“Fair value and demographic aspects of the insured loans”

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Fair value and demographic aspects of the insured loans

Abstract

The paper deals with the liability valuation of the insured loan in compliance of the fair value requirements for the financial assets and liabilities, as mapped out by the international boards engaged in this tool. Initially we propose a closed form for the fair valuation of the mathematical provision in a framework in which the randomness in the mortality is considered along with the financial risk component. Furthermore, the aim of the paper is to analyze the relevance of the risk arising from the demographic movements on the insured loan reserve.

The approach we follow implies the mathematical provision calculated as current values, this meaning at current interest rates and at current mortality rates. In these two variables the basic risk drivers of a life insurance business dwell and the many-sided risk system consists, in its systematic aspects, in the choice of the adequate models for forecasting the future scenarios. The relevance of the impact of the risk connected to the choice of the mortality table (*table risk*) on the fair value of the mathematical provision is pointed out and quantified using a measurement tool obtained by conditional expectation calculus. The risk mapping is performed analyzing the accidental risk impact on the insured loan portfolio liabilities. In all likelihood, insured loan portfolios are not large enough to be considered well diversified to the aim of the pooling risk reduction; this consideration makes interesting the measuring of the liability variability caused by the random events connected to mortality (*mortality risk*). Practical implications of assuming different mortality scenarios on the reserve fair value are presented, a graphic description of the model risk deriving from the choice of the demographic model is provided and numerical evidences of the accidental mortality risk are shown.

Keywords: mathematical provision, longevity risk, survival models, model risk, mortality risk.
JEL Classification: G22, G28, G13.

Introduction

In an economic scenario characterized by a general instability contextual with the increasing phenomenon of globalization, the need for new solvency requirements in the life insurance field becomes more pressing. A proper assessment of the risks, the homogenization through different countries and different companies and the correct information about company activity, have produced a wide ongoing debate among the international boards working in the accounting field of the life insurance business. To resolve these issues, the International Accounting Standards Board in Europe and the Financial Accounting Standards Board in the United States are carrying out a very precise work program.

The guidance up to now picked out by the international boards concerning the mathematical provision valuation can be synthesized in the request of the insurance liabilities in each period expressed in the mark-to-market valuation known as *fair value*. This request makes a change in the liability valuation outlined with the aim of depicting the life insurance business in its realistic economic profile (Jorgensen, 2004). In spite of the absence of an univocal settlement of the fair value definition and, in particular, on how the fair value has to be calculated, the minds seem to converge towards an estimate of an exit-price determined by market interactions (as the Financial Accounting Standard Board settles), leading to a fair value of the mathematical provision expressed by an estimated market price and not necessarily an equilibrium price (Cocozza et al., 2007). In this new perspective, the insurance companies have to represent a real world in which they carry out the role of financial intermediaries, from the point of view of their capacity to be market makers.

The fair value form requested for the financial assets and liabilities, based on the current values, implies the mathematical provision calculated at current interest rates and at current mortality rates, too. In these two variables the basic risk drivers of a life insurance business dwell and the many-sided risk system can be regarded, in its more relevant characteristics, in the choice of the "right" discounting process and of the "right" mortality table for forecasting the future scenarios (Cocozza et al., 2007).

In this order of ideas, the basic role of the interactions between the solvency valuation and the risk sources affecting the life insurance portfolio flows and, as a consequence, the solvency assessment tool is exhaustively treated only if considering the impact of each risk source and of the interactions among them.

The paper concerns the fair valuation in the case of the insured loan, a contract particularly interesting in a system making the interlacement between strictly banking and strictly life insurance activities running up in the daily business of both sections. Two different aims are attainable by means of this kind of contract: the insured loan satisfies the borrower's requirement of dispensing the heirs from the not fulfilled obligations in case of his predecease and/or the bank need of transforming a risky loan into a risk-free loan, asking the borrower himself to underwrite a life insurance (Encyclopedia of Actuarial Science).

