“Basket cliquet options pricing with a dynamic dependence structure and stochastic interest rates”

AUTHORS	Giovanni Masala https://orcid.org/0000-0003-1719-641X

ARTICLE INFO	Giovanni Masala (2013). Basket cliquet options pricing with a dynamic dependence structure and stochastic interest rates. *Insurance Markets and Companies*, *4*(1)
RELEASED ON	Wednesday, 14 August 2013
JOURNAL	"Insurance Markets and Companies"
FOUNDER	LLC “Consulting Publishing Company “Business Perspectives”

Giovanni Masala (Italy)

Basket cliquet options pricing with a dynamic dependence structure and stochastic interest rates

Abstract

This paper presents a model for the pricing of an index linked insurance contract with a basket cliquet option embedded. The model moves from the seminal and widely accepted model of Brennan & Schwartz but uses a dynamic copula approach to describe the dependence between the two stochastic assets composing the underlying basket.

As no closed form is available for this kind of contract, the pricing is performed via Monte Carlo stochastic simulation; some useful algorithms are also described. For what concern the main features of the model, the time varying dependence structure between the correlated assets is given by a t-Student copula and the underlying assets are modelled through a AR(1)-GARCH(1,1) process. The time varying correlation between assets needed as input for the dynamic copula structure follows the DCC model developed by Engle. Parameters of the model are estimated through IFM (inference function of margins) method and Maximum Likelihood Estimation. The model also takes into account a stochastic instantaneous risk free interest rate driven by the CIR process.

Finally, a numerical application illustrates the model considering real data issued from the US markets. The author performs a sensitivity analysis with respect to some parameters of the basket cliquet option embedded and the CIR process.

Keywords: index linked policies, basket cliquet options, dynamic copula, CIR process, Monte Carlo simulation.

Introduction

The literature about the pricing of the index linked insurance policies starts with the seminal paper of Brennan & Schwartz (1976).

In their paper "The pricing of equity-linked life insurance policies with an asset value guarantee" the two authors recognize for the first time the presence of an embedded option in an ELPAVG ("equity linked life insurance policy with an asset value guarantee") contract and use the option theory to price the single and periodic premium of this kind of insurance products.

In particular, they find an explicit formula for the pricing of the single premium using the valuation model of Black & Scholes (1973) while they apply the finite difference equation numerical method to value the periodic one.

Almost all the successive papers analyzing these kind of insurance contracts (even more complex than the first one) used the contract decomposition proposed by Brennan & Schwartz (1976) to evaluate the implicit options and the price of the consequent premium.

Delbaen (1990), for instance, applies the martingale theory developed by Harrison & Kreps (1979) (instead of the Black & Scholes formula) to evaluate the periodic premium of policies with a minimum guarantee, while Bacinello & Ortu (1993) analyze the case of an insurance contract in which the minimum guarantees are endogenous, i.e. they are not fixed as data of the model, but depend on the premium (premia) paid. Moreover Bacinello & Ortu (1993) analyze insurance contracts not explicitly linked to a minimum amount guarantee but to a minimum number of units of the fund that must be bought each time the periodic premium is paid.

Among these models describing maturity guarantees, which are binding only at the expiration of the contract, there is an increasing literature analysing multiperiod guarantees (see, for example, Hipp, 1996) and multiasset options (see, for example, Stulz, 1982). Bacinello & Persson (2002) incorporate stochastic interest rates for the pricing of equity-linked life insurance contracts.

The models just described have a distinctive property: they all move from the typical hypotheses of the Black & Scholes model assuming the normality of underlying assets stochastic returns and linear correlation amongst them.

The main purpose of this work is to propose a model based on a dynamic copula dependence structure approach to price basket index linked insurance policies and, in particular, we present some simple algorithms useful to price the basket cliquet option embedded in a real policy where the basket is formed by two correlated stochastic assets. The marginal assets are then modelled through a $AR(1)$ process with dynamic volatility given by a $GARCH(1,1)$ process.

The model of valuation uses the traditional paradigms of quantitative finance (no arbitrage, risk neutral valuations) and introduces new features such as copula functions to model the dependence between the two sources of uncertainty. Indeed, copula functions provide a more flexible tool than the linear Pearson coefficient to describe dependence structures.

A fundamental feature of our model comes from the following remark. The constant correlation hypothesis among financial returns has been a challenging problem in recent years as the dependence structure plays in general a fundamental role in risk management. Several authors proposed statistical tests in order to verify the constant correlation hypothesis and they found evidence that correlations among assets tend to be time-varying. We can enumerate at this purpose Bera & Kim (2002), Engle & Sheppard (2001) and Tse (2000).

For these reasons, we take into account dynamic copulas which permit to consider a time varying dependence structure. At this purpose, we use the *DCC*(1,1) model developed by Engle (2002). The innovative aspect of this contribution concerns mainly the application of a dynamic dependence structure to index linked policies, which aims to generalize the surveys on this subject present in literature. The main steps of the model are thoroughly illustrated through a numerical application.

The paper is organized as follows. Section 1 describes briefly the theoretical backgrounds of the pricing model; section 2 introduces the economic framework and defines the insurance contract to evaluate; section 3 illustrates the model through a numerical application and the final section concludes.

1. The model specifications

1.1. Copula functions. Copula functions enable to deal with multivariate modeling by taking into account complex non linear dependence structures between the marginals. The fundamental feature of a copula function is given by the fact that a joint distribution can be factored into the marginals and a dependence function which is represented by the copula (indeed, the Latin term *copula* means "link" so that the copula joins the marginal distributions together in order to form a multivariate distribution). The dependence structure between the marginals is entailed in the copula function, while other characteristics such as mean, standard deviation, skewness and kurtosis are fully determined by the marginals. We can enumerate in literature a great variety of copula functions. Copulas can be used to obtain more general multivariate densities than the traditional joint normal. For example, we can maintain the normal dependence structure by considering the so-called Normal copula, but on the other side the marginals can be modelled separately with specific distributions.

