

# “Exotic options with Lévy processes: the Markovian approach”

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## Exotic options with Lévy processes: the Markovian approach

### Abstract

This paper proposes a simplified methodology to price exotic options when the log returns follow a Lévy process. The Markovian approach is simpler than others proposed in literature for these processes and it allows to define hedging strategies. In particular, the authors consider three Lévy processes (variance gamma, Meixner and normal inverse Gaussian) and show how to compute barrier, compound and lookback option prices. The article first discusses the use of a homogeneous Markov chain approximating the risk neutral log return distribution. Then, it describes the methodology to price exotic contingent claims. Finally, the paper compares the convergence results considering the three different distributional assumptions.

**Keywords:** Lévy processes, Markov processes, exotic options, variance gamma, Meixner, normal inverse Gaussian.

**JEL Classification:** C63, F31.

### Introduction

It is well known that log returns are not Gaussian distributed. In particular, exponential Lévy models, that assume asset log returns follow a Lévy process, have been widely used in the recent financial literature. This distributional assumption represents a possible choice to overcome some drawbacks that afflict the traditional Black-Scholes model. Many empirical researches show the lack of the geometric Brownian motion and highlight the necessity to model asset returns by stochastic processes with skew and fat-tails distributions (see, among others, Rachev and Mitnik, 2000). Lévy non-Gaussian processes meet this necessity because their distributions can take into account skewness and excess kurtosis. Black-Scholes model also suffers the so called *volatility smile*. Specifically, the implied volatility, which should be a constant value, when plotted against strike prices, displays graphs which are smile shaped, that is, in-the-money and out-the-money volatilities are higher than at-the-money volatilities. Instead, empirical studies (Eberlein et al., 1998; Rachev and Mitnik, 2000) show how exponential Lévy models can reduce, at least in part, this volatility behavior.

Geman et al. (2001) provide a theoretical motivation to the use of Lévy processes in financial applications. They define an economic model where price processes result to be differences of two increasing stochastic processes, representing, respectively, the up and down movements of the market. Then, the resulting price processes are of finite variation and with jumps. Thus, pure jump Lévy processes, such as the variance gamma (VG) and CGMY processes can be used as distributional assumptions in this economic modeling. Moreover, since Lévy processes

are semimartingale, their use can be justified by the studies on no-arbitrage assumption (see Harrison and Kreps, 1979; Harrison and Pliska, 1981; and Delbaen and Schachermayer 1994). Unfortunately, Lévy processes except the Brownian motion and the Poisson process do not satisfy the so-called *predictable representation property* of a martingale and thus there exist infinitely many equivalent martingale measures. The choice of the right equivalent martingale measure is still a delicate problem. Generally, it is the market to choose for us, that is, given a set of current option prices, one should select the equivalent martingale measure that approximates better this set of data.

Option pricing under exponential Lévy models can be performed in several ways and each method can be more suitable according to the chosen Lévy process and contingent claim to be priced. When the option is European and we know the Lévy subordinator process of the time-changed Brownian representation, then Monte Carlo method can be very fast and accurate. For example, Ribeiro and Webber (2003; 2004) developed simulation schemes for the Normal Inverse Gaussian (NIG) and VG processes on the base of inverse Gaussian and gamma bridges, respectively. Instead, more general Lévy processes can be simulated through a compound Poisson approximation (see Asmussen and Rosiński, 2001). When the option is American, Monte Carlo method is not so straightforward and an adjustment to optimal stopping problems has to be carried out. Thus, the least squares Monte Carlo method (see Longstaff and Schwartz, 2001; and Carrière, 1998) can be used to approximate conditional expectations. Vanilla options can be easily priced by Fast Fourier Transform methods, which only need the knowledge of the characteristic function of the risk-neutral stock price process. Since Fast Fourier Transform methods return option price surfaces within a second, they make computationally efficient the cali-

bration of exponential Lévy models to the market prices. Barrier and lookback options can be treated by Wiener-Hopf approaches, but this methods are generally time consuming and quite complex (see Boyarchenko and Levendorskii, 2002; and Yor and Nguyen, 2001). Moreover, Wiener-Hopf factors depend on the density function of the Lévy process, which is usually unknown. The partial-integro differential equation (PIDE) approach can be applied to price vanilla and European barrier options. This PIDE is derived by the Feynman-Kac formula for Lévy processes (see Nualart and Schoutens, 2001; and Raible, 2000). Numerical solutions are thus computed applying finite difference schemes for PIDE with boundary conditions (see Cont and Voltchkova, 2005; and Hirta and Madan, 2003). Even American options can be priced by PIDE approach, but, in this case, we have to solve a system of partial-integro differential inequalities. Numerical solution can be obtained by the analytic method of lines or Garlekin methods (see Meyer and Van Der Hoek, 1997; and Matache et al., 2003). The above list of numerical methods does not represent at all a complete review of all existing methodologies. It only constitutes a partial revision of some popular techniques. For example, recently, there has been a considerable development of quadrature methods, starting from the Sullivan's studies (see Sullivan, 2000; Andricopoulos et al., 2003; O'Sullivan, 2005; and Lord et al., 2008).

The main contributions of this paper are: (1) the extension of the Duan et al.'s approach (see Duan and Simonato, 2001; and Duan et al., 2003) to price exotic contingent claims when the underlying follows an exponential Lévy process; (2) a new methodology to price lookback type options; (3) an empirical analysis of the proposed methodology.

Thus we first describe how to extend the Duan et al.'s lattice scheme (see Duan and Simonato, 2001; and Duan et al., 2003) to price contingent claims when the log return follows a Lévy process. The possibility to use lattice schemes dates back to the Amin's work (Amin, 1993), where the author applies a lattice scheme to price Bermudan options under jump-diffusion processes. Kellezi and Webber (2004) described four methods to construct lattices that approximate a general Lévy process. The first construction is obtained from the density function of the Lévy processes, whereas the other three ones are obtained, respectively, from the generating triplet, from a subordinated Brownian representation, and from a time copula. In their paper, Kellezi and Webber use the density function construction to price vanilla and Bermudan options under NIG, VG processes. Differently from Kellezi and Webber (2004), we construct a sequence of Markov chains converging weakly to the underlying Lévy process

on a finite set of dates. Then, the option pricing problem is reduced to that one of pricing contingent claims under Markov chains. The discretization process presents the same advantages of the binomial model, since it permits us to price path dependent contingent claims. With this method it is simple to obtain prices of Bermudan options, whereas American prices can be computed as limits of Bermudan prices, doing the sets of dates more and more dense. In particular, we examine the Markovian approach to price compound, barrier, and lookback options assuming exponential Lévy models for the underlying. Doing so, on the one hand, we extend Duan et al.'s approach when applied to price compound and barrier options. On the other hand, we discuss a new approach for pricing fixed and floating strike price lookback options.

In the proposed empirical analysis we compare option pricing results under the assumption that the log return follows either a NIG process, or a VG process or a Meixner process or a standard Brownian motion. Then, we show the convergence of the compound option prices in the case analyzed by Geske (1979) for the Brownian motion and we extend the same analysis to the other three Lévy processes. For compound and barrier options we just adapt the Markovian approach to Lévy processes. Alternatively to classic methods (see Babbs, 2000; and Cheuk and Vorst, 1997) we propose a different approach to price fixed and floating strike price lookback options in a Markov chain framework. Moreover, discretizing the continuous Markovian models we can approximate very well the right prices of floating strike lookback contracts, since in these contracts the maximum and/or the minimum of the underlying asset price are computed over some dates only, such as daily, weekly or monthly.

The paper is organized as follows. Section 1 is a brief introduction to Lévy processes and their use in pricing problems. Section 2 discusses the Markovian approach and shows some convergence results for Bermudan, European options and their Greeks when the log return follows either a NIG process, or a VG process or a Meixner process. In Section 3 we deal with the compound, barrier and lookback options when we use the three different Lévy processes. Finally, we briefly summarize the results.

## 1. Pricing with exponential Lévy processes

In this Section we describe Lévy processes reporting their characterization by the Lévy triplet and we discuss the asset pricing with exponential Lévy processes. Let us assume in the market there are two assets: a riskless asset and a risky asset whose log return process follows a Lévy process. In particular,

we assume that the riskless asset has price process  $B_t = \exp\left(\int_0^t r(s)ds\right)$ , where the right continuous with left-hand limits time-dependent function  $r(t)$  defines the short term interest rate. While we assume the risky asset pays no dividends and presents price process  $S_t = S_0 \exp(X_t)$ , where the log return process  $X = (X_t)_{t \geq 0}$  (i.e.,  $X_t = \log(S_t / S_0)$ ) is an adapted RCLL Lévy process defined on a filtered probability space  $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{0 \leq t \leq \infty}, P)$ , that satisfies the usual conditions.

