credit spreads, cumulative default probabilities and recovery values under different levels of fraction ρ. Figure 8 shows the relationship between credit spreads and bond maturity. One can observe that if ρ = 0.9 credit spreads are higher than those of Model 1, whereas there are almost no differences for ρ = 0.8 and ρ = 0.6. When ρ is relatively low, the immediate default barrier can hardly affect default probabilities since the firm can default if the firm value stays below the caution barrier long enough. We graph the cumulative default probability and average recovery value to see which factor causes this phenomenon.



Note: (−) denotes Model 1; (--) denotes ρ = 0.9; (-·) denotes ρ = 0.8; (···) denotes ρ = 0.6.

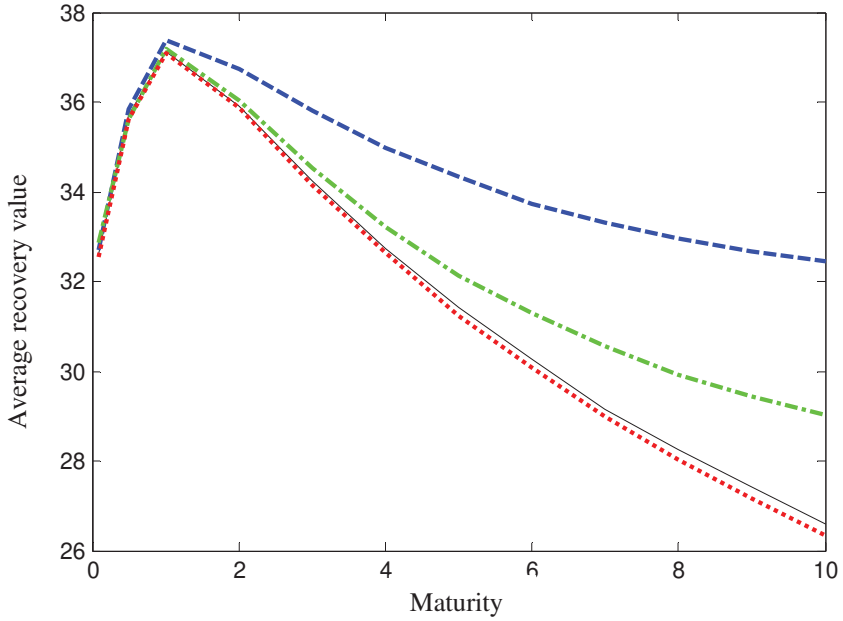
Fig. 8. Relationships between credit spread and maturity for various levels of *p*

Figure 9 depicts the relationship between cumulative default probability and maturity at different levels of ρ. It shows that higher values of ρ can lead to higher cumulative default probabilities. However, if ρ is lower than 0.6, default probabilities are almost no different from those in Model 1. As one knows that credit spreads are determined by both default probability and recovery value. Figure 10 presents the relationship between average recovery value and maturity at various levels of ρ This figure shows that a higher ρ results in higher average recovery value. In case of ρ = 0.9, as well as ρ = 0.8, average recovery values are higher than those in Model 1. Only when ρ = 0.6, the average recovery value is lower than Model 1. The intuition behind this is that the immediate default boundary can prevent the firm value from falling too low in the presence of the caution condition and thus make the average recovery higher. Note that a higher default probability leads to higher credit spreads, while higher recovery value results in smaller credit spreads. Accordingly, the interaction of these two effects determines the credit spread of the corporate bond. Combining the observations of Figures 9 and 10, our numerical results imply that the effect of default probability is larger than recovery value, on credit spreads. Therefore, credit spreads of Model 2 under ρ = 0.9 are higher than those of Model 1. Finally, we shall interpret this phenomanom with caution because other possible combinations of parameters may yield different results and have different implications.



Note: (−) denotes Model 1; (--) denotes ρ = 0.9; (-·) denotes ρ = 0.8; (···) denotes ρ = 0.6.

Fig. 9. Relationships between cumulative default probability and maturity for different values of *p*



Note: (−) denotes Model 1; (--) denotes ρ = 0.9; (-·) denotes ρ = 0.8; (···) denotes ρ = 0.6.

Fig. 10. Relationships between average recovery value and maturity for various levels of *p*

Conclusion

This paper develops two models for risky corporate bonds valuation and default risk analysis under the Parisian option framework and double exponential jump diffusion process. Our models have more flexibility in parameter settings than Merton's log-normal jump-diffusion model under a barrier option framework (see Zhou, 2011). We develop a Monte Carlo method that can efficiently produce accurate estimates of corporate bond prices under a Parisian option framework. Our modified uniform sampling simulation approach combines the uniform sampling approach of Metwally and Atiya (2002) and standard Brownian bridge simulation. This approach can significantly reduce computation time and bias of estimates compared to a standard approach. This is a substantial advantage over the standard Monte Carlo simulation approach where one needs more accurate and fast calculation of bond prices. Moreover, this method is also very efficient in pricing risky debt with complex safety covenant, and can be easily applied to other jump-diffusion processes.

We also demonstrate the shapes of credit spread curves under different caution time settings. Caution time leads to a variety of shapes of credit spread curves, default probability and recovery value for different maturities. Our results indicate that longer caution time results in lower cumulative default probability of corporate bonds. In general, credit spreads decrease as caution time increases. The only exception is that very short caution time (5 days) can result in credit spreads higher than under zero caution time for bonds with maturities longer than 2 years. The reason is that the average recovery value under this condition is lower than that under zero caution time. In addition, the features of two-side exponential jumps are capable of producing various shapes of credit spread curves compared with log-normal jumps. Furthermore, we also conduct default analysis when a second "immediate default boundary" is introduced into the model. The numerical results show that default probabilities and credit spreads generally increase as we incorporate the immediate default boundary.

There are still some possible improvements and interesting applications for future research to explore. First, variance reduction methods of Monte Carlo simulation could be explored for improving the estimates; for example, Ross and Ghamami (2010) improved the method of Metwally and Atiya (2002) by well-known variance reduction techniques. Second, it may be of interest to study corporate securities under complex capital structure such as the case where a firm issues both senior and junior debts. Finally, it would be a potentially fruitful research to calibrate the model to actual bond prices in the market, and examine if a more complex firm value process and bond safety covenant can better fit the real data.

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