Abe Sklar (1959) introduced copula functions in the framework of "probabilistic metric spaces". From 1986 on copula functions are intensively studied from a statistical point of view due to the impulse of Genest and MacKay's work "The joy of copulas" (1986).

Nevertheless, applications in financial and (in particular) actuarial fields are revealed only in the end of the 90's. We can quote for example the seminal papers of Frees and Valdez (1998) in actuarial field and Embrechts for what concerns financial applications (Embrechts et al., 2002, 2003).

We have just observed that Copula functions allow to model efficiently the dependence structure between variates, that's why they assumed in these last years an increasingly importance as a tool for investigating problems such as risk measurement in financial and actuarial applications.

In this paper we restrict to the bidimensional case. The next definition and the subsequent consequences can be adapted to the multivariate case.

Definition 1. A bidimensional copula ("2-copula") is a function C satisfying the following properties:

(1) domain $[0,1]\times[0,1]$;

(2) $C(0,u) = C(u,0) = 0$; $C(u,1) = C(1,u) = u$ for every $u\in[0,1]$;

(3) C is a 2-increasing function, that's to say:

$C(v_1, v_2) + C(u_1, u_2) \geq C(v_1, u_2) + C(u_1, v_2)$ for every $(u_1, u_2)\in[0,1]\times[0,1]$; $(v_1, v_2)\in[0,1]\times[0,1]$ such that $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$.

Consequences:

- C is a distribution function with uniform marginals;
- consider now two one-dimensional probability distributions F_1 and F_2, and a bidimensional copula C. It is clear that $F(x_1, x_2) = C(F_1(x_1),F_2(x_2))$ represents a bidimensional distribution with marginals F_1 and F_2.

The last result can be inverted; this conduces to the following fundamental theorem demonstrated by Sklar (1959).

Theorem 1. Let F be a bidimensional distribution, with marginals F_1 and F_2. Then there exists a 2-copula C such that

$F(x_1, x_2) = C(F_1(x_1),F_2(x_2))$.

If the marginals F_1 and F_2 are continuous, then the copula C is unique.

The previous representation is called *canonical representation* of the distribution. Sklar's theorem is then a powerful tool to construct bidimensional distributions by using one-dimensional ones, which represent the marginals of the given distribution. Dependence between marginals is then characterized by the copula C. Note moreover that the construction of multidimensional non-Gaussian models is particularly hard. An approach using copulas permits to simplify this problem; moreover one can construct multidimensional distributions with different arbitrary marginals.

Suppose that the bivariate $X = (X_1, X_2)$ possesses a density function $f(x_1, x_2)$. We can then express it by means of the marginal density functions $f_i(x)$ and the copula in the following manner:

$F(x_1, x_2) = c(F_1(x_1),F_2(x_2)) \times f_1(x_1) \times f_2(x_2)$ with

$$c(u_1,u_2) = \frac{\partial C(u_1,u_2)}{\partial u_1 \partial u_2}$$

and $F_i(x)$ are the c.d.f. of the marginals.

The definition 1 can be easily generalized to the n dimensional case.

1.2. Conditional copulas. Let us now generalize the notion of copula introduced before. We just observe that time series often involve random variables conditioned on some variables (we shall denote hereafter conditioning variables at time t by F_t). A typical example is furnished by lagged returns. The introduction of the so-called conditional copula appears then very natural. The definition in the bidimensional case is established as follows.

Definition 2. The conditional copula of $(x,y)|F_{t-1}$, where $x|F_{t-1} \sim F_t$ and $y|F_{t-1} \sim G_t$ is the conditional joint distribution function of $U_t \sim F_t(x|F_{t-1})$ and $V_t \sim G_t(y|F_{t-1})$ given F_{t-1}.

The meaning of conditional copulas is unchanged. Indeed, a two-dimensional conditional copula is the conditional joint distribution of the probability integral transforms of each marginal X_t and Y_t with respect to their marginal distributions F_t and G_t. Furthermore, it can be proved that the Sklar theorem stated before admits an obvious generalization.

Theorem 2. Sklar's theorem for the conditional copula.

Let F_t be the conditional distribution of $x|F_{t-1}$, given the conditioning set $|F_{t-1}$, G_t be the conditional distribution of $y|F_{t-1}$, and H_t be the joint conditional bivariate distribution of $(x,y)|F_{t-1}$. Assume that F_t and G_t are continuous in x and y. Then there exists a unique conditional copula C_t such that:

$$H_t(x,y|F_{t-1}) = C_t(F_t(x|F_{t-1}), G_t(y|F_{t-1})|F_{t-1}).$$

Conversely, if we let F_t and G_t be the conditional distributions of the two random variables X_t and Y_t, and C_t be a conditional copula, then the function H_t defined above is a conditional bivariate distribution function with conditional marginal distributions F_t and G_t (for a proof, see Patton, 2006).

The Sklar's theorem just stated for conditional distributions requires necessarily that the conditioning variable F_{t-1} must be the same for both marginal distributions and the copula. Otherwise, if the conditioning variable for F_t, G_t and C_t do not coincide, the function H_t will not be, in general, a joint conditional distribution function. As a particular case, it can be showed that H_t is the joint distribution of $(x,y|F_{t-1}) = (x,y|w_1,w_2)$ whenever $F_t(x|w_1) = F_t(x|w_1,w_2)$ and $G_t(y|w_2) = G_t(y|w_1,w_2)$, in other words when some variables influence the conditional distribution of one variable but not the other.