Non-Gaussian Lévy processes generally take into account the skewness and the heavy tails often observed in the log return distribution. As a matter of fact, Lévy processes are all the stochastic processes with stationary, independent increments and stochastically continuous sample paths. Since they have infinitely divisible distributions, their characteristic function  $\phi(u)$  is uniquely determined by the triplet  $[\gamma, \sigma^2, \nu]$  that identifies the so-called Lévy-Khintchine characteristic exponent  $\psi(u) = \log \phi(u)$  given by:

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux \mathbb{1}_{\{|x| < 1\}}) \nu(dx),$$

where  $\gamma \in R$ ,  $\sigma^2 > 0$  and  $\nu$  is a measure on  $R \setminus \{0\}$  with  $\int_{-\infty}^{+\infty} (1 \wedge x^2) \nu(dx) < \infty$ . In particular the Lévy triplet  $[\gamma, \sigma^2, \nu]$  identifies the three main components of any Lévy process: the deterministic component ( $\gamma$ ), the Brownian component ( $\sigma^2$ ) and the pure jump component ( $\nu$ ). For further details on the theoretical aspects we refer to Sato (1999). Under the assumption the log return process follows a Lévy process whose trajectories are neither almost surely increasing nor almost surely decreasing we can always guarantee that there exists at least one equivalent martingale measure. Since the market is generally incomplete, then more than one equivalent martingale measure could exist. Given the risk neutral probability measure  $\tilde{P}$ , we can use it to determine the free-arbitrage price of any contingent claims with maturity  $T$ . That is, given the contingent

$$\phi_{NIG}(u; \alpha, \beta, t\delta, tq) = \exp\left(- (t\delta)\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right) + iutq\right).$$

That is the density of the log return at time  $t$  is given by:

$$f_{NIG}(x; \alpha, \beta, t\delta, tq) = \frac{t\delta\alpha}{\pi} \exp\left(t\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - tq)\right) \frac{K_1\left(\alpha\sqrt{(t\delta)^2 + (x - tq)^2}\right)}{\sqrt{(t\delta)^2 + (x - tq)^2}},$$

where  $K_\lambda(x)$  denotes the modified Bessel function of the third kind with index  $\lambda$ .

claims function  $H : \Omega \rightarrow R$  ( $\mathfrak{F}_T$ -measurable function), then its price at time  $t$  is:

$$\Pi_t(H) = \exp\left(-\int_t^T r(s)ds\right) E^{\tilde{P}}(H | \mathfrak{F}_t). \quad (1)$$

There exist several techniques to determine a risk neutral martingale measure. However, the choice of the equivalent martingale measure is not subject of the present paper and we adopt the mean-correcting measure just to follow other financial studies (see, among others, Ribeiro and Webber, 2003, 2004; Shoutens, 2003). A two steps methodology commonly is used:

1. To determine a class of equivalent martingale measures.
2. To determine the risk neutral measure, among the equivalent martingale measures, that minimizes a distance with respect to some historical contingent claim prices.

Typically, in order to determine the optimal parameters that better approximate the risk neutral distribution, we minimize the root mean squared prediction error (RMSE) with respect to the observed prices. Therefore, we consider  $N$  historical contingent claim prices  $cc_i (i = 1, \dots, N)$  and we determine the risk neutral Lévy process parameters  $\varpi \in \Theta$  that minimize

$$RMSE = \min_{\varpi \in \Theta} \sum_{i=1}^N \left(cc_i - \overline{Lp}_i(\varpi)\right)^2, \quad (2)$$

where  $\overline{Lp}_i(\varpi)$  is the price of the  $i$ -th contingent claim obtained using the relation (1) under the equivalent martingale Lévy density with the parameters  $\varpi \in \Theta$ .

Next, we consider three Lévy processes alternative to the Brownian motion that present skewness and semi heavy tails: the normal inverse Gaussian process (NIG), the variance gamma process (VG) and the Meixner one.

**1.1. Normal inverse Gaussian.** Under the assumption that the log return follows a NIG process  $NIG(\alpha, \beta, \delta, q)$ , with parameters  $\alpha > 0$ ,  $\beta \in (-\alpha, \alpha)$ ,  $\delta > 0$ ,  $q \in R$ , we have that the characteristic function of the process at time  $t$  is given by:

**1.2. Variance gamma.** The variance gamma process can be also defined as the difference between two independent gamma processes. Under the assumption that the log return follows a VG process,  $VG(\sigma, \nu, \theta, q)$ , with parameters  $\sigma > 0, \nu > 0$  and  $q, \theta \in R$ , the characteristic function of the process

$$f_{VG}(x; \sigma\sqrt{t}, \nu/t, t\theta, tq) = \frac{2e^{\frac{\theta(x-qt)}{\sigma^2} \left(\frac{(x-qt)^2}{2\sigma^2/\nu + \theta^2}\right)^{\frac{t}{2\nu} - \frac{1}{4}}}}{\nu^{t/\nu} \sqrt{2\pi\sigma}\Gamma(t/\nu)} K_{\frac{t}{\nu} - \frac{1}{2}} \left( \frac{1}{\sigma^2} \sqrt{(x-qt)^2(2\sigma^2/\nu + \theta^2)} \right),$$

where  $K_{\frac{t}{\nu} - \frac{1}{2}}(x)$  is the modified Bessel function of the third kind with index  $\frac{t}{\nu} - \frac{1}{2}$ .

**1.3. Meixner.** Under the assumption that the log return follows a Meixner process,  $Meixner(\alpha, \beta, \delta, q)$ , with parameters  $\alpha > 0, \beta \in (-\pi, \pi), \delta > 0$ ,

at time  $t$  is given by:

$$\phi_{VG} \left( u; \sigma\sqrt{t}, \frac{\nu}{t}, t\theta, tq \right) = \left( 1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2 \right)^{\frac{t}{\nu}} e^{iuqt}$$

That is the density of the log return at time  $t$  is given by:

$m \in R$  the characteristic function of the process at time  $t$  is given by:

$$\phi_{Meixner}(u; \alpha, \beta, \delta t, qt) = \left( \frac{\cos(\beta/2)}{\cosh((\alpha u - i\beta)/2)} \right)^{2\delta t} e^{iuqt}.$$

That is the density of the log return at time  $t$  is given by:

Table 1. MLE of parameters and Kolmogorov-Smirnoff test of daily S&P500 log returns assuming a NIG process, or a VG process or a Meixner process.

NIG	$\alpha = 153.866$	$\beta = 7.603$	$\delta = 1.562$	$q = -0.00029$	$D = 0.0653$
VG	$\theta = 0.0756$	$\sigma = 0.0984$	$\nu = 0.0024$	$q = 0.00055$	$D = 0.0667$
Meixner	$\alpha = 0.0146$	$\beta = 0.1116$	$\delta = 94.676$	$q = -0.00026$	$D = 0.0661$

$$f_{Meixner}(x; \alpha, \beta, \delta t, qt) = \frac{(2\cos(\beta/2))^{2\delta t}}{2\pi\Gamma(2\delta t)\alpha} \exp\left(\frac{\beta(x-qt)}{\alpha}\right) \left| \Gamma\left(\delta t + \frac{i(x-qt)}{\alpha}\right) \right|^2.$$

In order to value the best approximation of these distributions, we consider quotations of the index S&P500 from January 2006 to March 2007. Then we compute the parameters maximizing the likelihood function when the log returns follow either a NIG process, or a Meixner process or a VG process (see Table 1). Finally, we consider the Kolmogorov-Smirnov test:

$$D = \sup_{x \in R} |F(x) - F_E(x)|,$$

where  $F_E$  is the empirical cumulative distribution and  $F$  the assumed distribution. Considering that the

Brownian motion hypothesis gives a value of the test  $D = 0.0766$ , then the other three distributional hypotheses present a better approximation. This empirical result is confirmed by the QQ-plot analysis of Figure 1.

Figure 1 reports a QQ-plot among the sample and the Gaussian, NIG and VG distributions (we get similar results with the Meixner distribution). Thus we can see how the empirical and theoretical distributions are closer on the whole real line when we use the NIG or VG distributions to model the log returns.

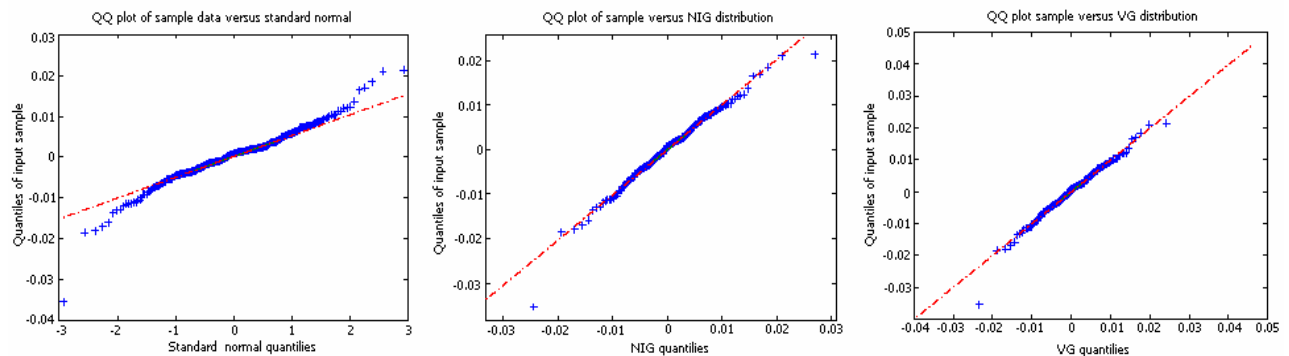


Fig. 1. QQ plot among the sample and the Gaussian, NIG and VG distributions

Here in the following we briefly recall the mean-correcting equivalent martingale measure that we use

in our following empirical analysis (see, among others, Cont and Tankov, 2003; Schoutens, 2003).

Mimicking the Black and Scholes model, the discounted price process  $\tilde{S}_t = \exp(-\int_0^t r(s)ds)S_t$  becomes a martingale if we change the price process  $S_t = S_0 \exp(X_t)$  with  $S_t = S_0 \exp(\int_0^t \mu(s)ds + X_t)$ , where  $\mu(s) = q + r(s) - \log \phi(-i)$  and  $q$  is the translation parameter previously introduced for the three distributions. Therefore, we have to define a new equivalent probability measure  $\tilde{P}$  on  $(\Omega, \mathfrak{F})$  under which the log returns follow the Lévy process  $\{\int_0^t \mu(s)ds + X_t\}$ . In the three processes defined above, we have:

$$\mu^{(NIG)}(s) = r(s) + \delta \left( \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2} \right), \quad (3)$$

$$\mu^{(VG)}(s) = r(s) + \frac{1}{\nu} \log \left( 1 - \theta \nu - \frac{1}{2} \sigma^2 \nu \right), \quad (4)$$

$$\mu^{(Meixner)}(s) = r(s) - 2\delta (\log(\cos(\beta/2))) - \log(\cos((\alpha + \beta)/2)). \quad (5)$$

**2. A lattice method to price Bermudan and European options with Lévy processes**

In this Section, we opportunely adapt to Lévy processes the Markovian methodology proposed by Duan et al. (2003). Since Lévy processes are Markov processes we suggest to use an approximating Markov chain in order to price exotic options when the log return follows a Lévy process. This discretization process provides the same ductility of the binomial model and for this reason it is possible to price almost every path dependent contingent claim once we know the risk neutral distribution of the underlying Markov process.

