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The pricing of structured notes with credit risk

Abstract

It is inappropriate to ignore counterparty risk when pricing structured products, especially after the financial tsunami that occurred in 2008. Motivated by these circumstances, we developed an exogenous model that embeds the concept of Moody's KMV model for evaluating the issuer's credit risk premium under the framework of American binary put option. We select equity-linked structured notes to illustrate that the model is applicable to any kind of financial derivative. The CIR model and GJR-GARCH model are employed to forecast both risk-free rate and variance paths. Fair price under issuer's credit risk can then be estimated by deducting the premium from the default-free price.

The default event can be triggered at any time point and the recovery rate is time-varying, depending on the capital structure of the issuer upon default. Our numerical example shows that the price of a 2-year USD equity-linked note issued by JP Morgan Chase is about 0.9% lower than otherwise identical the default-free note. The default probability within 2 years is 1.8%. Besides, based on the comparative static analysis, the initial asset to debt ratio and asset to strike ratio have a negative effect on default premium, while the asset volatility has a positive effect.

Keywords: default, credit risk, structured notes, CIR model, GJR-GARCH model.
JEL Classification: G12.

Introduction

It is well-known that the value of an over-the-counter option is affected by the credit risk of its writer. Options that are vulnerable to credit risk tend to have lower values than comparable non-vulnerable options since investors require higher expected payoffs to compensate for the credit risk they bear. There is no liquid secondary market that provides tradable prices nor is there any margin requirement to protect against the counterparties in the transaction. The risk of default on the issuer side should by no means be neglected.

Motivated by the notorious financial tsunami that occurred in the fourth quarter of 2008, and the following considerable credit losses due to unexpected bankruptcy events, we attempt to take the issuer's credit risk into account when pricing structured products. Expanding the concept embedded in Moody's KMV model, we estimate the fair premium for bearing issuer's default risk by forming an invented American binary put option. Refering to the literature about pricing vulnerable options, the default of the issuer is defined under a continuous framework that when the asset value of the issuer falls below the predetermined default boundary at any time during the life of the notes, the put option is exercised automatically with a payoff equal to the loss given default. Besides, the recovery rate, a key determinant for loss given default, is time-varying and related to the issuer's capital structure after liquidation at the time of default.

As equity-linked products account for more than 50% of the market of structured products, it will be selected as an example. The fair value and the default risk premium will be evaluated under Monte-Carlo simulation methods. Furthermore, we try to improve the pricing accuracy with the aid of CIR model and GJR-GARCH model.

Among the numerical methods we adopt in this article are the crude Monte-Carlo method, the Monte-Carlo method with antithetic variable, and the quasi Monte-Carlo method, the second one is used for comparative static analysis based on its high speed of convergence and time efficiency.

According to the outcome of the numerical example, a 2-year USD equity-linked note issued by JP Morgan Chase, the default premium is about 0.85% of the principal, and the fair value of this product under the issuer's credit risk concerned is 0.9% lower than its default-free counterpart. The probability of default of the issuer within 2 years converges to 1.8%. Furthermore, the comparative static analysis shows that the default risk premium rises as the volatility of asset increases. However, the asset to debt ratio and the asset to strike ratio have the opposite effect on the default premium.

The rest of the paper is outlined as follows. Section 1 reviews some academic literatures about pricing vulnerable financial products and credit risk models. Section 2 describes the model settings that incorporate CIR model, GJR-GARCH and credit risk premium estimation. Section 3 introduces different kinds of Monte-Carlo simulation methods. In section 4 we present the data, base-case parameters, and the pricing results. The comparative static analysis is performed in section 5. The final section provides concluding remarks.

1. Literature review

Structured notes are financial products that appear to be fixed income instruments containing embedded vulnerable options. In the 1980's, Johnson and

Stulz (1987) first consider the default risk for valuing options and define a word 'vulnerable' to describe such feature. To derive an analytic solution, Johnson and Stulz (1987) make some assumptions that a European option is the sole liability of the option writer; a restriction that a default occurs only at the exercise date and the option holder receives all the assets of the writer in default. They conclude that the effect of credit risk can be significant, and the value of a vulnerable European option can fall with the time to maturity, the interest rate, and the variance of the underlying asset. Furthermore, it may pay to exercise early a vulnerable American option on a non-dividend paying asset.

Hull and White (1995) assume that other than the option, the option writer could have some equal ranking claims on their assets upon default. The default can happen at any time before the option expiration date with a fixed default boundary. Default is set to occur once the assets of the option writer fall below that fixed level. Furthermore, both the probability of default and the size of the proportional recovery are random, that is, not related directly to the value of the writer's capital structure. The results of their numerical examples show that the impact of default risk on the price of an American option is less than that on the price of a European option.

Klein (1996) expands the flexibility of Johnson and Stulz (1987) model by easing the condition that the option writer does have other liabilities except the European option. The correlation between the option's underlying asset and the credit risk of the counterparty has been considered as well. In the model derived in Klein (1996), there exists an endogenous variable represented by the proportion of nominal claims paid out in default which can be estimated by the terminal value of assets of the option writer and the amount of other equally ranking claims. Most importantly, default is not restricted to occur only at maturity, and the recovery rate setting will be effective if the option writer's asset value hasn't rebounded above the default boundary at maturity date of the European option.

Klein and Inglis (2001) further extend the result of Johnson and Stulz (1987) and Klein (1996) by incorporating a default boundary which depends on the potential liability of the written option and the option writer's liabilities. Instead of deriving an analytic solution, a three-dimensional binomial tree approach is employed.

The literature discussed above mainly deals with vulnerable European options, especially call options. In Klein and Yang (2007), the pricing procedure for vulnerable American options is developed using the numerical method outlined in Hull and White (1990). Klein and Yang (2007) combine a default boundary at the time of maturity as in Klein and Inglis (2001) and a default barrier before maturity which is variable and linked to the payoff on the option. It is interesting to see that an increase in the volatility of the underlying assets has a mixed effect on the value of the vulnerable American put, in contrast to the always positive effect for default-free American options. Besides, the frequency of early exercise for vulnerable American options is relatively higher than for non-vulnerable American options, consistent with Hull and White (1995).