In this paper we intend to deepen two aspects inherent the fair valuation problem in the specific contractual case of the insured loan. In the first part we propose a closed form for the fair valuation of the mathematical provision of an insured loan in a framework in which the randomness in the mortality is considered together with the financial risk component. In the second part of the paper we highlight the relevance of the demographic uncertainty on the portfolio liability valuations.

The question is fronted in its systematic and unsystematic face. The first component originates from the deviations of the number of deaths from the expected values due to the betterment in the survival trend, taking place in the industrialized countries particularly in the last decades. The correct capital constraint, avoiding to reserve more than necessary, derives from the choice of the right mortality table, that is from the best mortality estimate. The risk source we consider comes true the survival description choice: the impact of the risk connected to the mortality model selection (*table risk*) on the fair value of the mathematical provision is measured. The second component seems to be a risk source particularly interesting in small portfolios, like the one at issue, for which a weak diversification can be supposed. As showed in Coppola et al. (2003), unlike the risks deriving from systematic variability, the risk due to the accidental deviations of the number of deaths from the expected values (*mortality* risk) is a pooling risk, for which the measure becomes negligible only when the number of contracts in portfolio tends to infinity. The impact of the mortality risk on the fair value of an insured loan portfolio is measured as well, in this way completing the risk map of the considered portfolio.

The layout of the paper is the following: in Section 1 we describe the insured loan in its contractual and financial details and in Section 2 the valuation framework is introduced proposing the cash flow related to the contract both in the deterministic formulation and proposing the fair valuation of the liabilities within stochastic assumptions. The contribution of the uncertainty in the choice of the mortality table usable for valuation is measured in Section 3 by means of the *table risk* calculation. In Section 4 the measuring tool leading to the quantification of the mortality risk impact on the fair value of the mathematical provision of an insured loan portfolio is provided. Section 5 is dedicated to several numerical results, in particular emphasizing the relevance of the demographic hypotheses in the link between the sign of the reserve fair value and the structure of the insurance product (i.e., the number of premiums) and of the two demographic risk components (both the table risk and the mortality risk) on the value itself. The last section presents some conclusions about the treated subjects.

1. The insured loan: contractual and financial details

1.1. Preliminary remarks. At present, the wide diffusion of the loan insurance industry leads to a better management of default risk and capital surcharges levied by the central banking regulator upon the mortgage institution. A default resulting in a loss for the lender typically occurs because of numerous unforeseen circumstances striking the borrower such as job loss, divorce or death.

In particular, we take into account that the event of loss under a loan insurance contract is the borrower's death, even though the analysis can be spread out, with the opportune changes, to the other cases.

The loan insurance industry is emerging in all the advanced financial markets and, in general, we can see an important evolution of loan guarantee instruments.

In order to cover the *outstanding loan balance* (the residual debt) in the event of the borrower – insured predecease, nowadays the banks provide a new case in point of the loan contract, the so-called exactly insured loan. The insured loan contract, underwritten by the person asking for a loan, is designed to give security to both the contractors, by ensuring that the loan is paid off in case of the borrowers' death.

It can alternatively assume two different technical forms, *joint* and *separated.*

- In the case of the *joint* insured loan contract, the debtor pays constant periodic anticipated amounts equal to P up to the expiration date n, or until his death, if he dies before the loan contract maturity. According to the principle of financial equilibrium between the two counterparties, supposing the insured's debt is one monetary unit, the following equation holds:

$$\sum_{k=0}^{n-1} P \; A_{x:\overline{k|}}^{\;\;\;1} = 1 , \tag{1}$$

where $A_{x:\overline{n|}}^{\;\;\;1}$ is the actuarial present value of a k-year pure endowment of 1 monetary unit paid in case of life of an insured aged x (Bowers et al., 1997).

- In this case, the payment P due at time s ($s=0,1,2,...,n-1$) incorporates:

- the capital instalment C_s;
- the interest instalment I_s on the residual debt D_{s-1};
- the actuarial premium covering the outstanding loan balance at the beginning of each period, if the death happens before maturity.

♦ In the case of the *separated* management of the insured loan, the borrower-insured supports the combination of the *n*-year term insurance and the basic loan contract.