**2.1. The Markovian approach.** Assume the maturity of the contingent claim is  $T$ . Our task is to approximate, under the risk neutral probability  $\tilde{P}$ , the log price process  $\{\ln(S_t)\}_{0 \leq t \leq T}$  at times  $\{0, \Delta t, 2\Delta t, \dots, s\Delta t = T\}$  by a sequence of Markov chains  $\{\tilde{Y}_{n\Delta t}^{(m)}, n = 0, 1, 2, \dots, s\}_{m=2i+1, i \in \mathbb{N}}$  with state space  $\{p_1, p_2, \dots, p_m\}$  and transition probability matrix  $Q_{(m)} = [q_{ij}]_{1 \leq i, j \leq m}$ , where  $m$  is an odd integer and  $p_{(m+1)/2} = \ln(S_0)$ . In order to fix the ideas, we adopt the mean correcting risk neutral valuation considering the riskless rate  $r(t) = r$  constant. Thus, we build a sequence of Markov chains  $\{\tilde{Y}_{n\Delta t}^{(m)}, n = 0, 1, 2, \dots, s\}_{m=2i+1, i \in \mathbb{N}}$  with state space  $\{p_1, p_2, \dots, p_m\}$ , converging weakly to the risk neutral Lévy process  $\{\ln(S_0) + \mu t + X_t, t = 0, \Delta t, 2\Delta t, \dots, T\}$  (here  $X = (X_t)_{t \geq 0}$  is the log return pro-

cess) as the state number  $m$  of the states tends to infinite, where  $\mu$  is defined (for the three processes introduced in the previous section) by equations (3), (4), (5). Therefore, given the current price  $S_0$ , we define an interval centered in  $\ln(S_0)$  such that the probability that  $\ln(S_T) + \mu T$  belongs to the interval is almost equal to 1, i.e.:

$$P(\ln(S_T) + \mu T \in [\ln(S_0) - I(m), \ln(S_0) + I(m)]) \approx 1,$$

where  $2I(m)$  is the length of the interval  $[\ln(S_0) - I(m), \ln(S_0) + I(m)]$ . The quantity  $I(m)$  depends on the number of the states of the Markov chain since to get the convergence we have to guarantee that  $I(m) \rightarrow \infty$  and  $I(m)/m \rightarrow 0$  as the number of the states converges to infinity ( $m \rightarrow \infty$ ) (see, among others, Pringent, 2002). For example, when the Markov process  $Y = \{\ln(S_t)\}_{0 \leq t \leq T}$  admits finite mean (i.e.,  $E(|\ln(S_{\Delta t})|) < \infty$ ), we can use  $I(m) = \max(|z_{1/m}|, |z_{1-1/m}|)$ , where  $z_k$  is the  $k\%$  quantile of  $\ln(S_T) + \mu T$ . Since  $I(m) \rightarrow \infty$  and  $I(m)/m \rightarrow 0$ , we can guarantee the convergence of the Markov chain sequence. However, the speed of convergence is strictly linked to the choice of  $I(m)$ . Thus, we have to choose opportunely  $I(m)$ . Duan et al. (2003) suggest to use  $I(m) = (2 + \ln(\ln(m)))\sigma\sqrt{T}$  for the Brownian motion. When we assume the mean correcting risk neutral valuation for the three processes introduced in the previous section, we observe an higher speed of convergence using  $I(m) = z + \frac{\log(\log(m))}{2}$ , where

$$p_i = \ln(S_0) + \frac{2i - m - 1}{m - 1} I(m), \quad i = 1, \dots, m.$$

Note that  $p_1 = \ln(S_0) - I(m)$ ,  $p_m = \ln(S_0) + I(m)$  and  $p_{(m+1)/2} = \ln(S_0)$ .

**Remark.** Fixed the ‘ $m$ ’ values  $p_i$ , we can always determine other ‘ $m$ ’ values starting by any other state  $p_k^i = p_i + \frac{2k - m - 1}{m - 1} I(m)$ . In particular, we get the transformation  $p_k^i = p_j$  if and only if  $k = j - i + \frac{m+1}{2}$ , that is:

$$p_k^i = p_i + \frac{2k - m - 1}{m - 1} I(m) = \ln(S_0) + \frac{2(i + k - \frac{m+1}{2}) - m - 1}{m - 1} I(m). \tag{6}$$

The transition probability between the  $i$ -th state and the  $k$ -th state is given by:

$$q_{ik} = P(\ln(S_{\Delta t}) + \mu\Delta t \in (c_k^i, c_{k+1}^i]), \tag{7}$$

where  $c_1^i = p_1^i - \frac{\log(\log(m))}{2}$ ,  $c_k^i = (p_k^i + p_{k-1}^i)/2$ ,

$k = 2, \dots, m$  and  $c_{m+1}^i = p_m^i + \frac{\log(\log(m))}{2}$ . Then

we deduce the convergence of the sequence of Markov chains  $\{\tilde{Y}_{n\Delta t}^{(m)}, n = 0, 1, 2, \dots, s\}_{m=2i+1, i \in \mathbb{N}}$  with state space  $\{p_1, p_2, \dots, p_m\}$ , to the risk neutral Lévy process  $\{\mu t + \ln(S_t), t = 0, \Delta t, 2\Delta t, \dots, T\}$  because

$$c_2^i - c_1^i = c_{m+1}^i - c_m^i = \frac{I(m)}{m - 1} + \frac{\log(\log(m))}{2m} \rightarrow 0,$$

$$\text{as } m \rightarrow \infty \text{ and } c_{k+1}^i - c_k^i = 2\left(\frac{I(m)}{m - 1}\right) \rightarrow 0, \text{ as } m \rightarrow \infty,$$

$$k = 2, m - 1.$$

Since  $p_k^i = p_j$  if and only if  $k = j - i + \frac{m+1}{2}$ , then

we have not to compute all the entries  $q_{ij}$  of the transition matrix  $Q_{(m)}$ . As a matter of fact, if we define  $k(j) = j - i + \frac{m+1}{2}$ ,  $j = 1, \dots, m$ , then the entries of the transition matrix  $Q_{(m)}$  are given by:

$$\text{if } i < \frac{m+1}{2} : q_{ij} = \begin{cases} \sum_{k=1}^{1+\frac{m+1}{2}-i} \int_{c_k^{i-p_i-\mu\Delta t}}^{c_{k+1}^{i-p_i-\mu\Delta t}} f_{X_{\Delta t}}(x) dx, & \text{if } j = 1, \\ \int_{c_{k(j)}^{i-p_i-\mu\Delta t}}^{c_{k(j)+1}^{i-p_i-\mu\Delta t}} f_{X_{\Delta t}}(x) dx, & \text{if } j = 2, \dots, i + \frac{m-1}{2}, \\ 0 & \text{if } j = i + \frac{m+1}{2}, \dots, m, \end{cases} \tag{8}$$

$$\text{if } i > \frac{m+1}{2} : q_{ij} = \begin{cases} 0, & \text{if } j = 1, \dots, i - \frac{m+1}{2}, \\ \int_{c_{k(j)}^{i-p_i-\mu\Delta t}}^{c_{k(j)+1}^{i-p_i-\mu\Delta t}} f_{X_{\Delta t}}(x) dx, & \text{if } j = i - \frac{m-1}{2}, \dots, m - 1, \\ \sum_{k=m-i+\frac{m+1}{2}}^m \int_{c_k^{i-p_i-\mu\Delta t}}^{c_{k+1}^{i-p_i-\mu\Delta t}} f_{X_{\Delta t}}(x) dx, & \text{if } j = m, \end{cases} \tag{9}$$

$$\text{if } i = \frac{m+1}{2} : q_{ij} = \int_{c_j^{i-p_i-\mu\Delta t}}^{c_{j+1}^{i-p_i-\mu\Delta t}} f_{X_{\Delta t}}(x) dx, \quad j = 1, \dots, m, \tag{10}$$

where  $f_{X_{\Delta t}}(\cdot)$  is the density function of the log return Lévy process. When  $m$  increases, the intervals  $(c_k^i, c_{k+1}^i]$  become so small that we can well approximate any integral with the area of only one rectangle, i.e.:

$$\int_{c_k^{i-p_i-\mu\Delta t}}^{c_{k+1}^{i-p_i-\mu\Delta t}} f_{X_{\Delta t}}(x) dx \approx f_{X_{\Delta t}}\left(\frac{c_k + c_{k+1}}{2}\right)(c_{k+1} - c_k). \tag{11}$$

**2.2. Pricing of European contingent claims.** When the maturity of an European contingent claim is  $T$  and we consider  $s$  steps (i.e.,  $s\Delta t = T$ ), then the price of the contingent claims is given by the  $((m + 1)/2)$ -th component of the price vector:

$$V(p, 0) = (Q_{(m)})^s Z, \tag{12}$$

where  $Z$  is the  $m$ -dimensional vector of payoff at the maturity correspondent to the vector of log prices  $p = [p_1, p_2, \dots, p_m]$ . So we can assume that

the payoff vector is given by  $Z = [g_{w,1}, \dots, g_{w,m}]'$  where  $g_{w,i} = \max\{w[\exp(p_i) - K], 0\}$ ,  $w$  is equal to 1 for a call and -1 for a put. Analogously, to the example reported by Duan and Simonato (2001) with the Black and Scholes model, in Table 2 we show the convergence of this methodology under the three different distributional assumptions. In this Table and in all the following ones we use the mean correcting risk neutral measure applied to the parameters estimated in Table 1. This choice is a simplification to the classic methodology that determines the risk neutral measure obtained from the market (as suggested by equations (1) and (2)). Moreover, this choice satisfies the main objectives of our empirical analysis consisting in showing the applicability of the proposed methodology and the convergence to a unique price. Clearly these objectives should be obtained with any assigned parameters. Finally, the chosen parameters refer to the same underlying stock process and thus the obtained option prices should be similar even for different distributional hypothesis.