When we consider the credit risk of the product writer, it's like attaching an embedded cash-or-nothing, path-dependent put option to the structured note, and the option can be exercised by the product writer at any time before the maturity of the structured note. The underlying of the embedded put option is the asset of the writer; the exercise boundary is predetermined and related to the capital structure of the issuer. Besides, under the default event condition, the recovery rate is defined as the asset to debt ratio at the default point, adjusted by a multiplier representing the liquidation cost. To obtain reasonable estimate of variables for computing the fair price of this put option, we could seek the aid from some authentic credit risk models.

Credit risk models can be divided mainly into two groups: one is "structural model" using financial variables as inputs, such as Merton (1974) and Moody's KMV (1995). It is particularly useful for credit portfolio analysis and credit risk management. The other is "reduced-form" model which assumes a firm's default time is driven by a default intensity that is a function of some latent state variables. Models derived by Jarrow, Lando, and Turnbull (1997), Duffie and Singleton (1999), and Hull and White (2000) are classified into this group.

Jarrow and Protter (2004) compare structural versus reduced-form credit risk models from an information based perspective. They indicate that the structural models assume complete knowledge of all the firm's assets and liabilities. Under reduced-form model, the modeler only has incomplete knowledge of the firm's condition, which is more realistic for pricing and hedging purposes.

Arora, Bohn and Zhu (2005) proposed an opposite view to Jarrow and Protter (2004). Instead of basing on the information theory, they emphasize the importance of empirical evaluation when evaluating the strengths or weaknesses of different types of credit risk models. Even though a reduced-form model is not compromising on the theoretical issue of complete information, it does suffer from other weakness including lack of clear economic rationale for defining the nature of the default process. One advantage that makes reduced-form models so attractive to modelers is the flexibility in their functional form, however, this advantage may result in a model with strong in-sample fitting properties, but poor out-of-sample predictive ability. To prove their view, three models (basic Merton model and MKMV for structural models; and HW model for reduced-form model) representing three key stages of the literature in credit risk modeling are chosen to analyze the power of the structural and the reduced-form models. According to their numerical results, the MKMV model consistently outperformed the other two models in terms of default predictive power especially across large and small firms, while the performance of the Merton and HW models worsens considerably across larger firms.

According to the above literature, for default predictability, model stability, and economic rationality concern, we employed the MKMV model to estimate the essential parameters, such as the expected return and volatility, for pricing the binary put option embedded in the structured products.

In the next section, we'll incorporate the first step of the KMV procedure to generate an initial estimate of asset volatility and market value of asset for valuing the credit risk premium. Besides, appropriate models for predicting time-varying interest rate paths and variance paths will be introduced as well.

2. The model

2.1. Equity-linked notes under default-free condition. Structured note is an investment vehicle which bundles fixed-income securities and financial derivatives. The regular coupon yield can be generated by the fixed-income component while the derivatives component provides leverage effect, yield enhancement, and different level of risk tolerance to the structured note investors. The value of the structured note will change according to the value of the underlying asset, the reference interest rate, and the reference index.

Structured notes can be mainly classified into two categories, namely principal guaranteed notes (PGN) and yield enhancement notes (YEN, also known as high-yield notes). The former is designed to provide normal return with certain level of principal guarantee, and the remaining capital is invested in options having similar underlings as the PGN to earn external return. On the other hand, the YEN is constructed for the purpose of seeking high yields by holding zero coupon bonds and selling short options with similar underlyings. Although the YEN could provide better rate of return, the relative risk is higher as well.

For simplicity, we select the most popular type of structured notes, the equity-linked notes, as the example in the following sections, but the model we define for evaluating the credit risk of issuer can be applied to structured notes with any kinds of reference assets, such as interest rate, currency rate, and commodity, by adjusting the payoff functions.

Equity-linked notes can be linked to various portfolios, such as single equity, a basket of equities, single equity index and a basket of equity indices. For notes with multi-assets, the coupon rate is determined on some observation dates and the payoff function at maturity can be determined on the worst performing one, the best performing one, or the average return of the underlying basket.

Assuming an equity-linked note with N underlying assets:

X: the notional principal.
S_i: the underlying equities (or indices), i = 1~N.
m: the number of observation dates.
T: the time to maturity of the product.
A: the degree of principal protection with the range of 0~100%.
B: the participation rate which is always positive.
C_j: the time-varying coupon rate depending on the performance of the underlyings, which is calculated on each observation date (j) specified in the contract. j = 1~m.
R_T: final fixing return based on the description of the contract, and it is used to calculate the repayment at the maturity.

The payoff function (P) at maturity can be presented as below:

$$Payoff\,(P)=X\times[A+B\times max(0,R_t)] \quad (1)$$

then the fair price of the equity-linked note is the present value of the interest payment and the final repayment paid to the investor:

$$Price = X\sum\nolimits_{j=1}^{m}\left(C_j\times\frac{T}{m}\right)e^{\frac{rT}{m}j}+P\times e^{-rT}, \quad (2)$$

where r is the constant risk-free rate set at the valuation date.

Since the changeable coupon rate and the final fixing return are determined by the performance of underlying equities or indices, it is difficult to find the analytic solution for pricing structured products. Instead, the numerical procedure is used to evaluate these products. Among three major types of numerical methods, namely, binomial tree method, finite difference method and Monte-Carlo method, Monte-Carlo method is the most efficient one in handling path-dependent and multi-assets problems; we apply Monte-Carlo simulation method in what follows.

Without loss of generality, we assume that underlying price process $S_{i,t}$ follows log-normal diffusion process. However, when we observe the term structure of risk-free interest rate (the drift term in the real world) and the fluctuation of the stock price (the diffusion term), the "constant" assumption definitely violates the real market condition. To alleviate the impact of this assumption, we apply an interest rate model to simulate time-varying interest rate path (and thus the drift term component), and a stochastic volatility model to allow random changes of volatility over time.

2.1.1. Interest rate model for time-varying risk-free rate. There are two main groups of interest rate models, i.e., general equilibrium model and no-arbitrage model. Since no-arbitrage models assume that the interest rate term structure is stable, therefore if the interest rate experiences very volatile movements, the model would be invalid. In equilibrium models, the Vasicek model is most common used. However, the volatility of interest rate is fixed, and the interest rate might be negative. As a result, the Cox-Ingersoll-Ross (CIR) model is chosen to generate interest rate paths.