As a consequence of the previous statements, we deduce that the implied conditional copula can be derivated from any bivariate conditional distribution. We just have to apply Sklar's theorem and to consider the well-known relation between the distribution and the density function. Thus, the bivariate copula density $C_t(F_t(x|F_{t-1}), G_t(y|F_{t-1})|F_{t-1})$ associated to a copula function $C_t(F_t(x|F_{t-1}), G_t(y|F_{t-1})|F_{t-1})$ is given by:

$$h_t(x,y)|F_{t-1} = c_t(F_t(x|F_{t-1}), G_t(y|F_{t-1})|F_{t-1}) \times$$
$$\times f_t(x|F_{t-1}) \cdot g_t(y|F_{t-1}).$$

We apply now these considerations to the Normal (Gaussian) copula and the Student t-copula.

We remind that the Normal copula is the copula of the bivariate Normal distribution, whose probability density function in the bivariate case is the following:

$$c_t(u_t, v_t; \rho_t|F_{t-1}) = \frac{1}{\sqrt{1-\rho_t^2}} \times \exp\left(-\frac{1}{2} \cdot \frac{\Phi^{-1}(u_t)^2 + \Phi^{-1}(v_t)^2 - 2\rho_t \cdot \Phi^{-1}(u_t) \cdot \Phi^{-1}(v_t)}{1-\rho_t^2}\right) \times \exp\left(\frac{1}{2} \cdot \left(\Phi^{-1}(u_t)^2 + \Phi^{-1}(v_t)^2\right)\right),$$

where ρ_t is the conditional linear correlation, given the conditioning set F_{t-1} and Φ^{-1} is the inverse of the standard univariate Normal distribution.

Regards the Student's t-copula, which is the copula of the bivariate Student's t-distribution, the density function is:

$$c_t(u_t, v_t; \rho_t, \upsilon_t|F_{t-1}) = \frac{\Gamma\left(\frac{\upsilon_t+2}{2}\right) \cdot \Gamma\left(\frac{\upsilon_t}{2}\right)}{\Gamma\left(\frac{\upsilon_t+1}{2}\right)^2 \cdot \sqrt{1-\rho_t^2}} \times \left[1 + \left(\frac{t_{\upsilon_t}{}^{-1}(u_t)^2 + t_{\upsilon_t}{}^{-1}(v_t)^2 - 2\rho_t \cdot t_{\upsilon_t}{}^{-1}(u_t) \cdot t_{\upsilon_t}{}^{-1}(v_t)}{(1-\rho_t^2) \cdot \upsilon_t}\right)\right]^{-\frac{\upsilon_t+2}{2}} \times$$

$$\times \left[\left(1 + \frac{t_{\upsilon_t}{}^{-1}(u_t)^2}{\upsilon_t}\right) \cdot \left(1 + \frac{t_{\upsilon_t}{}^{-1}(v_t)^2}{\upsilon_t}\right)\right]^{\frac{\upsilon_t+1}{2}},$$

where ρ_t is the conditional linear correlation, υ_t are the conditional degrees of freedom and $t_{\upsilon_t}^{-1}$ is the inverse of the Student's t cumulative distribution function. Some recent applications of dynamic copulas can be found in Ausin & Lopes (2010), Fantazzini (2008) and Manner & Reznikova (2011).

1.3. Marginal modeling and estimation. We have just stated that the joint density function in the bivariate conditional case is:

$$h(x,y|F_{t-1};\vartheta_h) = f_t(x|F_{t-1};\vartheta_f)\times g_t(y|F_{t-1};\vartheta_g)\times$$
$$\times c_t(u,v|F_{t-1};\vartheta_c),$$

where $u \equiv F_t\left(x|F_{t-1};\vartheta_f\right), v \equiv G_t\left(y|F_{t-1};\vartheta_g\right)$ and ϑ_h, ϑ_f, ϑ_g, ϑ_c denote respectively the joint density, marginals and copula parameters' vectors, with $\vartheta_h = \left[\vartheta_f^{'}, \vartheta_g^{'}, \vartheta_c^{'}\right]$.

We deduce by applying Maximum likelihood method that:

$$L_{xy}\left(\vartheta_h\right) = L_x\left(\vartheta_f\right) + L_y\left(\vartheta_g\right) + L_c\left(\vartheta_f, \vartheta_g, \vartheta_c\right),$$

where $L_{xy}\left(\vartheta_h\right) = \log h_t\left(x,y|F_{t-1};\vartheta_h\right)$,

$L_x\left(\vartheta_f\right) = \log f_t\left(x|F_{t-1};\vartheta_f\right)$,

$L_y\left(\vartheta_g\right) = \log g_t\left(y|F_{t-1};\vartheta_g\right)$ and

$L_c\left(\vartheta_f, \vartheta_g, \vartheta_c\right) = \log c_t\left(u,v|F_{t-1};\vartheta_c\right)$.

In order to estimate all the parameters, we use the inference functions for margins (IFM) method. According to this method, the parameters of the marginal distributions are estimated separately from the parameters of the copula. Thus, the estimation process consists in the following two steps:

1. Estimate the parameters ϑ_f and ϑ_g of the marginal distributions F_t and G_t using the Maximum Likelihood method:

$$\hat{\vartheta}_f = \arg\max L(\vartheta_f) = \arg\max \sum_{t=1}^{T} \log f_t(x_t;\vartheta_f)$$

$$\hat{\vartheta}_g = \arg\max L(\vartheta_g) = \arg\max \sum_{t=1}^{T} \log g_t(y_t;\vartheta_g)$$

if the parameters' vectors are variation free; otherwise we set

$$\left\{\hat{\vartheta}_f, \hat{\vartheta}_g\right\} = \arg\max\left[L(\vartheta_f) + L(\vartheta_g)\right] =$$
$$= \arg\max \sum_{t=1}^{T}\left[\log f_t(x_t;\vartheta_f) + \log g_t(y_t;\vartheta_g)\right].$$