Table 2. European put option prices under NIG, VG, and Meixner process

States	NIG process		VG process		Meixner process	
	Weekly	Daily	Weekly	Daily	Weekly	Daily
m = 101	1.7428	1.7984	1.6795	1.7489	1.7343	1.8022
m = 501	1.7442	1.7442	1.6809	1.6852	1.7357	1.7357
m = 1001	1.7442	1.7442	1.6810	1.6840	1.7358	1.7358
m = 1501	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m = 2001	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m = 2501	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m = 3001	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358

Table 2 reports European put option prices at the money under NIG, VG, and Meixner processes on a stock price with current value  $S_0 = 100$  euro, maturity  $T = 0.5$  years, short interest rate  $r = 5\%$  a.r. Moreover, we consider that the temporal horizon is shared either in 24 periods or in 126 periods (i.e.,  $\Delta t$  is equal respectively either to one week or to one day). In both cases we observe the convergence of the option prices when the number of the states  $m$  increases. The convergence price is the same we obtain approximating the integral that defines the risk neutral put option price:

$$S_0 \exp(-rT) \int_{-\infty}^0 (1 - e^x) f_{\tilde{X}_T}(x) dx.$$

**2.3. Pricing and hedging of Bermudan contingent claims.** Let us consider an Bermudan option with maturity  $T$  and strike price  $K$ . We assume that the contract may be exercised at times  $\{0, \Delta t, 2\Delta t, \dots, s\Delta t\}$ , where  $T = s\Delta t$  and the pre-determined exercise dates are given every  $\Delta t$ . If  $\Delta t$  is very small the Bermudan option price approximates the American one. For several contingent claims it is sufficient to consider daily exercise dates to get a good approximation of American type options if the log returns are Gaussian distributed. However, when we use non-Gaussian Lévy processes the convergence to the American type price is slow (see, among others, Ribeiro and Webber, 2003; 2004). For this reason in the following we will always refer to Bermudan type options (with daily exercise dates). By fixing the number of states  $m$ ,

$$\Delta = \frac{\partial V_w}{\partial \ln S_0} \frac{1}{S_0} \approx \frac{V_w(p_{\frac{m+1}{2}} + \varepsilon, 0) - V_w(p_{\frac{m+1}{2}} - \varepsilon, 0)}{2\varepsilon} \frac{1}{S_0},$$

$$\Gamma = \frac{\partial}{\partial S_0} \left( \frac{\partial V_w}{\partial \ln S_0} \frac{1}{S_0} \right) \approx \left( \frac{V_w(p_{\frac{m+1}{2}} - \varepsilon, 0) - V_w(p_{\frac{m+1}{2}} + \varepsilon, 0)}{2\varepsilon} + \frac{V_w(p_{\frac{m+1}{2}} + \varepsilon, 0) + V_w(p_{\frac{m+1}{2}} - \varepsilon, 0) - 2V_w(p_{\frac{m+1}{2}}, 0)}{\varepsilon^2} \right) \frac{1}{S_0^2}.$$

Table 3. Delta, gamma and Bermudan put option prices with daily exercise dates under the assumption the log returns follow NIG, VG, Meixner processes and their Monte Carlo valuation

	NIG process		VG process		Meixner process	
	K = 98	K = 102	K = 98	K = 102	K = 98	K = 102
m = 501	1.2419	3.0101	1.2067	2.9527	1.2349	3.0025

we build the vector of the state values  $p = [p_1, p_2, \dots, p_m]$  of an approximating Markov chain  $\{\tilde{Y}_{n\Delta t}^{(m)}, n = 0, 1, 2, \dots, s\}_{m=2i+1, i \in N}$ , with risk neutral transition matrix  $Q_{(m)}$ . Since the states remain the same for all the time steps, then at each time  $\{0, \Delta t, 2\Delta t, \dots, s\Delta t\}$  there is an unique payoff vector:

$$g_w(p, K) = [g_{w,1}, \dots, g_{w,m}]', \tag{13}$$

where  $g_{w,i} = \max\{w[\exp(p_i) - K], 0\}$ ,  $w$  is equal to 1 for a call and -1 for a put. For every couple of vectors  $a = [a_1, \dots, a_m]'$ ,  $b = [b_1, \dots, b_m]'$  we assume the vectorial notation:

$$\max[a, b] := [\max(a_1, b_1), \max(a_2, b_2), \dots, \max(a_m, b_m)]'.$$

Therefore, the price of the Bermudan option can be computed using the recursive vectorial formula:

$$V_w(p, T) = g_w(p, K),$$

$$V_w(p, t_i) = \max\left[g_w(p, K), e^{-r\Delta t} Q_{(m)} V_w(p, t_{i+1})\right] \tag{14}$$

$$i = 0, \dots, s - 1, t_i = i\Delta t, s\Delta t = T.$$

The option price at time 0 is given by the  $((m + 1)/2)$ -th element of  $V_w(p, 0)$ . When we price a contingent claim with the Markovian approach we get the vector  $V_w(p, 0)$  whose elements are option prices corresponding to discrete values of the stock price. Thus, we can compute the Greeks in a way very similar to the finite-difference approach using the option prices adjacent to the  $((m + 1)/2)$ -th element of  $V_w(p, 0)$ . However, as suggested by Duan et al., in order to obtain higher quality Greeks it is advisable to have adjacent prices very close to the initial stock price. This approximation problem can be easily solved considering the states  $p_{\frac{m+1}{2}} + \varepsilon$ , and  $p_{\frac{m+1}{2}} - \varepsilon$  in the Markov chain with  $\varepsilon$  opportunitely small. In this way we can use the following approximation of delta and gamma values:

Table 3 (cont.). Delta, gamma and Bermudan put option prices with daily exercise dates under the assumption the log returns follow NIG, VG, Meixner processes and their Monte Carlo valuation

	NIG process		VG process		Meixner process	
	K = 98	K = 102	K = 98	K = 102	K = 98	K = 102
Delta	-0.2919	-0.5686	-0.2914	-0.5739	-0.2914	-0.5692
Gamma	0.0560	0.0816	0.0572	0.0829	0.0561	0.0820
$m = 1001$	1.2419	3.0101	1.1882	2.9529	1.2349	3.0025
Delta	-0.2919	-0.5686	-0.2881	-0.5732	-0.2914	-0.5692
Gamma	0.0560	0.0816	0.0571	0.0847	0.0561	0.0820
$m = 1501$	1.2419	3.0101	1.1869	2.9509	1.2349	3.0025
Delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
Gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
$m = 2001$	1.2419	3.0101	1.1868	2.9507	1.2349	3.0025
Delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
Gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
$m = 2501$	1.2419	3.0101	1.1868	2.9508	1.2349	3.0025
MC	1.2431	3.0121	1.1891	2.9552	1.2356	3.0047
Delta	-0.2919	-0.5686	-0.2879	-0.5732	0.2914	-0.5692
MC delta	-0.2911	-0.5682	-0.2886	-0.5741	-0.2911	-0.5688
Gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
MC gamma	0.0559	0.0814	0.0574	0.0852	0.0560	0.0818

Consider Bermudan put options with predetermined daily exercise dates and exercise prices  $K = 98$  euro or  $K = 102$  euro under the assumption the log returns follow either a NIG, or a VG, or a Meixner process. Moreover let us consider the mean correcting risk neutral measure applied to the parameters estimated in Table 1 for puts on a stock price with current value  $S_0 = 100$  euro, maturity  $T = 0.5$  years (120 working days), short interest rate  $r = 5\%$  a.r.

In order to simulate Lévy processes recall that any semi-martingale  $X = \{X_t\}_{t \geq 0}$  can be represented as time changed Brownian motion, i.e.:

$$X_t = \mu t + \theta I_t + \sigma W_{I_t}, \quad (15)$$

where  $\{I_t\}$  and  $\{W_t\}$  are respectively: a positive intrinsic time process and, a standard Brownian motion. Lévy processes are particular semi-martingales. In particular, the intrinsic time process  $\{I_t\}$  is: an inverse Gaussian process when  $X = \{X_t\}_{t \geq 0}$  is NIG, a gamma process when  $X$  is variance gamma and it is defined by a proper process (see Madan and Yor, 2008) when  $X$  is a Meixner process. The equation (15) is generally used to simulate Lévy processes using the antithetic variates method as variance reduction technique. In the literature there exist several variance reduction techniques (see, among others, Ribeiro and Webber, 2003; 2004; Kawai, 2008), but in this paper we use only the antithetic variates method. Let us explain the procedure when the underlying asset follows one of the three Lévy processes. A sample path on

time points  $\{k\Delta t : k = 0, 1, \dots, n\}$  can be generated as follows:

- ◆ generate  $n$  independent random numbers  $\{I_k, k = 1, \dots, n\}$  from a the intrinsic time distribution;
- ◆ generate  $n$  independent random numbers  $\{Z_k, k = 1, \dots, n\}$  from a standard normal distribution;
- ◆ a sample path on time points  $\{k\Delta t, k = 0, 1, \dots, n\}$  is given by:

$$X_0 = 0, \quad X_{k\Delta t} = X_{(k-1)\Delta t} + \mu\Delta t + \theta I_k + \sigma\sqrt{I_k}Z_k, \quad k = 1, \dots, n.$$

Now, we can use this sample path to calculate the final payoff of the derivative. Let us denote this value by  $f_1$ . According to the antithetic variates method, we can compute a second value  $f_2$  of the final payoff by using the following sample path:

$$X_0 = 0, \quad X_{k\Delta t} = X_{(k-1)\Delta t} + \mu\Delta t + \theta I_k - \sigma\sqrt{I_k}Z_k, \quad k = 1, \dots, n.$$

Finally, an estimate of the final payoff is given by the mean:

$$\hat{f} = \frac{f_1 + f_2}{2}.$$

Say we have generated  $M$  final payoff  $\hat{f}$ , then the standard error is much smaller than that obtained with  $2M$  standard simulations.

In Table 3 we report the option prices and the values of delta and gamma obtained either with Monte Carlo (MC) simulations or with the Markovian ap-

proximation when we assume  $\varepsilon = 10^{-6}$ . Even in this case we observe the convergence of these values for a number of states  $m$  greater than 500.

Observe that the prices and the Greeks obtained with Meixner and NIG processes are very similar and are always higher than the prices obtained with the VG process. These results confirm the analogous obtained for the European put prices of Table 2. Clearly these differences are essentially due by the different evolution of the processes and by the risk neutral measure used in this analysis. We compare the results using Monte Carlo simulations with variance reduction techniques and we need 5 millions simulations to get similar results to those given by the Markovian approach. The prices difference between the Markovian approach and those obtained by generating Monte Carlo simulations is generally of order  $10^{-3}$ .