In addition, we add the market price of risk in the CIR model, that is, transfer the Q measure into P measure. According to Brigo et al. (2001), the following formulation is adopted to express the concept:

$$dr_t = [\alpha\theta - (\alpha + \lambda k r_t)]dt + k\sqrt{r_t}\,dw_{r,t}^P, \tag{3}$$

where θ is the long-term mean of interest rate, α is the mean reversion rate, k is the volatility of the interest rate process, $dw_{r,t}$ is a standard wiener process, λ is a factor for market price process.

For the characteristic of square-root process the interest rate is always positive and if $2\alpha\theta > k^2$ then it cannot reach zero.

2.1.2. GJR-GARCH for volatility forecasting. There are two important stylized facts with equity return data that the returns often exhibit volatility clustering and leptokurtosis. Consequently, a generalized autoregressive conditional heteroskedasticity (GARCH) is applied to forecast the volatility through time.

Since the product we evaluate here is an equity-linked note, we should consider the "leverage" effect of the equity return when selecting a suitable GARCH model. The "leverage" effect, or the volatility asymmetry is a well-known stylized fact first discussed by Black (1976), who observed that the extent of relative price fluctuations of a stock tends to increase when its price drops. In other words, non-negative and negative returns allow causing different level of effect to the variances. Schwert (1989a, b) also presents evidence that stock volatility is higher during recessions and financial crises.

Both the EGARCH (Nelson, 1991) and GJR-GARCH (Glosten, Jagannathan and Runkle, 1993) models contain the variables for taking the leverage effect into account. Duan et al. (2006) document that the GJR-GARCH has higher accuracy and preferred computation speed as compared to EGARCH model when used in option pricing, therefore GJR-GARCH is applied to estimate the variance process of the underlying equity return of the ELN product.

According to Glosten et al. (1993), the GJR-GARCHM (1,1) model can be expressed as follows:

the mean equation is

$$R_t = \alpha_0 + \frac{1}{2}\alpha_1 h_t^{\frac{1}{2}} + \varepsilon_t \tag{4}$$

and the variance process is set as

$$h_t = \omega_0 + \omega_1 + \varepsilon_{t-1}^2 I_{(\varepsilon_{t-1<}0)} + \beta h_{t-1} \tag{5}$$

with constraints that $\beta > 0$.

Besides, the indicator function $I_{(\varepsilon_{t-1<0})}$ is equal to one when ε_{t-1} is positive; and equal to zero otherwise. The conditional variance is a predictable process because h_t is expressed only in terms of variables known at time *t-1*.

2.2. Premium for the issuer's default risk. To access the premium for bearing the default risk of the issuer, we transfer the concept of equity being a call option on the issuer's asset into an American put option. The underlying asset of the put option is also the asset of the issuer, but the strike price is different from the book value of the corporate debt. In general, a firm is regarded to be in default if any scheduled payment were not met on time.

Since a financial institution is thought to be in default if the expected (the short-term debts mature in the near future and the interest payment) and unexpected (deposit withdrawals) payments exceed the asset level, the default boundary, and, therefore the strike price of the put option are defined as:

Default boundary (K) = short-term borrowings and liabilities + interest payment of total liabilities + demand deposits + saving deposits.

Besides, the put option is an American option, since it can be exercised immediately once the value of asset falls below the strike price (default boundary) during the life of the structured product.

We assume that all liabilities have the identical level of priority, with the recovery rate defined as $\xi = (1-v)V_{A,t} / V_{D,t}$, where $V_{A,t}$ and $V_{D,t}$ denote the market value of the issuer's assets and the book value of the issuer's liabilities, v represents the deadweight costs of default expressed as a percentage of $V_{A,t}$. Once the issuer defaults at time τ, $\tau \le T$, all the following payments are cancelled, and only part of the notional principal (X) will be repaid.

The payoff of (Q) the put option can be regarded as the loss of the investor if the issuer turns to default. It should be the expected amount that the investor would receive under normal situation (no default occurs), that is,

$$Q = \sum_{t=\tau}^{T}\left(XC_t \frac{T}{m}\right) e^{-r(t-\tau)} + Pe^{-r(t-\tau)} - \left[(1-v)V_{A,t} / V_{D,t}\right]x,\ if\ \tau < T \tag{6}$$

where C_t = the annual coupon rate at t, P = the final payment, m = the frequency of coupon payment per year, T = the time to maturity of the ELN, in years. Equation (6) would reduce to equation (7) if $\tau < T$,

$$Q = (XC_T)e^{-r(T-\tau)} + Pe^{-r(T-\tau)} - \left[(1-v)V_{A,t} / V_{D,t}\right]X. \tag{7}$$

To evaluate the American put option, we assume that under Q measure, the underlying asset ($V_{A,t}$) and the book value of total liabilities follow the stochastic processes described by equations (8) and (9)

$$dV_A = rV_A dt + \sigma_A V_A dw_{A,t}, \tag{8}$$

$$dV_D = rV_D dt, \tag{9}$$

$$dw_{A,t} \cdot dw_{r,t}^P = \rho_{A,r} dt \ and \ dw_{A,t} \cdot dw_{S_i,t} = \rho_{S_i,r} dt.$$

To estimate the initial market value and volatility of assets, we apply the technique described in Crosbie and Bohn (2003). According to the option pricing theory derived in Black and Scholes (1973) and Merton (1974),

$$V_E = V_A N(d_1) - e^{-rT} V_D N(d_2) \tag{10}$$

where $d_1 = \dfrac{ln\left(\dfrac{V_A}{V_D}\right) + \left(r + \dfrac{\sigma_A^2}{2}\right)T}{\sigma_{A\sqrt{T}}}$, $d_2 = d_1 - \sigma_{A\sqrt{T}}$.

The relationship between asset volatility (σ_A) and equity volatility (σ_E) can be expressed as in equation (11):

$$\sigma_E = \frac{V_A}{V_E}\sigma_A N(d_1). \tag{11}$$

The variables V_E and σ_E can be obtained from public market, and the book value of liabilities (V_D) is recorded on the issuer's financial report. Thus, there are only two unknown variables, V_A and σ_E, with two equations ((10) and (11)), we can then solve for these two unknowns.

Finally, the value of this American put option can be calculated as the mean of discounted payoff among numerical iterations. The value of the ELN for considering issuer's default risk is expected to be lower than the value of default-free counterpart by the amount equal to the put option premium.

3. Numerical method

We apply three Monte-Carlo methods to construct the simulation, and the one with the most preferable rate of convergence and accuracy will be adopted for comparative static analysis.