2. Estimate the copula parameters ϑ_c, using the results of step 1:

$$\hat{\vartheta}_c = \arg\max L(\vartheta_c) =$$
$$= \arg\max \sum_{t=1}^{T} \log\left[c_t(F_t(x_x;\hat{\vartheta}_f), G_t(y_t;\hat{\vartheta}_g);\vartheta_c)\right].$$

For what concerns the marginal distributions time series, we use a general $AR(1)$-$GARCH(1,1)$ model for the continuously compounded log-returns y_t given by:

$$y_t = \mu + \phi \times y_{t-1} + \varepsilon_t,$$

$$\varepsilon_t = \eta_t \times \sqrt{h_t},\ \eta_t \overset{i.i.d.}{\sim} f(0,1),$$

$$h_t = \omega + \alpha \times \varepsilon_{t-1}^2 + \beta \cdot h_{t-1}.$$

Such a choice has been carried out by Guégan & Zhang (2008), Hafner & Reznikova (2010), Palaro & Hotta (2006). We estimate the given model assuming two typical density functions $f(0,1)$ for η_t namely the Normal and the Student's t distributions.

After having estimated the parameters of the marginal distributions $\{F_t, G_t\}$ in the first step through the $AR(1)$-$GARCH(1,1)$ model just described, we finally estimate the copula parameters in the next step, as previously explained.

We remind that in the framework of dynamic copulas, the Sklar's theorem for conditional distributions requires that the conditioning variable must be the same for both marginal distributions and the copula.

We suggest that the dynamic evolution of the correlation parameter ρ_r evolves through time as in the $DCC(1,1)$ model of Engle (2002):

$$\rho_t = (1-\lambda-\gamma)\cdot \overline{Q} + \lambda \cdot \Psi_{t-1} + \gamma \cdot \rho_t - 1,$$

where $\overline{Q}$ is the sample correlation and Ψ_t is a sample correlation of a moving window of arbitrary size κ. The parameter constraints for the DCC are the same as for the univariate $GARCH(1,1)$ models, namely:

$$\lambda + \gamma < 1 \quad \lambda, \gamma \in (0,1).$$

This model has been also proposed by Patton (2006), Jondeau & Rockinger (2006) and Embrechts & Dias (2010). Furthermore, Vogiatzoglou (2010) illustrates an algorithm written in Matlab which permits to estimate these parameters.

The final goal is then to generate a large number of daily log-returns $\{r_1(t), r_2(t)\}$ for the two assets through a classical Monte Carlo simulation.

We describe hereafter the detailed steps of the whole procedure.

1. In order to generate pseudo-random numbers from the t-Student copula we use the following algorithm (we assume that the correlation is time-varying and the degree of freedom υ is fixed):

- set initial value of the correlation and determine recursively the time-varying correlation through $DCC(1,1)$ model: $\rho_t = (1-\lambda-\gamma)\times\overline{Q}+\lambda\times\Psi_{t-1}+$ $+\gamma\times\rho_t - 1$;
- find the Cholesky decomposition A_t of the correlation matrix Σ_t;
- simulate two independent random variates $z = (z_1, z_2)$ from the standard normal distribution;
- simulate a random variate s from χ^2_υ distribution, independent of z;
- determine the vector $y_t = A_t \times z$;
- set $x_t = \frac{\sqrt{\upsilon}}{\sqrt{s}} y_t$;
- determine the components $R_i(t) = t_\upsilon(x_{i,t})$ $i = 1,2$. The resultant vector is $(R_1(t), R_2(t))^T \sim C^2_{\upsilon,\Sigma_t}$.

The algorithm for the Gaussian copula is simpler and very similar so that we omit it for sake of brevity.

2. Regards the simulation of the marginal distributions, we describe the algorithm for the $AR(1)$-$GARCH(1,1)$ process:

- fix an initial value for $r_i(t)$ and $h_i(t)$;
- set recursively (for $t = 1,\ldots,T$) the following relations (for $i = 1,2$)

$$\begin{vmatrix} \varepsilon_i(t) = \sqrt{h_i(t)} \times R_i(t), \\ h_i(t+1) = \omega_i + \alpha_i \times \varepsilon_i(t)^2 + \beta_i \times h_i(t), \\ r_i(t) = \mu_i + \phi_i \times r_i(t-1) + \varepsilon_i(t), \end{vmatrix}$$

where ω_i, α_i, β_i, μ_i and $\O_i$ are the estimated parameters of the $AR(1)$-$GARCH(1,1)$ model and $R_i(t)$ are the random numbers obtained from the copula algorithm determined at the previous step.

3. We finally deduce the simulated time series' prices through the relation $P_i(t) = P_i(t-1)\times\exp(r_i(t))$ after assigning the starting value $P_i(0)$, for $i = 1,2$.

The next step will be the simulation of the pay offs' contracts, which will be revealed in the next section.

1.4. Stochastic risk free rate. The Cox, Ingersoll, Ross (CIR) model (Cox et al., 1985) is a standard one-factor model for term structure of interest rates, which ensures the positivity of rates at any time-to-maturity. In the CIR model the instantaneous short term interest rate $r(t)$ satisfies the following diffusion equation:

$$dr(t) = a \times (b - r(t))dt + \sigma \times \sqrt{r(t)}dW_t,$$

where $a \in R^+$ is the mean-reverting parameter, $b \in R^+$ is the long run parameter, $\sigma \in R^+$ is the volatility and W_t is a standard Brownian motion. The parameters of this rocess can be estimated by several standard methods (for example least square method or MLE).

We assume that the risk free rate process is independent from the asset portfolio returns.

In order to draw Monte Carlo simulations from the risk free process we will consider the following discretization form (coming from the Euler's scheme):

$$r(t+\Delta t) = r(t) + a \times (b - r(t)) \times \Delta t + \sigma \times \sqrt{\Delta t} \times \sqrt{r(t)} \times \varepsilon,$$

where ε is a pseudo-random number extracted from the Normal distribution.