### 3. Compound, barrier and lookback option prices with Lévy processes

In this Section we propose to value exotic option prices assuming that a sequence of Markov chains  $\{\tilde{Y}_{n\Delta t}^{(m)}, n=0,1,2,\dots,s\}_{m=2i+1, i \in \mathbb{N}}$  describes the risk neutral behavior of  $\ln(S_t)$  at times  $\{0, \Delta t, 2\Delta t, \dots, s\Delta t = T\}$ . We compute compound, barrier, and lookback option prices under the three distributional assumptions. In particular, the methodology proposed is innovative for compound, and lookback options that have not been dealt by Duan and Simonato (2001) and Duan et al. (2003).

**3.1. Compound options.** Compound options are options written on options and can be of four types: a call on call, a put on call, a call on put, and a put on put. Consider a call on call. At the first maturity  $T_1$  the compound option holder has the right to pay

the first exercise price  $K_1$  and get a call. Then, the call gives to the compound option holder the right to buy the underlying asset at the second maturity  $T_2$  paying the second exercise price  $K_2$ . The Markovian approach allows to price easily compound options. Using the recursive system to price an option with maturity  $T_2 - T_1$  and exercise price  $K_2$ , we find a vector which represents the possible prices at time  $T_1$  of the European option on which the first option is written. Denote this vector as:

$$\tilde{V}_{w_1}(p, T_1) = [\tilde{V}_{w_1,1}, \dots, \tilde{V}_{w_1,m}]', \tag{16}$$

where  $w_1$  is equal to 1 for a call and -1 for a put. The payoff at time  $T_1$  of the compound option is given by the vector

$$V_{w_2}(p, T_1) = \max \{w_2[\tilde{V}_{w_1}(p, T_1) - K_1 \bar{1}], \bar{0}\}, \tag{17}$$

where  $\bar{1}$  and  $\bar{0}$  are respectively vectors of ones and zeros,  $w_2$  is equal to 1 for a call and -1 for a put. Thus, using again the recursive system with  $s$  steps (i.e.,  $s\Delta t = T_1$ ), the price at time 0 of an European option on an European option is given by the  $((m + 1)/2)$ -th element of the vector

$$V_{w_2}(p, 0) = e^{-rT_1} Q_{(m)}^s V_{w_2}(p, T_1).$$

Table 4 and Figure 2 exhibit the prices of compound options obtained under Brownian motion, NIG, VG, and Meixner processes (considering different number of states  $m$ ). Figure 2 shows that the biggest differences are between the prices obtained either with the VG process or with the Brownian motion. Generally the prices obtained with the VG process are lower than those obtained with the Brownian motion as also observed in the previous Tables 2 and 3.

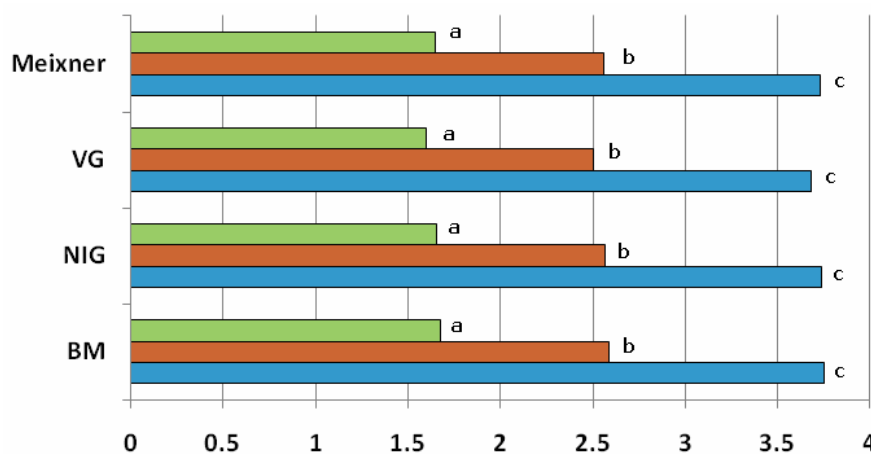


Fig. 2. Compound option prices under Brownian motion, NIG, VG, and Meixner processes when  $K_2$  is respectively 102 (a), 100 (b), 98 (c) and  $K_1 = 2$ .

Table 4 describes the evolution of the estimates prices when we use different number of states  $m$ .

In particular, we compare the results we get under the Brownian motion and those given by Geske's closed formula (see Geske, 1979). These prices concern European calls on European calls, where the current asset price is  $S = 100$ , the first call has strike price  $K_1$  and maturity  $T_1 = 0.25$  years, and the second

call has strike price  $K_2$  and maturity  $T_2 = 0.25$  years. We consider two possible strike prices  $K_1 \cdot (K_1 = 1.5, 2)$  and three possible strike prices  $K_2 \cdot (K_2 = 98, 100, 102)$ . Moreover, the short interest rate is  $r = 5\%$ , the annual volatility of the Brownian motion is  $\sigma = 10.14\%$ , and the parameters of the NIG, Meixner and VG processes are (for simplicity) those ones of Table 1.

Table 4. Compound option prices under Brownian motion, NIG, VG, and Meixner processes.

$K_1 = 2$	Brownian motion			$K_1 = 1.5$	Brownian motion		
	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$		$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
$m = 101$	3.7530	2.5803	1.6764	$m = 101$	4.1629	2.9332	1.9609
$m = 501$	3.7540	2.5851	1.6747	$m = 501$	4.1637	2.9381	1.9598
$m = 1001$	3.7542	2.5851	1.6747	$m = 1001$	4.1637	2.9385	1.9598
$m = 1501$	3.7542	2.5852	1.6746	$m = 1501$	4.1637	2.9386	1.9598
$m = 2001$	3.7542	2.5852	1.6747	$m = 2001$	4.1637	2.9386	1.9597
Geske	3.7542	2.5852	1.6747	Geske	4.1637	2.9386	1.9597
$K_1 = 2$	NIG process			$K_1 = 1.5$	NIG process		
	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$		$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
$m = 101$	3.7380	2.5584	1.6607	$m = 101$	4.1479	2.9127	1.9438
$m = 501$	3.7360	2.5655	1.6577	$m = 501$	4.1459	2.9189	1.9415
$m = 1001$	3.7359	2.5660	1.6574	$m = 1001$	4.1459	2.9190	1.9413
$m = 1501$	3.7359	2.5660	1.6275	$m = 1501$	4.1459	2.9191	1.9414
$m = 2001$	3.7359	2.5660	1.6575	$m = 2001$	4.1458	2.9191	1.9414
$K_1 = 2$	Meixner process			$K_1 = 1.5$	Meixner process		
	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$		$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
$m = 101$	3.7304	2.5519	1.6552	$m = 101$	4.1394	2.9065	1.9365
$m = 501$	3.7289	2.5578	1.6494	$m = 501$	4.1389	2.9107	1.9330
$m = 1001$	3.7288	2.5578	1.6496	$m = 1001$	4.1388	2.9108	1.9329
$m = 1501$	3.7287	2.5580	1.6495	$m = 1501$	4.1388	2.9110	1.9329
$m = 2001$	3.7287	2.5580	1.6495	$m = 2001$	4.1387	2.9110	1.9330
$K_1 = 2$	VG process			$K_1 = 1.5$	VG process		
	$K_2 = 98$	$K_2 = 100$	$K_2 = 102$		$K_2 = 98$	$K_2 = 100$	$K_2 = 102$
$m = 101$	3.6634	2.4874	1.5795	$m = 101$	4.0738	2.8397	1.8610
$m = 501$	3.6800	2.5043	1.5965	$m = 501$	4.0904	2.8564	1.8776
$m = 1001$	3.6805	2.5048	1.5971	$m = 1001$	4.0909	2.8570	1.8781
$m = 1501$	3.6806	2.5049	1.5971	$m = 1501$	4.0910	2.8571	1.8782
$m = 2001$	3.6807	2.5050	1.5972	$m = 2001$	4.0911	2.8571	1.8783

Notes: We consider European calls on European calls, where the current asset price is  $S = 100$ , the first call has strike price  $K_1$  and maturity  $T_1 = 0.25$  year, and the second call has strike price  $K_2$  and maturity  $T_2 = 0.25$  years.

**3.2. Barrier options.** Barrier options may be of two types, knock-out and knock-in. We proceed explaining how to use the Markovian approach to price knock-out options and we refer to Duan et al. (2003) for knock-in options. An option is said knock-out when it becomes worthless if the underlying asset touches or crosses a constant barrier  $H$  at any monitoring time. The barrier can be lower or upper (i.e.,  $H$  or  $H^*$ ). A barrier option is double when

there are two barriers and the underlying asset must remain between these two barriers at the monitoring days. Following Duan et al. (2003), we introduce an auxiliary variable  $a_t$  which takes the value 1 if the barrier condition is triggered before or at time  $t$ , and the value 0 otherwise. If we denote with  $v(p_i, t; a_t)$  the option price at time  $t$ , for a knock-out option we have:

$$\begin{aligned}
 &\text{for every time } v_w(p_i, t_k; a_{t_k} = 1) = 0, \\
 &\text{if } t_s = s\Delta t = T, v_w(p_i, T; a_T = 0) = \max\{w[\exp(p_i) - K], 0\}, \\
 &\text{if } t_k = k\Delta t, k = 0, s - 1,
 \end{aligned}$$

$v_w(p_i, t_k; a_{t_k} = 0) = \max [g_w(p_i, K, a_{t_k} = 0), e^{-r\Delta t} \sum_{j=1}^m \tilde{P}(X_{t_{k+1}} = p_j, a_{t_{k+1}} = 0 | X_{t_k} = p_i, a_{t_k} = 0) v(p_j, t_{k+1}; a_{t_{k+1}} = 0)]$ ,  
 where  $w$  is equal to 1 for a call and -1 for a put, and

$$g_w(p_i, K, a_{t_k} = 0) = \begin{cases} \max\{w[\exp(p_i) - K], 0\} & \text{if Bermudan} \\ 0 & \text{if European.} \end{cases}$$