3.1. Crude Monte-Carlo method. The most common use of Monte-Carlo simulations in finance is to calculate an expected value of a function f(x) given a specified distribution density φ(x) over $x \in R^n$:

$$payoff\ V(x) = E_{\varphi(x)}[f(x)] = \int f(x)\varphi(x)dx^n. \tag{12}$$

After drawing variate x from the target distribution φ (x), the function value can be computed. With N-times iteration, the Monte-Carlo estimator of the payoff *(V)* is equal to

$$\hat{V}_N \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i). \tag{13}$$

In a risk-neutral measure, the value of the financial security is the discounted value of its expected payoff at the termination:

$$Price = e^{-rT}E_{\varphi(x)}[f(x)] \approx e^{-rT}\left[\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right]. \tag{14}$$

Furthermore, for an M-dimensional domain and N-point evaluation, the crude Monte-Carlo method has absolute error of estimate that decreases as $1/\sqrt{N}$, that is the rate of convergence is $1/\sqrt{N}$ independent of the dimensionality. For comparison purposes, the rate of convergence of lattice methods is $1/\sqrt[M]{N^2}$.

3.2. Monte-Carlo method with variance reduction technique. "Antithetic variate" is one of the most appropriate techniques for variance reduction. It says that for any one drawn path its mirror image has equal probability, that is, if a single evaluation driven by Gaussian variance vector draw x_i is given by $V_i = V(x_i)$, and $\widetilde{V}_i = V(-x_i)$ is used as well.

Then the pairwise average for an individual iteration is calculated as

$$\widetilde{V}_i = \frac{1}{2}\left[V(x_i) + V(-x_i)\right]. \quad (15)$$

The antithetic sampling procedure provides a variance reduction that:

$$E[\overline{V}_i] = E\left[\tfrac{1}{2}\left(V_i + \widetilde{V}_i\right)\right] = \frac{E[V_i] + E[\widetilde{V}_i]}{2} = E[V_i]. \quad (16)$$

$$Var[\overline{V}_i]Var\left[\frac{1}{2}\left(V_i + \widetilde{V}_i\right)\right] = \frac{1}{2}Var[V_i] + \frac{1}{2}Cov[V_i, \widetilde{V}_i] < \frac{1}{2}Var[V_i], \quad (17)$$

since $Cov[V_i, \widetilde{V}_i] = Cov[V(x), V(-x_i)] < 0. \quad (18)$

3.3. Quasi Monte-Carlo methods. In numerical analysis, a quasi Monte-Carlo method is an extension of the regular Monte-Carlo method that computes the integral or other pricing problems based on low-discrepancy sequences instead of pseudo-random numbers. The standard operation procedures of both regular and quasi Monte-Carlo methods are executed in a similar way.

Furthermore, it has been shown the number sequences, such as Halton, Faure, Sobol and Niederreiter, can be generated that enables us to do quasi Monte-Carlo calculations which give certain smoothness conditions of the function to be integrated. The spread of convergence is not as $1/\sqrt{N}$ for the crude Monte-Carlo; instead, it is much more close to 1/N, namely $c(M)[(lnN)^M/N]$. Thus, even for a large dimensionality M, the quasi Monte-Carlo method is asymptotically much faster than $1/\sqrt{N}$. The only problem is that the coefficient $c(M)$ depends on the dimensionality. The most common low-discrepancy sequence people used in empirical pricing is the Halton sequence, and it's also the one we use for evaluating numerical examples.

The idea behind Halton sequences is to use one different prime base for each dimension. The number bases are chosen to be the prime numbers of which one has to be precalculated for each dimension in order to prevent any asymptotic pairwise periodicity. A sequence corresponding to the prime y has cycles of length y in which numbers increase monotonically. Since the larger base means longer cycle, correlation problems could emerge between sequences generated from higher primes if the dimensionality is high. For instance, if there exist two prime bases y_1 and γ_2 ($\gamma_1 > \gamma_2$), then it's suggested to drop the first ($\gamma_1 + 1$) numbers to avoid the perfect linear correlation problem.

The algorithm to construct a Halton sequence is as follows: For an assigned integer $J>1$, it could be expanded in a base γ, where γ is any prime number. The integer J can be expressed as:

$$J = \sum_{j=0}^{k} \alpha_j \gamma^j \quad (19)$$

with integers $\alpha_j \in [0, \gamma - 1]$ chosen large enough to make sure that all non-zero digits of J in the number base γ are accounted for.

The sequence of calculated coefficients is now inverted and used as multipliers of fractions in the number base γ to construct the coordinate number U located in the interval [0,1).

$$U = \sum_{j=0}^{k} \alpha_j \gamma^{-(j+1)}. \quad (20)$$

4. Numerical example

4.1. Terms of the structured note. The product we choose is a "2-Year USD Equity Linked Note" issued by J.P. Morgan International Derivatives Ltd and it is guaranteed by JPMorgan Chase Bank. The itemized description is presented in Table 1.

Table 1. Definitive term sheet & simplified prospectus

Notional amount	1,000 USD
Issue price	100% of principal
Principal guarantee	100%, if the issuing-institution remains non-default
Term	2 year
Participant rate	0%
Launch date	July 18th, 2007
Initial fixing date	July 19th, 2007
Issue date	July 25th, 2007
Maturity date	July 27th, 2009
Initial fixing price	The closing price of underlying stocks on initial fixing date
Business day	New York

The underlying basket contains two equities and one benchmark in the same industry, as shown in Table 2.

Table 2. Underlying basket and the benchmark

Symbol	Stock name	Ticker	Currency	Initial fixing price
S1	General Motors Corp.	GM UN	USD	35.38
S_2	Ford Motor Co.	F UN	USD	8.63
S_{bench}	Toyota Motor Corp.	7203 JT	JPY	7,540

There is an automatic early termination scheme attached to this ELN. The automatic early termination event is deemed to have occurred if the aggregate coupon payment is greater than or equal to a specific amount on some coupon valuation date, illustrated in Table 3.