Even if both the insured loan forms are technically practicable, the separated one appears to be the most commonly used.

An insured loan contract balances opposite needs of counter-parties: the *bank-lender* and the *borrower-insured*. In fact, in the event of *borrower-insured* predecease as regards the loan maturity, the *bank-lender* will have a non-performing loan, because he receives by the *insurer* the outstanding balance, calculated according to the amortization scheme. On the other hand, the debtor avoids heirs to take an obligation of extinction upon themselves.

1.2. Basic characteristics of a typical Italian insured loan. The borrower of a loan stipulates a new separate policy with the insurance company for the bank lender. The policy is the legal contract stating all the provisions of the insurance coverage. In particular, it guarantees the outstanding debt, if the death happens. The beneficiary is the bank-lender to whom the benefit will be paid when a claim is made on the policy. The coverage consists in paying residual debt plus the interest at the end of the death period, to the designated beneficiary (the bank-lender).

The face amount is the initial loan balance, usually subject to the inferior limit of €10.000,00 and to a variable superior limit.

The insurance coverage begins when the insurance application has been approved, the first premium has been paid and the loan to which the insurance applies is into effect.

The policy term of coverage is based on client's loan term and age. The issue age plus the term cannot exceed 75 years old. The contract can be stipulated to 18 years old until 60 years old, anyway the policy terminates at the expiration date following the insured's 75th birthday.

1.3. Amortization schedule and premium valuation. In what follows some details on the amortization schedule and on the premium valuation are reported in the *separated* management case.

Let us consider that a borrower (aged x) will repay 1 monetary unit to the lender in n years by means of instalment constant payments at the end of each year, at a given fixed annual rate of interest i. The constant annual payment amount and the outstanding loan balance at time h are respectively:

$$R = 1/a_{\overline{n}|i}\,, \qquad O_h = a_{\overline{n-h}|i}\,/\,a_{\overline{n}|i}\,, \tag{2}$$

where $a_{\overline{n}|i}$ represents the present value of n unitary annual payments at the end of each year, calculated using the annual interest rate i.

In the insured loan contract, the insurer will repay to the lender the obligations due by the borrower, if this one dies during the contract duration; at time h they consist in the outstanding balance at time $h-1$ plus the annual interest on this sum. The value B_h of the benefit payable at time h ($h=1,2,\ldots,n$) if the insured-borrower aged x at issue dies during the h-th year and the probability of this event are respectively:

$$B_h = \left(1/a_{\overline{n}|i}\right)\ddot{a}_{\overline{n-h+1}|i}\,, \qquad {}_{h-1/1}q_x\,, \tag{3}$$

where the two dots in the symbol $\ddot{a}_{\overline{n}|i}$ indicate the payments are due at the beginning of each year. The single premium of the insurance contract is:

$$S = \left(1/a_{\overline{n}|i}\right)\sum_{j=0}^{n-1} {}_{j/}a_{\overline{n-j}|i}\;{}_{j/1}q_x\,, \tag{4}$$

while the premium (constant in our case) payable at the beginning of the first m years ($0<m\leq n,\ 0\leq h<m-1$) if the insured is alive, is given by:

$${}_{/m}P_{x,h+1} = {}_{/m}P_x = \left(1/a_{\overline{n}|i}\right){}_{/m}\pi_x\,, \tag{5}$$

where ${}_{/m}P_{x,h+1}$ represents the $(h+1)$-th constant premium payable at the beginning of the h-th year, and:

$${}_{/m}\pi_x = \left(1/\ddot{a}_{\overline{x:m}|}\right)\sum_{j=0}^{n-1} {}_{j/}a_{\overline{n-j}|i}\;{}_{j/1}q_x\,. \tag{6}$$

2. The valuation framework in a portfolio case

2.1. The cash flow structure in deterministic hypotheses. We consider a portfolio of c homogeneous insured loans, each contract being issued on an insured aged x, with premiums payable at the beginning of each period (one year) while the insured is alive or up to the contractual duration n, and benefit payable at the end of the period of the insured's death, if this event occurs before n.