To compute the transition probability, we define the set of the states for which the option is knocked out and becomes worthless:

$$\Lambda = \begin{cases} \{i \in \{1, \dots, m\} : \exp(p_i) \leq H\} & \text{down - and - out option,} \\ \{i \in \{1, \dots, m\} : \exp(p_i) \geq H^*\} & \text{up - and - out option,} \\ \{i \in \{1, \dots, m\} : \exp(p_i) \leq H \text{ or } \exp(p_i) \geq H^*\} & \text{double option.} \end{cases}$$

When the states  $p_i$  and  $p_j$  do not belong to  $\Lambda$ , the conditional probabilities are the same of the matrix  $Q_{(m)} = [q_{ij}]$  as described in the previous section, otherwise they are equal to zero. Therefore, the probability to transit from state  $p_i$  to state  $p_j$  are given by:

$$\pi_{ij} = \tilde{P}\{X_{t_{i+1}} = p_j, a_{t_{i+1}} = 0 | X_{t_i} = p_i, a_{t_i} = 0\} = \begin{cases} q_{ij} & \text{if } i \in \Lambda^c \text{ and } j \in \Lambda^c, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\Lambda^c$  is the complement of  $\Lambda$ . Therefore the matrixes that define the conditional probabilities (that we call quasi-transition probabilities matrices) for the down-and-out, up-and-out, and double barrier-out options are respectively given by:

$$\Pi_{DO} = \begin{bmatrix} 0_{k-1, k-1} & 0_{k-1, m-k+1} \\ 0_{m-k+1, k-1} & Q(k, m; k, m) \end{bmatrix},$$

$$V_w(p, t_k, a_{t_k} = 0) = [v_w(p_1, t_k, a_{t_k} = 0), \dots, v_w(p_m, t_k, a_{t_k} = 0)] = \max [g_w(p, K, a_{t_k} = 0), e^{-r\Delta t} \Pi V_w(p, t_{k+1}; a_{t_{k+1}} = 0)], \quad (19)$$

where  $g_w(p, K, a_{t_k} = 0) = [g_w(p_1, K, a_{t_k} = 0), \dots, g_w(p_m, K, a_{t_k} = 0)]'$ ,

and  $\Pi$  is either  $\Pi_{DO}$ , or  $\Pi_{UO}$ , or  $\Pi_{DBO}$ , depending on the nature of the knock-out option. The knock-out option price at time 0 is given by the  $((m + 1) / 2)$ -th element of  $V_w(p, 0; a_0 = 0)$ . Barrier

$$\Pi_{UO} = \begin{bmatrix} Q(1, l; 1, l) & 0_{l, m-l} \\ 0_{m-l, l} & 0_{m-l, m-l} \end{bmatrix},$$

$$\Pi_{DBO} = \begin{bmatrix} 0_{k-1, k-1} & 0_{k-1, l-k+1} & 0_{k-1, m-l} \\ 0_{l-k+1, k-1} & Q(k, l; k, l) & 0_{l-k+1, m-l} \\ 0_{m-l, k-1} & 0_{m-l, l-k+1} & 0_{m-l, m-l} \end{bmatrix}.$$

where  $k$  is the index number of the log price located immediately above the lower barrier  $H$ ,  $l$  is the index number of the price located immediately below the upper barrier  $H^*$ ,  $0_{i,j}$  is an  $i \times j$  matrix of zeros, and  $Q(i,j; k,l)$  is the sub-matrix of  $Q_{(m)}$  taken from rows  $i$  to  $j$  and from columns  $k$  to  $l$  inclusively. Thus the knock-out option price with maturity  $T$  and strike price  $K$  can be computed using the recursive vectorial formula:

$$V_w(p, T; a_T = 0) = [v_w(p_1, T; a_T = 0), \dots, v_w(p_m, T; a_T = 0)] \quad (18)$$

and for  $t_k = k\Delta t, k = 0, s - 1$ ,

option prices are very sensitive to the position between discrete asset prices and barrier value. Thus, to reduce this effect it is important to define the cells of the Markovian approach so that the barrier value correspond exactly to a cell's border.

Table 5. European barrier option prices under NIG, VIG, and Meixner processes. The current asset price, the short interest rate and the maturity are respectively  $S = 100, r = 5\%$  and  $T = 0.5$

Strike price	European down-out call options under NIG process				European down-out call options under VG process				European down-out call options under Meixner process			
	Weekly		Daily		Weekly		Daily		Weekly		Daily	
K = 100	H = 94	H = 98	H = 94	H = 98	H = 94	H = 98	H = 94	H = 98	H = 94	H = 98	H = 94	H = 98
m = 501	4.1358	3.1026	4.1059	2.8162	4.0826	3.0813	4.0536	2.7955	4.1288	3.0986	4.0993	2.8123
m = 1001	4.1359	3.1033	4.1059	2.8183	4.0825	3.0820	4.0625	2.8071	4.1288	3.0993	4.0993	2.8145
m = 1501	4.1359	3.1031	4.1058	2.8177	4.0825	3.0812	4.0553	2.7991	4.1288	3.0991	4.0991	2.8139
m = 2001	4.1359	3.1029	4.1059	2.8171	4.0825	3.0815	4.0544	2.7996	4.1288	3.0989	4.0990	2.8132
m = 2501	4.1359	3.1028	4.1059	2.8168	4.0825	3.0813	4.0546	2.7991	4.1288	3.0988	4.0991	2.8129

Table 5 (cont.). European barrier option prices under NIG, VG, and Meixner processes. The current asset price, the short interest rate and the maturity are respectively  $S = 100$ ,  $r = 5\%$  and  $T = 0.5$ 

Strike price	European up-out call options under NIG process				European up-out call options under VG process				European up-out call options under Meixner process			
	Weekly		Daily		Weekly		Daily		Weekly		Daily	
K = 100	H = 102	H = 106	H = 102	H = 106	H = 102	H = 106	H = 102	H = 106	H = 102	H = 106	H = 102	H = 106
m = 501	1.1594	4.1648	0.9289	3.8133	1.1844	4.2817	0.9680	4.0230	1.1610	4.1780	0.9301	3.8265
m = 1001	1.1563	4.1616	0.9203	3.8025	1.1847	4.2820	0.9439	3.9200	1.1579	4.1730	0.9210	3.8096
m = 1501	1.1568	4.1607	0.9217	3.7997	1.1847	4.2818	0.9420	3.9126	1.1579	4.1735	0.9210	3.8115
m = 2001	1.1565	4.1607	0.9206	3.7996	1.1849	4.2820	0.9425	3.9123	1.1580	4.1730	0.9214	3.8099
m = 2501	1.1564	4.1604	0.9204	3.7995	1.1847	4.2818	0.9420	3.9119	1.1579	4.1732	0.9211	3.8103

Table 6. Bermudan down-out and up-out put option prices, where both early exercise and monitoring are on daily basis under NIG, VG, and Meixner processes.

Strike price	Bermudan down-out put with daily monitoring		Bermudan down-out put with daily monitoring		Bermudan down-out put with daily monitoring	
	NIG		VG		Meixner	
K = 101	H = 96	H = 99	H = 96	H = 99	H = 96	H = 99
m = 501	2.2453	1.1477	2.2568	1.1579	2.2496	1.1452
m = 1001	2.2453	1.1462	2.2394	1.1540	2.2496	1.1438
m = 1501	2.2454	1.1459	2.2382	1.1535	2.2497	1.1436
m = 2001	2.2454	1.1458	2.2380	1.1534	2.2497	1.1434
m = 2501	2.2454	1.1455	2.2380	1.1533	2.2498	1.1432
Strike price	Bermudan up-out put with daily monitoring		Bermudan up-out put with daily monitoring		Bermudan up-out put with daily monitoring	
	NIG		VG		Meixner	
K = 101	H = 101	H = 104	H = 101	H = 104	H = 101	H = 104
m = 501	1.1425	2.0802	1.1174	2.0635	1.1302	2.0747
m = 1001	1.1334	2.0800	1.1165	2.0417	1.1308	2.0744
m = 1501	1.1335	2.0793	1.1164	2.0407	1.1309	2.0736
m = 2001	1.1341	2.0793	1.1164	2.0405	1.1316	2.0736
m = 2501	1.1337	2.0795	1.1165	2.0404	1.1312	2.0737

Notes: The current asset price, the short interest rate and the maturity are respectively  $S = 100$ ,  $r = 5\%$  and  $T = 0.5$ .

Table 5 exhibits European barrier option prices. We consider two possible strike prices  $K = 100$  and  $K = 90$  for different fixed barriers and different distributional assumptions (NIG, VG, and Meixner). Even for this Table we assume that the temporal horizon is shared either in 24 periods or in 126 periods (i.e.,  $\Delta t$  is equal respectively either to one week or to one day). These prices refer to European down-out and up-out call options on a stock price with current value  $S_0 = 100$  euro, maturity  $T = 0.5$  years, short interest rate  $r = 5\%$  a.r. We also compare some of these results for European barrier options with those obtained with Monte Carlo simulations with variance reduction techniques. In this case we need more than 10 millions simulations to get the same results we get with the Markovian approach. In particular, the Monte Carlo approximation appears more time consuming for the VG process. Similarly, Table 6 displays Bermudan barrier option prices with daily exercise dates on a stock with the same current asset price, short interest rate and maturity. In this case, we consider Bermudan down-out and up-out put option prices assuming a strike price  $K = 101$  and that the early exercise and the monitoring are on daily basis. As for Bermudan and European vanilla

options Tables 5 and 6 show a good tendency towards a specific price when we increase the number of states of the Markov chain.

**3.3. Lookback options.** Lookback options belong to the class of path-dependent options and can be of two types, fixed and floating strike price. In the case of European fixed strike lookback option, the strike price is fixed at purchase but the option is not exercised at the market price. For a call, the option holder can look back over the life of the option and choose the highest price of the underlying asset, whereas, for a put, the option holder can choose the lowest price. Thus, the fixed lookback option is exercised at the selected market price against the fixed strike. If American, the right of the option holder is extended to the whole time to maturity. In the case of European floating strike lookback option, the strike price is fixed at maturity. For a call, the strike price is fixed at the lowest price reached by the underlying asset during the life of the option, whereas, for a put, it is fixed at the highest price. At maturity the floating lookback option is exercised at the market price against the floating strike. If American the lookback option can be exercised at any time during its life.