There are eight coupon valuation (payment) dates, the coupon rate may be zero or equal to 2.5% depending on the relative performance of the underlying basket to the benchmark, that is, whether the outperformance rate (i) is greater than or equal to 0%. The outperformance rate (i) on coupon valuation date (i) is defined as:

$$OR(i) = outperformance\ rate\ (i) = \frac{S_{bench}(i)}{S_{bench}(0)} - Max\left[\frac{S_1(i)}{S_1(0)}, \frac{S_2(i)}{S_2(0)}\right].$$

Table 3. The automatic early termination mechanism

The early redemption information				
i	Valuation date	Payment date	Coupon rate	Early redemption amount
1	2007/10/18	2007/10/25	0%, if OR(i)< 0 2.5%, ifOR(i)>0	100% x Principal amount per note
2	2008/01/17	2008/01/25		
3	2008/04/18	2008/04/25		
4	2008/07/18	2008/07/25		
5	2008/10/20	2008/10/27		
6	2009/01/16	2009/01/25		
7	2009/04/20	2009/04/27		
8	2009/07/20	2009/07/27		

Note: An automatic early termination event will occur if the aggregate coupon payment is greater than or equal to 10% on the i^{th} valuation date. And the issuer will pay the note holder the 100% principal amount per note.

4.2. Parameters estimation. According to Hull (2009), the sample period used for evaluating a financial derivative product under simulation should be as long as the life of the product. Thus, the sample period is set from 2005/07/08 to 2007/07/18, and the data source is Bloomberg database. The initial values of key variables are presented in Table 4.

Table 4. Initial values of variables

	S_1	S_2	S_{bench}
Initial price	35.38	8.63	7,540
Dividend yield	0	0.02688	0.01471
Volatility	0.39410	0.33362	0.20981
Initial rate	0,04980	0,04980	0,006780

Since this ELN was evaluated on the initial fixing date 2007/7/19, the latest financial report we can obtain from J.P. Morgan is the annual report of 2006. In order to match the time frame of the product, we assume the process of default boundary (part of the liabilities) to be identical to the book value of debt. Furthermore, the price and the shares outstanding of equity on 2007/7/19 is used to calculate the market value of equity.

The values of related variables for estimating the default premium are shown in Table 5. Besides, the rate of liquidation cost (v) is set to be 0.2.

Table 5. Initial values of variables

	V_A*	V_E	V_D
Initial value	1.37129E+12	1.65237E+11	1.26764E+12
Volatility	0.02554	0.27711	-
Risk-free rate	0.04980	0.04980	0.04980
Default boundary**	8.96822E+11		

Notes: * The initial value and volatility of asset are calculated from the KMV model. ** The default boundary is the summation of short-term borrowings and liabilities, interest payment of total liabilities, demand deposits, and saving deposits.

Since CIR model and GJR-GARCH (1,1) are applied to simulate the path of interest rate and volatility, the Maximum Likelihood Estimate (MLE) is conducted to obtain estimates of the parameters. The yield of 3-month Treasury bill is used in solving parameters of CIR, and the discount rate is the yield of the 2-year U.S. Treasury note on the valuation date. Parameter estimates are presented in Tables 6 and 7.

Table 6. Parameters for CIR model

		α	θ	κ	λ
U.S.	OLS	1.95999	0.05097	0.02321	0.00347
	MLE	1.9337	0.05099	0.0234	0.00412
Japan	OLS	1.22386	0.00018	0.03583	0.00021
	MLE	1.07108	0.0006	0.03334	0.00025

Table 7. Parameters for GJR-GARCH (1,1)

	α_0	α_1	ω_0	ω_1	ω_2	β
GM (S_1)	-0.00014	0.08321	0.00002	0.04562	0.02743	0.91446
Ford (S_2)	-0.00012	0.06458	0.00002	0.05856	0.03283	0.91861
Toyota (S_{bench})	0.00009	0.05273	0.00003	0.07312	0.03982	0.73788

At the 5% significance level, the estimated coefficients of GR-GARCH in mean process for the underlyings are statistically significantly different from zero.

4.3. Simulation result. After thousands of simulation, the default-free fair price of the ELN converges to \$931~\$932. The premium of American put option which represents the compensation for bearing issuer's default risk is about \$7~\$8. As a result, the value of this ELN product with issuer's default risk is \$923~\$924, about 0.86% lower than the default-free price. Furthermore, we can find that the estimated probability of default for J.P. Morgan is about 1.80% for the next 2 years, starting on July 19, 2007.

Table 8. Information on the convergence of simulation methods

	CMC[1]	MCAV[2]	QMC[3]
Computation time per simulation	0.0589	0.0959	0.2006
Standard deviation	60.2980	38.4335	30.5435

Notes: [1] Crude Monte Carlo Simulation, [2] Monte Carlo with antithetic variable, [3] Quasi Monte Carlo Simulation.

The computation time and the standard deviation of each simulation model are provided in Table 8. Monte-Carlo with variance reduction technique (MCAV) seems to be the best model based on its low level of standard deviation and 50% time saving compared to the low discrepancy sequence (QMC). Besides, both Monte-Carlo with variance reduction technique and quasi Monte-Carlo methods exhibit outstanding rate of convergence.

5. Comparative static analysis

Since interest rate and variance follow certain time-varying processes, we shift the entire path by a multiple instead of adjusting the initial value when executing comparative static analysis. As Monte-Carlo simulation method with antithetic variable exhibits great convergence with more than 2,500 iterations, it is robust to run 5,000 iterations for sensitivity analysis.

5.1. Product base. *5.1.1. Correlation.* First of all, we change the correlation coefficients between the two underlyings and the benchmark to observe the resulting price and default premium. As shown in Figure 1, the price is sloping upward with respect to the correlation between the two underlyings (S1 and S2). When the two underlyings are perfectly positive correlated, there is about 50% chance that the best performer will be beaten by the benchmark, and the coupon is delivered. On the contrary, when the two are perfectly negatively correlated, a diffusion term would have opposite effect on them which in turn reduces the probability of coupon delivery and early termination.

When we change the correlation coefficient between the benchmark and one of the underlying equities, the outcome is quite different. If the correlation is positive, the price movements of them are similar and there will be less probability that the benchmark outperforms the best performing underlying. Consequently, we observe a negative relationship between the product price and the correlation coefficient as shown in Figure 2.