2. The index linked contract and the economic framework

In this section we introduce our assumptions and notations concerning the economic framework and the contract to evaluate. The contract is an index linked policy with a basket cliquet option embedded; the basket is composed of two US assets included in the Dow Jones Index: AT&T Inc. and Microsoft Corporation. We remind that a cliquet (or ratchet) option is a particular type of exotic option in which the strike price periodically resets at specified dates before the final expiration time is reached. The application described in this section aims to explain the different steps of the model.

2.1. Notations and assumptions. As usual in financial literature, we assume a perfectly competitive and frictionless market, no arbitrage and rational operators all sharing the same information revealed by a filtration.

In this economic framework, we introduce the following variables:

T is the expiration date of the contract; $r(t)$ is the instantaneous risk-free interest rate (stochastic CIR model); $x(t)$ is the value of the first asset at time t (AT&T asset); $y(t)$ is the value of the second asset at time t (MSFT asset); $b(t)$ is the benefit payable at time t; D is the reference capital invested at time $t = 0$; $v(\tau,t)$ is the price at $\tau \le t$ of a unitary zero coupon bond with maturity time t.

We now remind the characteristics of the state variables characterizing our model:

- first asset (AT&T): $AR(1)$-$GARCH(1,1)$ model;
- second asset (MSFT): $AR(1)$-$GARCH(1,1)$ model;
- the dependence between $x(t)$ and $y(t)$: dynamic copula (Gaussian and Student) model where the correlation parameter evolves through a $DCC(1,1)$ dynamic.

Figure 1 is a scatter plot of the historical return values of the AT&T versus the MSFT asset (the data are referred to the period September 2007-September 2012; 1,260 records are available); the global correlation coefficient ρ between the two returns vectors is 59.24%.

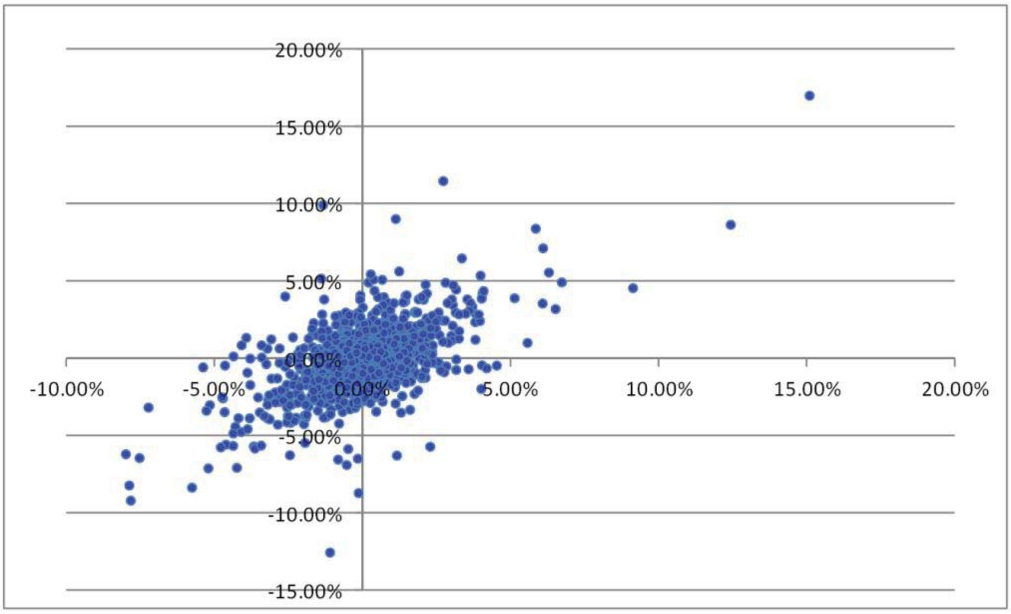

Fig. 1. Scatter plot of AT&T vs. MSFT returns

As stated before, it's a well-known fact from literature that the correlations between stock assets are generally time-variable. For this reason, the pricing of multivariate financial products based on the hypothesis of constant correlation may be flawed. In order to highlight this feature, we exhibit in the next Figure 2 the correlation coefficient between the two assets estimated on a rolling window of 50 trading days.

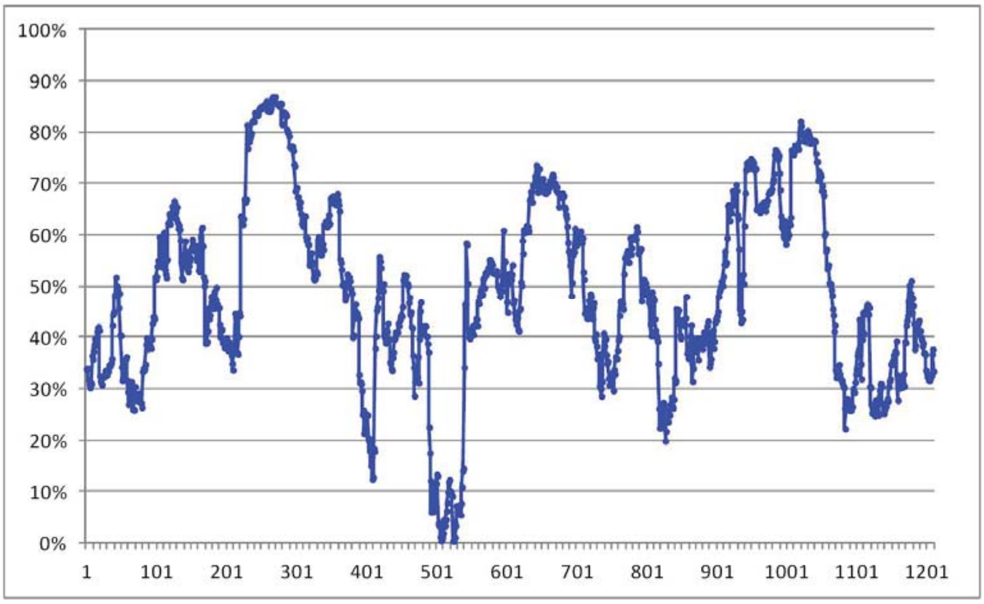

Fig. 2. Correlation dynamic

We deduce from this figure the variability of the correlation coefficient, which justify the use of the DCC model. We also assert that the time-variability correlation can be emphasized by changing the width of the rolling windows.