The benefit is the sum of the outstanding balance at the beginning of the year and the annual interest due to that amount.

Within a deterministic scenario for all the variables, let us indicate by k_x the curtate future lifetime of the insured aged x at issue.

In the case of a single premium payment, the flow at time h related to each insured results:

$$X_h = \begin{cases} \left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i} & h-1 \le k_x < h \\ 0 & k_x \ge h \end{cases}$$

$h = 1,2,...,n$ and with $X_0 = -S$.

In the case of anticipated annual payments, the flow at time h is given by the following scheme:

$$X_h = \begin{cases} -{}_{/m}P_{x,h+1} & k_x \ge h & 0 \le h \le m-1 \\ 0 & k_x \ge h & h \ge m \\ \left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i} & h-1 \le k_x < h & 1 \le h \le n \end{cases}$$

with $h = 1,2,...,n,\quad P_{n+1} = 0$ *and* $X_0 = -P_1$.

The generic cash flow connected to the entire portfolio, consisting in c homogeneous insured loan contracts, can be written as follows:

$f_0 = -c\,{}_{/m}P_{x,1}$ if h=0

$f_h = -{}_{/m}P_{x,h+1}n_h + \left[\left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i}\right](n_{h-1} - n_h)$ if *h=1,2,…,n*

in which n_h represents the number of survivors at time h.

2.2. The fair valuation of the insured loan portfolio. As usually assumed (cf. Coppola et al., 2005), we consider the probability space $\{\Omega,\Im,\wp\}$ originated by the two probability spaces $\{\Omega,\Im',\wp'\}$ and $\{\Omega,\Im'',\wp''\}$, referred respectively to the financial and the demographic events.

In particular, the σ-algebra $\Im$ is represented by the filtration $\{\Im_t\} \subset \Im$, with $\Im'_t \cup \Im''_t = \Im_t$, containing the information flow at time t about both the financial and the mortality events.

The market in which we operate is frictionless with continuous trading, no restrictions on borrowing and short sales, in which securities are perfectly divisible.

In what follows we indicate by:

- $\tilde{N}_h$ the random variable representing the number of survivors at time h belonging to the group of those, among the c initial insureds at time 0, living at time t;
- $v(t,h)$ the stochastic present value at time t of one monetary unit at time h;
- F_h the stochastic flow at time h;
- L_t the stochastic loss in t of the portfolio of c contracts;
- $K_{x,t}$ the random variable *curtate future lifetime* at time t of the insured aged x at issue.

On the basis of the market completeness hypotheses, we will write the stochastic loss at time t in its fair value form, replicating the stochastic flow F_h at time *h (h>t)* by a trading strategy consisting in

$$\tilde{N}_h\ {}_{/m}P_{x,h+1} + \left(\tilde{N}_{h-1} - \tilde{N}_h\right)\left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i}$$

units of Zero Coupon Bonds issued at time t and maturing at time h.

The stochastic value of the reserve results in:

$$U_t = \sum_{h=t+1}^{n}\left[-{}_{/m}P_{x,h+1}\ \tilde{N}_h + \left(\tilde{N}_{h-1} - \tilde{N}_h\right)\left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i}\right]v(t,h) \tag{7}$$

while its fair value is:

$$V_t = E\left\{\sum_{h=t+1}^{n}\left[-{}_{/m}P_{x,h+1}\ \tilde{N}_h + \left(\tilde{N}_{h-1} - \tilde{N}_h\right)\left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i}\right]v(t,h)/\Im_t\right\} \tag{8}$$

calculated under the opportune risk neutral probability measure $\wp$. Further considerations on this point will follow in Section 3.