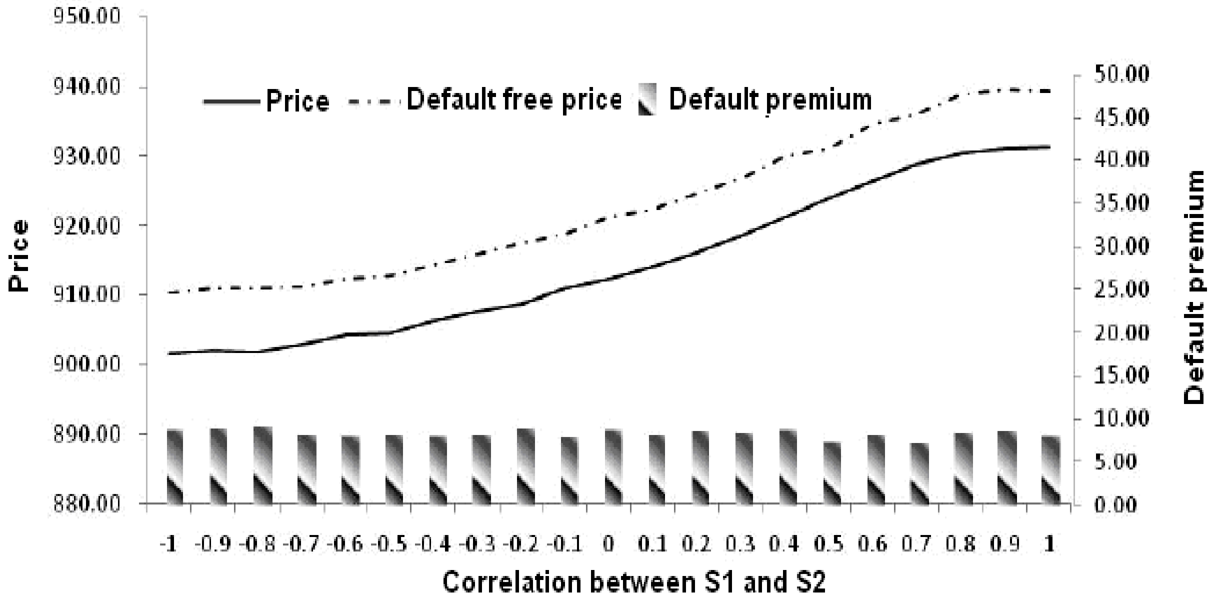

Fig. 1. Comparison under different levels of correlation between S1 and S2

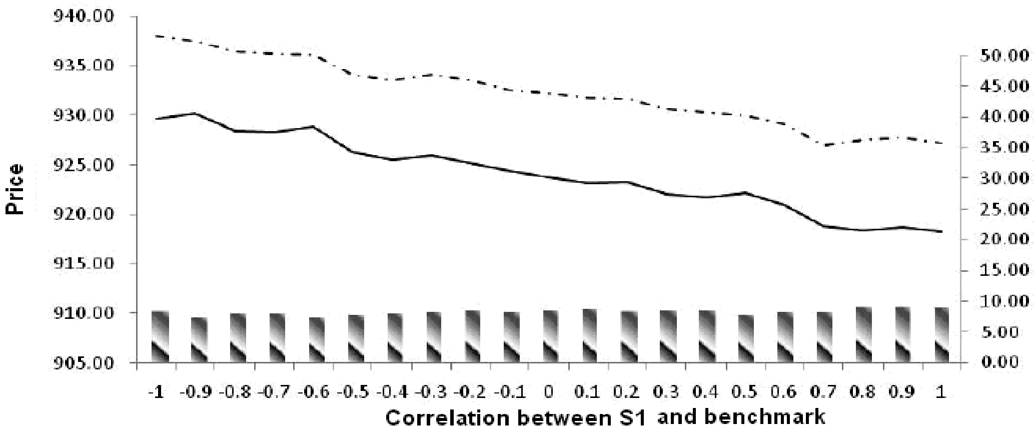

Fig. 2. Comparison under different levels of correlation between S1 and benchmark

5.1.2. Volatility of the underlying assets. It is observed from Figure 3 that the higher the volatility (variance multiplier), the higher the price of the product. The condition for coupon delivery is easier to be met and the possibility of early termination also increases when the volatility of the underlying gets larger. The positive relationship holds for both the two underlying equities.

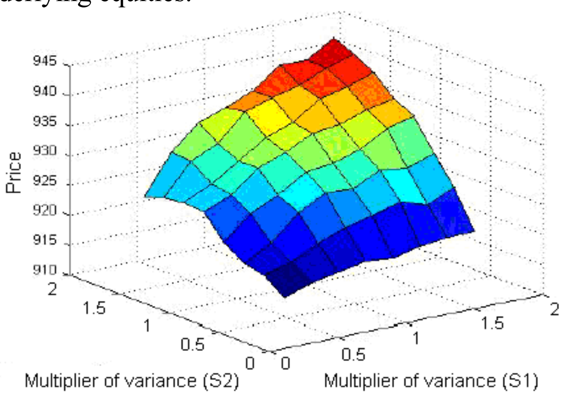

Fig. 3. Price under different levels of variances of underlyings

On the other hand, since the default-free price is greatly affected by the level of volatility, the loss given default (LGD) also changes accordingly. As a result, even the default probability fluctuates within a stable interval, the effect on the default premium is ambiguous among various pairs of variances' multipliers, as shown in Figure 4.

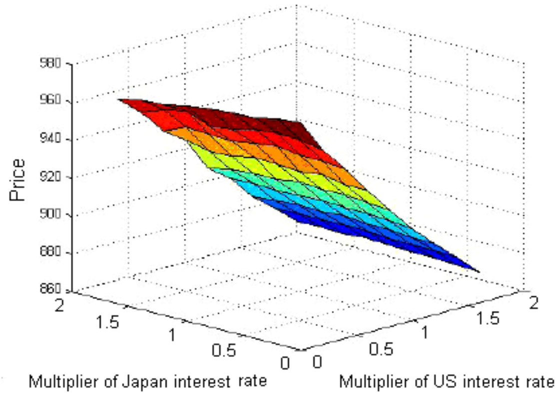

Fig. 4. Default premium under different levels of variances of underlyings

5.1.3. Interest rate. We shift the entire paths upward and downward to investigate the reaction of price and default premium. With a multiple smaller than one, the expected rates of return of these two underlyings decrease due to the level down interest rate path. Consequently, both the aggregate coupon rate and the frequency of early termination event rise. Furthermore, the price is monotonically decreasing with the increasing US interest rate. Compared to the US interest rate, the effect on price caused by the change of Japan interest rate is tiny, as depicted in Figure 5.