2.2. Definition of the index linked insurance contract. We consider an index linked insurance contract which pays at time $T = 4$ a benefit $b(T)$ consisting in the reference capital equal to a notional amount D (conventionally 100 Euro) plus the payoff of a basket cliquet option; the benefit $b(T)$ can then be expressed as:

$$b(T) = D + D \times \mathrm{Max}\left(F_g ; \sum_{k=1}^{8} P_k \right),$$

where

$$P_k = \mathrm{Max}\left[F_l ; \min\left(C_l ; \frac{x(k)}{x(k-1)} - 1 \right) \times \frac{1}{2} \right] + \\ + \mathrm{Max}\left[F_l ; \min\left(C_l ; \frac{y(k)}{y(k-1)} - 1 \right) \times \frac{1}{2} \right]$$

and the symbol k denotes the k-th semester between date 0 (the starting date of the contract) and

8 (the maturity date equal to four years). We also denote F_g the global floor, F_l the local floor and C_l the local cap.

Therefore, the price of the policy depends on the value at time 0 of a *zcb* with maturity $T = 4$ and on the pricing of the basket cliquet option whose evaluation will be discussed in the next section.

Due to the presence of a bivariate risk neutral distribution with copulas, in order to price the option embedded in the contract we will perform a Monte Carlo simulation as no closed form is available to evaluate this kind of derivative.

3. The evaluation model

According to the standard results in Harrison & Kreps (1979) and Harrison & Pliska (1981) and to the generalization of the option pricing with a bivariate risk neutral distribution proposed by Rapuch & Roncalli (2004), the price of the proposed contract is given by:

$$V_0(x,y) = E_0^C\left(b(T)\cdot e^{-\int_0^T r(u)du} \right),$$

where $E_0^C(\cdot)$ is the date 0 expectation of $b(T)$ taken under the bivariate risk neutral distribution with a copula dependence structure.

The expression of $V_0(x,y)$ will be obtained thanks to numerical methods because no pricing formula is available. The next subsection will then be devoted to the pricing of $V_0(x,y)$ through Monte Carlo simulation.

3.1. A Monte Carlo approach. The evaluation of the basket cliquet option embedded in the benefit proposed in the previous sections requires the pricing a derivative written on two correlated assets, namely AT&T denoted $x(t)$ and Microsoft denoted $y(t)$.

Let us simulate a large number M of the bivariate assets $(x_s(t), y_s(t))$. The price of the benefit

$$V_0(x,y) = E_0^C\left(b(T)\cdot e^{-\int_0^T r(u)du} \right)$$

can be estimated by the sample mean:

$$\hat{V}_0(x,y) = e^{-\int_0^T r(u)du} \cdot \frac{1}{M} \cdot \sum_{s=1}^{M} \hat{b}_s(T),$$

where

$$\hat{b}_s(T) = D + D\cdot \text{Max}\left(F_g; \sum_{k=1}^{8} \hat{P}_k^{(s)} \right)$$

and

$$\hat{P}_k^{(s)} = \text{Max}\left[F_l; \min\left(C_l; \frac{x_s(k)}{x_s(k-1)} - 1 \right) \times \frac{1}{2} \right] +$$

$$+ \text{Max}\left[F_l; \min\left(C_l; \frac{y_s(k)}{y_s(k-1)} - 1 \right) \times \frac{1}{2} \right].$$

We remind that the interest rate $r(t)$ follows a stochastic CIR process. We simulate random trajectories of this process thanks to the discretized equation given in subsection 1.4. The discount factor $e^{-\int_0^T r(u)du}$ is then discretized and estimated through the simulated values of $r(t)$ previously determined.

3.2. The value of the policy. In this subsection we present a numerical application of the model described.

The application has been carried out with the following parameters: $T = 4$, $\Delta t = 1/2$, $F_g = 4\%$, $F_l = 2\%$, $C_l = 10\%$ and $D = 100$.

The parameters of the CIR model have been estimated from the US Daily Treasury Yield Curve Rates for the period January 2, 2009 to July, 27, 2010. This estimation, performed with Matlab software through maximum likelihood techniques, furnished the following values: $a = 0.01$, $b = 0.001$, $\sigma = 0.0074$ and $r(0) = 0.0016$.

The parameters of the $AR(1)$-$GARCH(1,1)$ model and the test statistics for the two marginal assets are illustrated in Table 1 and Table 2.

Table 1. AT&T parameters

Parameter	Value	St. error	t-stats
μ	0.0007	0.000	2.2368
ϕ	-0.0116	0.013	-0.9209
ω	$8.842\cdot 10^{-7}$	0.000	1.6683
α	0.0829	0.019	4.3375
β	0.9141	0.018	51.6216
dof	9.2647	1.561	5.9357
Akaike: -7,348.0656			
BIC: -7,317.2372			
Log likelihood: -3,680.033			

Table 2. MSFT parameters

Parameter	Value	St. error	t-stats
μ	0.0005	0.000	1.1576
ϕ	-0.0307	0.024	-1.2948
ω	$8.504\cdot 10^{-7}$	0.000	0.9079
α	0.0553	0.016	3.4993
β	0.9447	0.013	73.2363
dof	5.3073	0.806	6.5873
Akaike: -6,686.1450			
BIC: -6,655.3165			
Log likelihood: -3,349.072			

Finally, the Student copula estimation and the parameters of the DCC model are revealed in the Table 3.