In this Section we explain how to price and hedge such types of contracts by the lattice built in the previous section. Given the time interval  $[0, T]$ , where  $T$  is the temporal horizon, we remember that  $[0, T]$  is discretized by the set of times  $\{0, \Delta t, 2\Delta t, \dots, s\Delta t = T\}$  and that the log-price process  $\{\ln(S_{n\Delta t}), n = 0, 1, 2, \dots, s\}$  is approximated, under the mean-correcting martingale measure  $\tilde{P}$ , by a Markov chain  $\{Y_{n\Delta t}, n = 0, 1, 2, \dots, s\}$  with state space  $\{p_1, p_2, \dots, p_m\}$  and transition probability matrix  $Q = [q_{i,j}]_{1 \leq i, j \leq m}$ .

Consider an European floating strike lookback put, then the payoff at maturity  $T$  is given by:

$$M_T - S_T,$$

where  $M_T = \max\{S_{n\Delta t} : n = 0, 1, 2, \dots, s\}$ . The evolution of the price process  $\{S_{n\Delta t} : n = 0, 1, 2, \dots, s\}$  is approximated under  $\tilde{P}$  by the Markov chain  $\{\hat{Y}_{n\Delta t} : n = 0, 1, 2, \dots, s\}$  with state space  $\{\hat{y}_i = \exp(p_i) : i = 1, \dots, m\}$  and transition probability matrix  $Q = [q_{i,j}]_{1 \leq i, j \leq m}$ . We define the function  $Z_n^p(h, w)$ ,  $h, w = 1, \dots, m$ , and  $n = 0, 1, 2, \dots, s$ , as the value at time  $n\Delta t$  of the contingent claim with final payoff  $M_T - S_T$ , when the current asset price is equal to  $\hat{y}_w$  and the maximum asset price up to time  $(n-1)\Delta t$  has been  $\hat{y}_h$ . Therefore, at time  $s\Delta t = T$  we consider the final payoff matrix:

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ Z_s^p(2,1) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Z_s^p(m,1) & Z_s^p(m,2) & \dots & 0 \end{bmatrix},$$

where, by definition,  $Z_n^p(h, w) = \max(\hat{y}_h - \hat{y}_w, 0)$ . According to the risk-neutral pricing, at time  $(s-1)\Delta t$  we have:

$$Z_{s-1}^p(h, w) = \sum_{j=1}^m q_{wj} Z_s^p(h, j) e^{-r\Delta t}, \text{ if } h > w, \quad (20)$$

$$Z_{s-1}^p(h, w) = \sum_{j=1}^m q_{wj} Z_s^p(w, j) e^{-r\Delta t}, \text{ if } h \leq w. \quad (21)$$

Equations (20) and (21) have a quite immediate explanation:  $q_{wj}$  is just the probability to move from the state  $\hat{y}_w$  to the state  $\hat{y}_j$ ; on the right-hand

of (20) we have  $Z_s^p(h, j)$  because  $\hat{y}_h > \hat{y}_w$  and thus the maximum at time  $(s-1)\Delta t$  is  $\hat{y}_h$ , whereas on the right-hand of (21) we have  $Z_s^p(w, j)$  because  $\hat{y}_h \leq \hat{y}_w$  and the maximum is  $\hat{y}_w$ ;  $e^{-r\Delta t}$  is just the discount factor. Iterating the procedure, at time  $n\Delta t$  we obtain:

$$Z_n^p(h, w) = \sum_{j=1}^m q_{wj} Z_{n+1}^p(\max(h, w), j) e^{-r\Delta t}. \quad (22)$$

After  $s$  backward steps we have a matrix whose element  $Z_0^p(h, w)$  is the value at time 0 of the contingent claim with payoff  $M_T - S_T$ , when the current asset price is  $\hat{y}_w$  and the maximum before time 0 has been  $\hat{y}_h$ . Therefore, the price of the contingent claim is given by any value  $Z_0^p(h, \frac{m+1}{2})$  with  $h \leq \frac{m+1}{2}$ . Bermudan (American) style options can be priced by:

$$Z_n^p(h, w) = \max \left\{ \sum_{j=1}^m q_{wj} Z_{n+1}^p(\max(h, w), j) e^{-r\Delta t}, \hat{y}_h - \hat{y}_w \right\}$$

for  $n = 0, 1, \dots, s-1$ ,

and then taking the element  $Z_0^p(h, \frac{m+1}{2})$  with  $h \leq \frac{m+1}{2}$ .

**Example.** Let us describe better the method showing a simple numerical example. Assume that we have only three times,  $t = 0, 1, 2$ , that is, the maturity of the European lookback put is  $T = 2$  and  $\Delta t = 1$ . Then, the current asset price is  $S_0 = 100$  and its evolution is described by the Markov chain  $\{\hat{Y}_n : n = 0, 1, 2\}$  with state vector and transition matrix given respectively by:

$$\hat{y} = \begin{bmatrix} 97 \\ 98 \\ 100 \\ 102 \\ 103 \end{bmatrix}; \quad Q = \begin{bmatrix} 2/5 & 3/10 & 1/5 & 1/10 & 0 \\ 1/5 & 2/5 & 1/5 & 3/20 & 1/20 \\ 1/10 & 1/5 & 2/5 & 1/5 & 1/10 \\ 1/20 & 3/20 & 1/5 & 2/5 & 1/5 \\ 0 & 1/10 & 1/5 & 3/10 & 2/5 \end{bmatrix}.$$

The function  $Z_2^p(h, w)$  is given by:

$$Z_2^p(h, w) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 5 & 4 & 2 & 0 & 0 \\ 6 & 5 & 3 & 1 & 0 \end{bmatrix},$$

where  $Z_2^p(h, w) = \max\{\hat{y}_h - \hat{y}_w, 0\}$ ,  $h, w = 1, \dots, 5$ ,  $\hat{y}_h$  and  $\hat{y}_w$  are the  $h$ -th and  $w$ -th element of the state vector  $\hat{y}$ . Now, assuming that the short interest rate is  $r = 5\%$  and using equation (22), we obtain:

$$Z_1^p(h, w) = \begin{bmatrix} 0 & 0.1902 & 0.6659 & 1.1890 & 1.3317 \\ 0.3805 & 0.1902 & 0.6659 & 1.1890 & 1.3317 \\ 1.7122 & 1.3317 & 0.6659 & 1.1890 & 1.3317 \\ 3.4244 & 2.8537 & 1.9976 & 1.1890 & 1.3317 \\ 4.3757 & 3.7574 & 2.8537 & 1.9500 & 1.3317 \end{bmatrix},$$

and

$$Z_0^p(h, w) = \begin{bmatrix} 0.2941 & 0.5044 & 1.0225 & 1.6559 & 1.9635 \\ 0.4388 & 0.5044 & 1.0225 & 1.6559 & 1.9635 \\ 1.2713 & 1.9121 & 1.0225 & 1.6559 & 1.9635 \\ 2.6105 & 2.3503 & 1.9816 & 1.6559 & 1.9635 \\ 3.4655 & 3.1466 & 2.7145 & 2.2825 & 1.9635 \end{bmatrix}.$$

Then, in this simple example, we have that the price of the lookback put is given by  $Z_0^p(h, \frac{m+1}{2}) = 1.0255$  for  $h \leq \frac{m+1}{2}$  (i.e.,  $Z_0^p(1, 3) =$

$$\Gamma \approx \left( \frac{Z_0^p(\frac{m+3}{2}, \frac{m+3}{2}) + Z_0^p(\frac{m-1}{2}, \frac{m-1}{2}) - 2Z_0^p(\frac{m+1}{2}, \frac{m+1}{2})}{\varepsilon^2} - \frac{Z_0^p(\frac{m+3}{2}, \frac{m+3}{2}) - Z_0^p(\frac{m-1}{2}, \frac{m-1}{2})}{2\varepsilon} \right) \frac{1}{S_0^2}, \quad (24)$$

where  $\varepsilon = P_{(m+1)/2} - P_{(m-1)/2} = P_{(m+3)/2} - P_{(m+1)/2} =$   
 $= \frac{2 \times I}{m-1}.$

In the case of European floating strike lookback call, we have at maturity  $T$  the payoff  $S_T - M_T$ , where  $M_T = \min\{S_{n\Delta t} : n = 0, 1, \dots, s\}$ .

Then, we define the function  $Z_n^c(h, w)$ ,  $h, w = 1, \dots, m$ , and  $n = 0, 1, \dots, s$ , as the value at time  $n\Delta t$  of the contingent claim with final payoff  $S_T - M_T$ , when the current asset price is equal to  $\hat{y}_w$  and the minimum asset price up to time  $(n-1)\Delta t$  has been  $\hat{y}_h$ .

In this case our final payoff matrix becomes:

$$\begin{bmatrix} 0 & Z_s^c(1, 2) & \dots & Z_s^c(1, m) \\ 0 & 0 & \dots & Z_s^c(2, m) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

where  $Z_s^c(h, w) = \max\{\hat{y}_w - \hat{y}_h, 0\}$ , and, with an argument similar to one for floating lookback put options, at time  $n\Delta t$  we have:

$$Z_n^c(h, w) = \sum_{j=1}^m q_{wj} Z_{n+1}^c(\min(h, w), j) e^{-r\Delta t}.$$

$= Z_0^p(2, 3) = Z_0^p(3, 3) = 1,0255$ ). In this example we have done a little abuse of notation, that is, we have used  $Z_n^p(h, w)$  to denote whether the same function  $Z_n^p(h, w)$  or the matrix of the values of  $Z_n^p(h, w)$ .