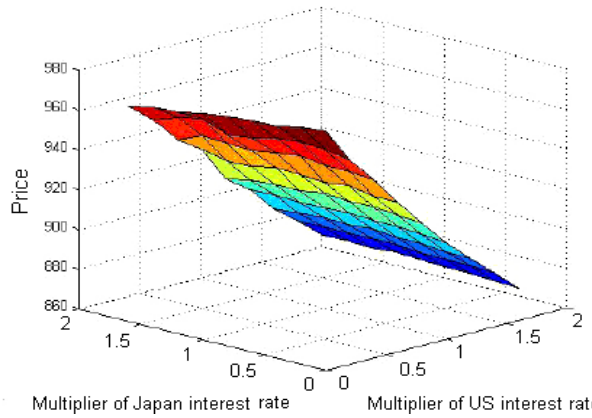

Fig. 5. Price under different levels of interest rates

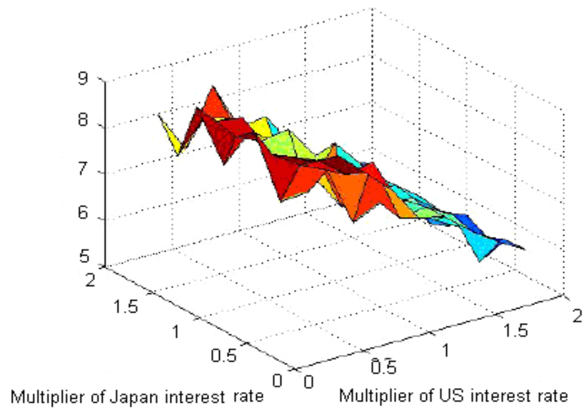

Fig. 6. Default premium under different levels of interest rates

With similar explanation, the probability of default decreases as the interest rate rises, and so does the default premium, shown in Figure 6.

5.2. Issuer base. *5.2.1. Asset to default boundary ratio.* Figure 7 shows that as the asset to default boundary ratio reaches the level of about 1.8, both the default probability and the default risk premium are close to zero. The defaulttable and default-free prices are almost identical.

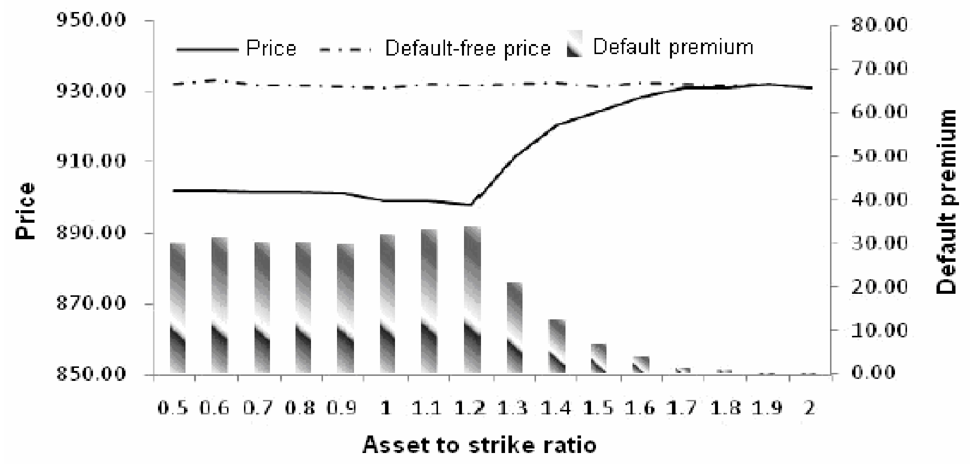

Fig. 7. Comparison under different levels of asset to strike ratio

5.2.2. Asset to debt ratio. Before discussing the effect on default premium caused by different initial asset to debt ratio, we assume that the default boundary, i.e., the strike price of the American binary put option, is independent to the initial asset to debt ratio.

Since the time-varying recovery rate is mainly determined by the asset to debt ratio at the time of default, the lower the ratio, the smaller the recovery rate and the larger the loss given default (LGD). Conversely, when the asset to debt ratio is quiet large, the total amount of debt needed to be repaid is very low. Thus, even when the issuer defaults suddenly, the asset value of the issuer after liquidation would be more than enough to cover debt payment. The investors would receive the same expected cash flow as they would under default-free assumption. It can be observed from Figure 8 that when asset to debt ratio reaches the level of 2, default premium becomes negligible.