Table 3. *t*-copula & DCC parameters

Parameter	Value	St. error	t-stats
λ	0.0532	0.019	2.8381
γ	0.9237	0.033	28.1328
dof	4.4676	0.838	5.3339
Akaike: -391.8522			
BIC: -376.4379			
Log likelihood: -198.926			

The parameters for the Gaussian copula are revealed in Table 4.

Table 4. Gaussian copula & DCC parameters

Parameter	Value	St. error	t-stats
λ	0.0658	0.025	2.6313
γ	0.9057	0.054	16.7169
Akaike: -361.9579			
BIC: -351.6817			
Log likelihood: -182.979			

The value of the policy is 122.71 for the Student copula and 122.70 for the Gaussian copula. We can also examine the price sensitivity with respect to the local floor. At this purpose, we let F_l vary in the range 0%-4%. The results are given in the Figure 3 for the Student copulas. We obtain for the Gaussian copula very similar results.

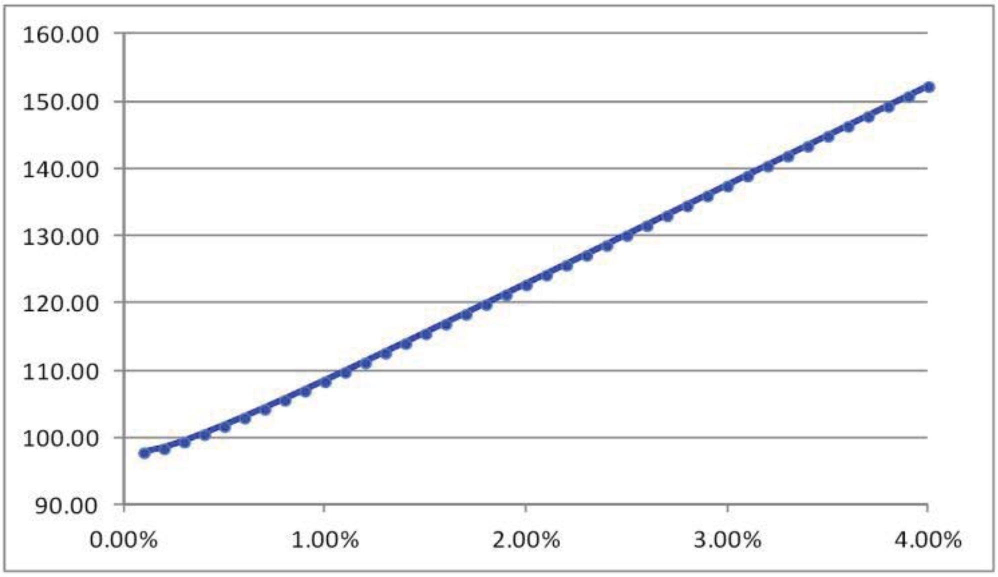

Fig. 3. Price sensitivity vs. F_l

The numerical application also highlights that the value of the policy depends on the parameters of the CIR process. For example, let us change the value of the mean-reverting parameter *a*. We exhibit in Figure 4 the increasing behavior of the policy value with respect to this parameter.

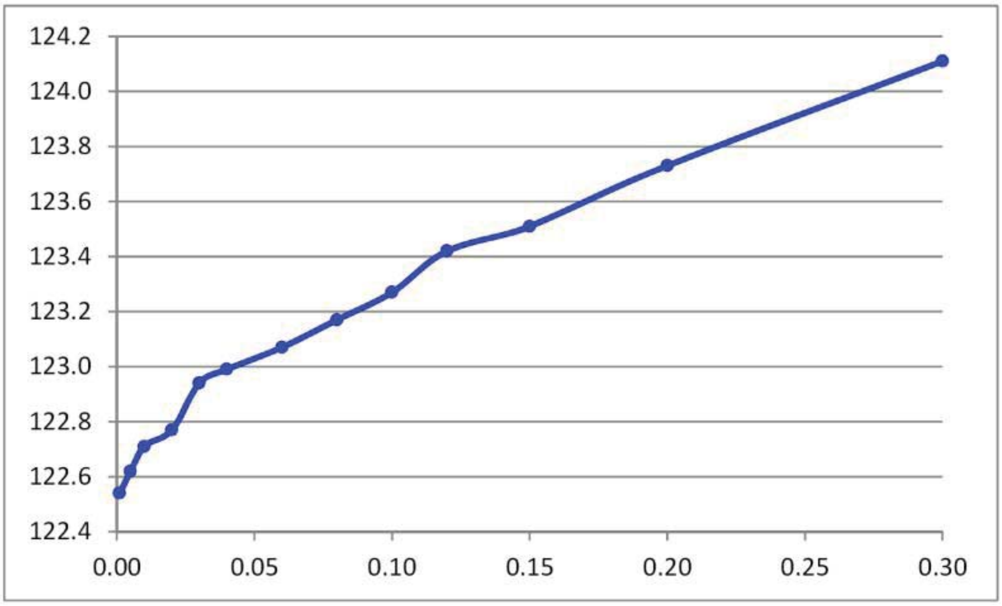

Fig. 4. Price sensitivity vs. *a*

Conclusions

In this paper we propose a procedure useful to perform the pricing of an index linked insurance policy with a basket cliquet option embedded. This basket is composed of two assets included in the Dow Jones Index (AT&T Inc. and Microsoft Corporation).

The scheme considers that the dependence between the two risky assets can be expressed and modeled through a dynamic copula in order to consider the realistic hypothesis of time varying dependence structure and the pricing procedure is based on the Monte Carlo method. At this purpose the two marginal assets have been carefully modeled through a $AR(1)$-$GARCH(1,1)$ model.