As the finite-difference approach, where the final output is a vector of option prices corresponding to discrete values of the asset price, with the lattice scheme above we also obtain a vector of option prices corresponding to asset prices. This vector is exactly given by the principal diagonal of the matrix  $Z_0^p(h, w)$  (we are continuing with our abuse of notation). Then, the Greek letters delta and gamma can be computed from the option prices adjacent to  $Z_0^p(\frac{m+1}{2}, \frac{m+1}{2})$  along the principal diagonal. Indeed, remembering that the price partition is constructed on the logarithmic asset price, the following equations can be used to approximate the delta and gamma (see Duan et al., 2003):

$$\Delta \approx \frac{Z_0^p(\frac{m+3}{2}, \frac{m+3}{2}) - Z_0^p(\frac{m-1}{2}, \frac{m-1}{2})}{2\varepsilon} \frac{1}{S_0}, \quad (23)$$

After  $s$  backward steps we obtain a matrix whose element  $Z_0^c(h, w)$  represents the value at time 0 of the contingent claim with final payoff  $S_T - M_T$ , when the current asset price is  $\hat{y}_w$  and the minimum before time 0 has been  $\hat{y}_h$ . Then, the price of the contingent claim is given by  $Z_0^c(h, \frac{m+1}{2})$  with  $h \geq \frac{m+1}{2}$ . Bermudan floating lookback call options can be priced by the equation:

$$Z_n^c(h, w) = \max \left\{ \sum_{j=1}^m q_{wj} Z_{n+1}^c(\min(h, w), j) e^{-r\Delta t}, \hat{y}_w - \hat{y}_h \right\},$$

for  $n = 0, 1, \dots, s-1$ , and then taking the element  $Z_0^c(h, \frac{m+1}{2})$  with  $h \geq \frac{m+1}{2}$ .

Let us now study the pricing and hedging of fixed strike lookback options. We begin with the case of fixed strike lookback put options which have at maturity  $T$  the payoff  $K - M_T$ , where  $M_t = \min\{S_{n\Delta t} : n = 0, 1, \dots, s\}$  and  $K$  is the fixed strike price. Then, we define the function  $W_n^p(h, w)$ ,  $h, w = 1, \dots, m$ , and  $n = 0, 1, \dots, s$ , as the value at time  $n\Delta t$  of the contingent claim with final payoff  $K - M_T$ , when the current asset price is  $\hat{y}_w$  and the minimum asset price up to  $(n-1)\Delta t$  has been  $\hat{y}_h$ .

The final payoff matrix is:

$$\begin{bmatrix} W_s^P(1,1) & W_s^P(1,2) & \cdots & W_s^P(1,m) \\ W_s^P(2,1) & W_s^P(2,2) & \cdots & W_s^P(2,m) \\ \vdots & \vdots & \ddots & \vdots \\ W_s^P(m,1) & W_s^P(m,2) & \cdots & W_s^P(m,m) \end{bmatrix},$$

where, by definition,

$$W_s^P(h, w) = \max(K - \hat{y}_{\min(h,w)}, 0).$$

In order to compute at time  $n\Delta t$ ,  $n = 0, 1, \dots, s - 1$ , the vales of the function  $W_n^P(h, w)$  we can use the recursive equation:

$$W_n^P(h, w) = \max\left(\sum_{j=1}^m q_{wj} W_{n+1}^P(\min(h, w), j) e^{-r\Delta t}, K - \hat{y}_{\min(h,w)}\right) \quad n = 0, 1, \dots, s - 1,$$

and the price at time 0 is given by  $W_0^P(h, \frac{m+1}{2})$  with  $h \geq \frac{m+1}{2}$ .

European fixed strike lookback call options have at maturity  $T$  the payoff  $M_T - K$ , where  $M_T = \max\{S_{n\Delta t} : n = 0, 1, \dots, s\}$ , and  $K$  is the fixed strike price. For this type of options we define the function  $W_n^C(h, w)$ ,  $h, w = 1, \dots, m$ , and  $n = 0, 1, \dots, s$ , as the value at time  $n\Delta t$  of the contingent claim with final payoff  $M_T - K$ , when the current asset price is  $\hat{y}_w$  and the maximum asset price up to  $(n - 1)\Delta t$  has been  $\hat{y}_h$ . Then, at maturity  $T$  our payoff matrix is:

$$\begin{bmatrix} W_s^C(1,1) & W_s^C(1,2) & \cdots & W_s^C(1,m) \\ W_s^C(2,1) & W_s^C(2,2) & \cdots & W_s^C(2,m) \\ \vdots & \vdots & \ddots & \vdots \\ W_s^C(m,1) & W_s^C(m,2) & \cdots & W_s^C(m,m) \end{bmatrix},$$

$$W_n^C(h, w) = \max\left(\sum_{j=1}^m q_{wj} W_{n+1}^C(\max(h, w), j) e^{-r\Delta t}, \hat{y}_{\max(h,w)} - K\right) \quad n = 0, 1, \dots, s - 1,$$

and the price at time 0 is given by  $W_0^C(h, \frac{m+1}{2})$  with  $h \leq \frac{m+1}{2}$ .

Observe that the equations (23) and (24), which return

$$W_n^P(h, w) = \sum_{j=1}^m q_{wj} W_{n+1}^P(\min(h, w), j) e^{-r\Delta t}.$$

Finally, after  $s$  backward steps, we obtain a matrix whose element  $W_0^P(h, w)$  represents the value at time 0 of the contingent claim with final payoff  $K - M_T$ , when the current asset price is  $\hat{y}_w$  and the minimum before time 0 has been  $\hat{y}_h$ . Then, the price at time 0 is given by  $W_0^P(h, \frac{m+1}{2})$  with  $h \geq \frac{m+1}{2}$ . For Bermudan (American) fixed lookback put options, we have:

$$W_s^P(h, w) = \max(K - \hat{y}_{\min(h,w)}, 0),$$

where  $W_s^C(h, w) = \max(\hat{y}_{\max(h,w)} - K, 0)$ . In this case we can compute the function  $W_n^C(h, w)$  by the recursive formula

$$W_n^C(h, w) = \sum_{j=1}^m q_{wj} W_{n+1}^C(\max(h, w), j) e^{-r\Delta t}, \quad n = 0, 1, \dots, s - 1,$$

and, at time 0,  $W_0^C(h, w)$  represents the value of the contingent claim with final payoff  $M_T - K$ , when the current asset price is  $\hat{y}_w$  and the maximum before 0 has been  $\hat{y}_h$ . Then, the price of the fixed lookback call is given by  $W_0^C(h, \frac{m+1}{2})$  with  $h \leq \frac{m+1}{2}$ . In the case of Bermudan fixed strike lookback call options, we have:

$$W_s^C(h, w) = \max(\hat{y}_{\max(h,w)} - K, 0)$$

delta and gamma values, continue to be valid not only for floating strike lookback put options, but even for all other types of lookback options, European and Bermudan (American).

Table 7. European and Bermudan (early exercise on daily basis) floating strike lookback put option prices, where monitoring is on daily and weekly basis using NIG, VG and Meixner processes

	European lookback put							
	Brownian motion		NIG process		VG process		Meixner process	
	Weekly	Daily	Weekly	Daily	Weekly	Daily	Weekly	Daily
m = 501	2.7121	3.1344	2.6680	3.0511	2.5998	3.0058	2.6605	3.0439
m = 801	2.7125	3.1355	2.6683	3.0524	2.6000	2.9866	2.6609	3.0452
m = 1001	2.7126	3.1358	2.6684	3.0528	2.6001	2.9843	2.6610	3.0456
m = 1501	2.7127	3.1361	2.6685	3.0531	2.6002	2.9832	2.6611	3.0459

Table 7 (cont.). European and Bermudan (early exercise on daily basis) floating strike lookback put option prices, where monitoring is on daily and weekly basis using NIG, VG and Meixner processes

	Bermudan lookback put							
	Brownian Motion		NIG process		VG process		Meixner process	
	Weekly	Daily	Weekly	Daily	Weekly	Daily	Weekly	Daily
m = 501	2.8587	3.2919	2.8176	3.2253	2.7528	3.1780	2.8113	3.2195
m = 801	2.8695	3.3216	2.8180	3.2266	2.7532	3.1646	2.8117	3.2209
m = 1001	2.8696	3.3218	2.8181	3.2269	2.7533	3.1630	2.8118	3.2212
m = 1501	2.8697	3.3221	2.8182	3.2273	2.7534	3.1624	2.8119	3.2215

In Table 7 we show the prices of European and Bermudan floating strike lookback put options, based on daily and weekly monitoring under the Brownian motion, NIG, VG and Meixner processes. The current asset price, the short interest rate and the maturity are respectively  $S = 100$ ,  $r = 5\%$  and  $T = 0.25$ . We compare the results for the European put with NIG and the VG processes (for all the processes we use the same parameters of Table 1). We compare part of these results using Monte Carlo simulations with variance reduction techniques. Even in this case we need more than 5 millions simulations to get the similar results for pricing European floating strike lookback put options.

In order to value the differences among computational times of the different option valuations we propose to compute the average times and the root mean squared errors (RMSE) of the times needed to compute the above European Bermudan lookback put options. In particular we repeat the 100 times the computation of the values with the Markov and the Monte Carlo simulation and then we compute the average times and the RMSE of these times. Table 8 reports the main differences observed among the average computational times of European floating strike lookback put option prices. From the comparison it appears evident the better performance of the Markovian approach in terms of computational time.

Table 8. Average computational times for European Bermudan lookback put options obtained with Markovian (m = 801 states) and Monte Carlo valuation

	NIG		VG		Meixner	
	Daily	Weekly	Daily	Weekly	Daily	Weekly
Markov average time	977 sec	186 sec	1445 sec	275 sec	1116 sec	212 sec
Markov RMSE in %	0.0004	0.00001	0.007	0.00067	0.0009	0.00007
MC average time	1140 sec	217 sec	1693 sec	322 sec	1398 sec	266 sec
MC RMSE in %	0.014	0.003	0.103	0.0097	0.09	0.007

## Conclusions

The paper shows the simplicity of the Markovian approach to price vanilla options and some types of exotic options when the log return follows a Lévy process. Clearly, we couldn't be exhaustive since this approach can be used to price many other Markovian processes and exotic options. In particular, the discretization process with Markov chains permits to price

path dependent options once we are able to approximate the risk neutral distribution of the underlying Markovian log return process. Moreover, we believe that further possible applications and extensions of the Markovian approach should be discussed for GARCH(1,1) processes with infinitely divisible distributions of the residuals, stochastic volatility Lévy processes and subordinated Lévy processes.

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