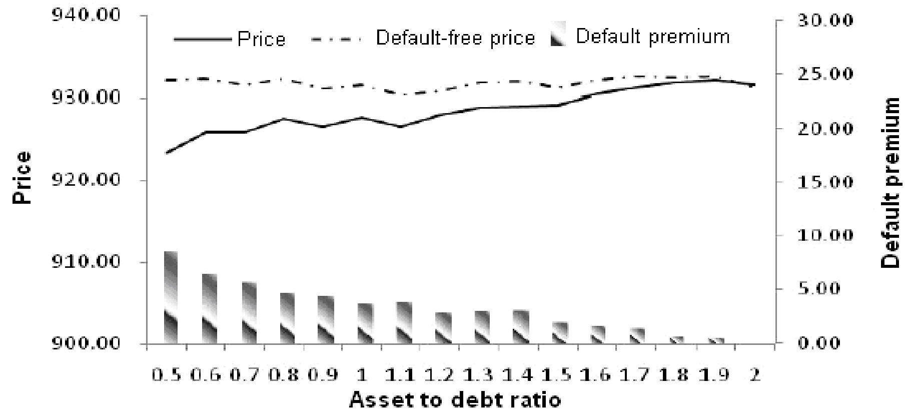

Fig. 8. Comparison under different levels of asset to debt ratio

5.2.3. Asset volatility.

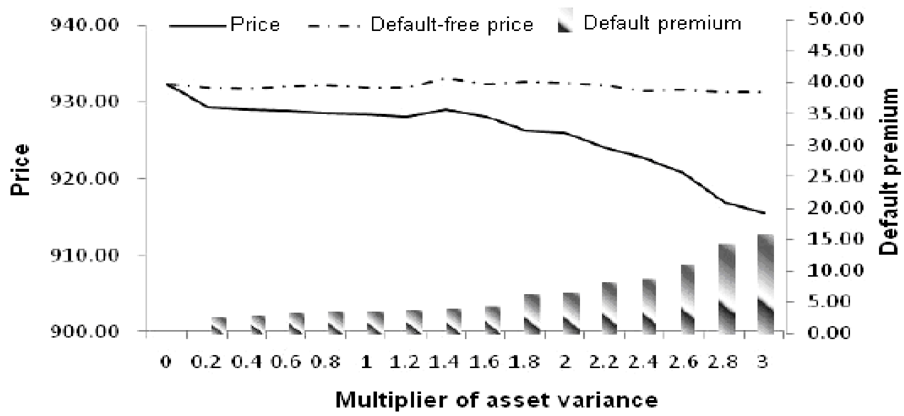

Fig. 9. Comparison under different levels of asset variance

The process of asset value consists of two parts: the drift term and the diffusion term. When the volatility of asset gets larger, the asset value walks in a more volatile fashion, the coefficient of the drift term fr—$o^2 A$ becomes smaller, leading to a lower expected growth rate of asset value. Accordingly, the probability of default is higher as the volatility of asset value increases. The premium to compensate for the default risk is especially higher when the variance multiplier gets beyond 1.4, as shown in Figure 9.

Conclusion

Incorporating the idea behind the Moody's KMV, one of the structural credit risk models, we develop a methodology for estimating the risk premium corresponding to the default risk originated from the structured product issuer. The framework of this method can be applied to all kinds of structured products, including interest rate related, foreign exchange rate related, equity related ones.

Similar to the nature of American binary put option, the credit risk is priced as an option sold by the investor (ELN holder) to the issuer (ELN writer). The option premium can be taken as the compensation for bearing issuer's credit risk, which has been largely ignored. The default premium is estimated based on the contingent claim of the structured product and the capital structure of the issuer at the time of evaluation.

Taking the "2-Year USD Equity Linked Note" issued by J.P. Morgan for illustration, the default probability within 2 years is about 1.70%, and the price of the structured notes is 0.9% lower than the default free counterpart. Furthermore, according to the outcome of comparative static analysis, the default probability rises as the asset to default boundary ratio becomes lower, and so does the asset to debt ratio. In addition, the asset volatility has a positive effect on the probability of default, and, hence, the default premium.

The default boundary is determined by the financial report of the issuer. For further research, we suggest two alternative definitions of default boundary to improve the information efficiency. First, a multiplier can be estimated by observing the frequency and the quantity of the debt issuance within a certain period (e.g., the last five years) of the issuer to adjust the book value of the debt to reflect the possible off-balance sheet effect. Secondly, we may incorporate the market data of the issuer's credit default swap (CDS) to reflect market expectation of the default risk. With the extension of the default boundary setting mechanism, the model can reflect timely information and corporate movement promptly.

References

1. Arora, N., J. Bohn, and F. Zhu (2005), Reduced Form vs. Structural Models of Credit Risk: A Case Study of Three Models, *Journal of Investment Management*, Vol. 3, No. 4, pp. 43-67.
2. Black, F. (1976), Studies of stock price volatility changes, Proceedings of the American Statistical Association, pp. 177-181, Business and Economics Statistics Section.
3. Boyle, P.P. (1977), Options: A Monte Carlo Approach, *Journal of Financial Economics*, Vol. 4, No.3, pp. 323-338.
4. Brigo, D., and F. Mercurio (2001), Interest Rate Models – Theory and Practice, Springer Finance, pp. 56-60.
5. Crosbie, P., and J. Bohn (2003), Modeling Default Risk – Modeling Methodology, Moody's KMV Company White Paper.
6. Duffie, D., and K. Singleton (1999), Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12, pp. 687-720.
7. Glosten, L.,R., R. Jagannathan, and D.E. Runkle (1993), On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance*, Vol. 48, No. 5, pp. 1779-1801.
8. Hull, J.C., and A. White (1990), Valuing derivative securities using the explicit finite difference method, *Journal of Financial and Quantitative Analysis*, Vol. 25, pp. 87-99.
9. Hull, J.C., and A. White (1995), The impact of default risk on the prices of options and other derivative securities, *Journal of Banking and Finance*, Vol. 19, pp. 299-322.
10. Hull, J.C., and A. White (2000), Valuing Credit Default Swap I: No Counterparty Default Risk, *Journal of Derivatives*, Vol. 8 (1), pp. 29-40.
11. Jarrow, R., and S. Turnbull (1995), Pricing derivatives on financial securities subject to credit risk. *Journal of Finance,* 50, pp. 53-86.
12. Jarrow, R., D. Lando, and S. Turnbull (1997), A Markov model for the term structure of credit risk spreads. *The Review of Financial Studies*, Vol. 10, No. 2, pp. 481-523.
13. Jarrow, R., and P. Protter (2004), Structural versus Reduced Form Models: A New Information Based Perspective*, Journal of Investment Management*, Vol. 2, No. 2, pp. 1-10.
14. Johnson, H., and R. Stulz (1987), The pricing of options with default risk, *Journal of Finance*, Vol. 42, pp. 267-280.
15. Kealhofer, S. (2003a), Quantifying Credit Risk I: Default Prediction, *Financial Analysts Journal*, Vol. 59, No. 1, pp. 33-44.
16. Kealhofer, S., 2003b, Quantifying Credit Risk II: Debt Valuation, *Financial Analysts Journal*, Vol. 59, No.3, pp. 78-92.
17. Klein, P. (1996), Pricing Black-Scholes options with correlated credit risk, *Journal of Banking and Finance,* Vol. 50, pp. 1211-1229.

18. Klein, P., and M. Inglis (2001), Pricing vulnerable European options when the options payoff can increase the risk of financial distress, *Journal of Banking and Finance*, Vol. 25, pp. 993-1012.
19. Klein, P., and J. Yang (2007), Pricing vulnerable American options, working paper, Simon Fraser University.
20. Merton, R.C. (1974), On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance*, Vol. 29, No. 2, pp. 449-470.
21. Schwert, G.W. (1989a), Business Cycles, Financial Crises and Stock Volatility, Carnegie-Rochester Conference Series on Public Policy, 31, pp. 83-126.
22. Schwert, G.W. (1989b), Why does Stock Market Volatility Change over Time, Journal of Finance, 44, pp. 1115-1154.
23. Vasicek, O. A. (1984), Credit Valuation, KMV Corporation, March.