We have carried out a numerical simulation to estimate the value of the policy and of the embedded option. Besides, we considered a stochastic interest rate whose dynamic is driven by the CIR process.

One of the main challenges is to highlight that copula functions can represent a useful tool to realize more refined risk management strategies for the financial risk managers of insurance companies, following the traditional scheme of risk neutral valuations. Copula functions permit to model correctly the dependence structure and the algorithms involved are easily implemented. Furthermore, dynamic copulas permit to take into account the more realistic features of time-varying dependence structure. At this purpose, we also pretend that the pricing of such policies under the hypothesis of constant correlation may be particularly misleading. As a further generalization, we should consider an embedded basket composed with more than two risky assets. The dynamic correlation between the risk free interest rate and the underlying index could also be investigated.

References

1. Ausin M.C., H.F. Lopes (2010). Time-varying joint distribution through copulas, *Computational Statistics and Data Analysis*, 54, pp. 2383-2399.
2. Bacinello A.R., F. Ortu (1993). Pricing equity-linked life insurance with endogenous minimum guarantee, *Insurance: Mathematics and Economics*, 12, pp. 245-257.
3. Bacinello A.R., S.A. Persson (2002). Design and Pricing of Equity-Linked Life Insurance under Stochastic Interest Rates, *The Journal of Risk Finance*, 3 (2), pp. 6-21.
4. Bera A.K., S. Kim (2002). Testing constancy of correlation and other specifications of the BGARCH model with an application to international equity returns, *Journal of Empirical Finance*, 9 (2), pp. 171-195.
5. Black F., M. Scholes (1973). The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, pp. 637-654.
6. Brennan M.J., E.S. Schwartz (1976). The pricing of equity-linked life insurance policies with an asset value guarantee, *Journal of Financial Economics*, 3 (3), pp. 195-213.
7. Cox J., J. Ingersoll, S.A. Ross (1985). Theory of the Term Structure of Interest Rates, *Econometrica*, 53, pp. 385-408.
8. Delbaen F. (1990). Equity linked policies, *Bulletin Association Royal Actuaries Belge*, 80, pp. 33-52.
9. Dias A., P. Embrechts (2010). Modeling exchange rate dependence dynamics at different time horizons, *Journal of International Money and Finance*, 29, pp. 1687-1705.
10. Embrechts P., A. McNeil, D. Straumann (2002). Correlation and dependence in risk management: properties and pitfalls, *Risk Management: Value at Risk and Beyond* (ed.), M.A.H. Dempster, Cambridge University Press, No 176, pp. 2-23.
11. Embrechts P., A. Hoing, A. Juri (2003). Using copulae to bound the value at risk for functions of dependent risks, *Finance and Stochastics*, 7 (2), pp. 145-167.
12. Engle R.F., K. Sheppard K. Theoretical and empirical properties of dynamic conditional correlations multivariate GARCH, NBER working paper 8554, 2001.
13. Engle R. (2002). Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models, *Journal of Business & Economic Statistics*, 20 (3), pp. 339-350.
14. Fantazzini D. (2008). Dynamic copula modelling for value at risk, *Frontiers in Finance and Economics*, 5 (2), pp. 72-108.
15. Frees E., E. Valdez (1998). Understanding relationships using copulas, *North American Actuarial Journal*, 2, pp. 1-25.
16. Genest C., J. MacKay (1986). The joy of copulas: bivariate distributions with uniform marginals, *The American Statistician*, 40, pp. 280-283.
17. Guégan D., J. Zhang (2008). Pricing bivariate option under GARCH processes with time-varying copula, *Insurance: Mathematics and Economics*, 42 (3), pp. 1095-1103.
18. Hafner C.M., O. Reznikova (2010). Efficient estimation of a semiparametric dynamic copula model, *Computational Statistics & Data Analysis*, 54 (11), pp. 2609-2627.
19. Harrison J.M., J.M. Kreps (1979). Martingales and arbitrage in multi-period security markets, *Journal of Economic Theory*, 20, pp. 381-408.
20. Harrison J.M., S.R. Pliska (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading, Stochastic Processes and their Applications, pp. 215-260.

21. Hipp C. (1996). Options for guaranteed index-linked life insurance, Proceedings of VI AFIR International Colloquium, Nurnberg, Germany, No. 2, pp. 1463-1483.
22. Jondeau E., M. Rockinger (2006). The Copula-GARCH model of conditional dependencies: An international stock market application, *Journal of International Money and Finance*, 25, pp. 827-853.
23. Manner H., O. Reznikova (2011). A survey on time-varying copulas: Specification, simulations and application, *Econometric Reviews*, 31 (6), pp. 654-687.
24. Palaro H.P., L.K. Hotta (2006). Using Conditional Copula to Estimate Value at Risk, *Journal of Data Science*, 4, pp. 93-115.
25. Patton A.J. (2006). Modelling asymmetric exchange rate dependence, *International Economic Review*, 47 (2), pp. 527-556.
26. Rapuch G., T. Roncalli (2004). Some remarks on two asset options pricing and stochastic dependence of asset prices, *Journal of Computational Finance*, 7 (4), pp. 23-33.
27. Sklar A. (1959). Fonction de répartition à n dimensions et leurs marges, Publ. Inst. Stat. Université de Paris, No. 8, pp. 229-231.
28. Stulz R.M. (1982). Options on the minimum or the maximum of two risky assets, *Journal of Financial Economics*, 10 (1), pp. 161-185.
29. Tse, Y.K. (2000). A test for constant correlations in a multivariate GARCH model, *Journal of Econometrics*, Vol. 98 (1), pp. 107-127.
30. Vogiatzoglou M. (2010). Statistical analysis of multivariate dependence structures among financial time series, Ph.D. Thesis, University of Macedonia, Thessaloniki.