We suppose the variables $K_{x,t}$ independent and identically distributed, the random variables F_j independent and identically distributed conditioning on the interest rate process and the independence of the two risk sources $K_{x,t}$ and $v(t,j)$.
Indicating by:

- $1_{\{K_{x,t}>h\}}$ the random variable indicator taking value *1* if the curtate future lifetime of the insured aged x at issue is greater than $t+h$, that is if the insured aged $x+t$ is alive at time $t+h$, and *0* otherwise;
- $1_{\{h-1<K_{x,t}<h\}}$ the random variable indicator taking value 1 if the insured dies between the ages $x+t+h-1$ and $x+t+h$, and *0* otherwise

on the basis of the conditional expectation calculus, we can write:

$$V_t = E\left\{\sum_{h=t+1}^{n}\left[-_{/m}P_{x,h+1}\ c\ 1_{\{K_{x,t}>h\}} + \left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i}\ c\ 1_{\{h-1<K_{x,t}<h\}}\right]v(t,h)/\Im_t\right\} = \quad (9)$$

$$= \sum_{h=t+1}^{n}\left[-_{/m}P_{x,h+1}\ c\ {}_tp_x\ {}_{h-t}p_{x+t} + \left(1/a_{\overline{n}|_i}\right)\ddot{a}_{\overline{n-h+1}|_i}\ c\ {}_tp_x\ {}_{h-1-t/1}q_{x+t}\right]E\left[v(t,h)/\Im_t\right]. \quad (10)$$

In formula (10) ${}_tp_x$ is the probability that an insured aged x at issue is alive at the age $x+t$ and ${}_{h-1/1}q_{x+t}$ is the probability that an insured aged $x+t$ dies during the h-th year of the contract. These probabilities are calculated according to the mortality law chosen to the aim of the better description of the phenomenon.

It's opportune to observe that no indications about mortality systematic deviations from the expected value come from the market, so that the obtained liability values are "marked to model" values.

By means of known operations with survival probabilities and recalling formulas (5) and (6), formula (10) can be written as follows:

$$V_t = \left(c / a_{\overline{n}|_i}\right)\ {}_tp_x \sum_{h=t+1}^{n}\left[\ddot{a}_{\overline{n-h+1}|_i}\ {}_{h-1-t}p_{x+t} - \left({}_{/m}\pi_{x,h+1} + \ddot{a}_{\overline{n-h+1}|_i}\right){}_{h-t}\,p_{x+t}\right]E\left[v(t,h)/\Im_t\right]. \quad (11)$$

3. The table risk

The insured loan in this particular case, so as all the mortality dependent contracts, are not *tradeable* in the market in the complete sense of the word, not existing a secondary market referable to this kind of products. Despite an increasing interest in the longevity bonds, these products are at the moment not sufficiently diffused for completely describing the insurance market, both from the mortality term structure point of view (the term contracts being extremely different and not all represented in the longevity bond offer) and for the liquidity aspects (Hari et al., 2007).

As a consequence, the market we refer to is incomplete concerning the demographic component and, for the practical fair valuation item, it gives no indications about the dynamic of the mortality measure. This circumstance reveals to be substantial in the light of the general criterion established by FASB2004 in the valuation technique issue, focusing on the connection between cash flows and products effectively traded in the markets as the basic principle. In the current valuation approach, we describe the demographic phenomenon using the expectation of its best estimate, considering the market neutral with respect to the two aspects, systematic and unsystematic, of the demographic risk (Ballotta et al., 2006).

The time horizons relating to contracts linked to the human life are often long enough to put in evidence the possibility of the contract mispricing and the cash flow incorrect valuations due to the wrong choice of the mortality description law. A sort of mortality hedging comes out from these considerations, realized by the best estimate of future mortality rates.

In the current section we consider this problem introducing the *table risk*, meaning the risk due to the randomness in the choice of the mortality rate set, analyzing it by means of the measure of its impact on the portfolio of insured loans. The calculation is placed in a scenario in which random stochastic rates of interest and random deviations of mortality are taken into account.

On the basis of the hypotheses in Section 2, we introduce the *table risk* considering the index RT_t representing its measure at time t, given by the variance due to the randomness in the choice of the mortality table T_t chosen for valuation at that time:

$$TR_t = Var\left[E\left(\boldsymbol{U}_t / T_t\right)\right] = Var\left[c\ {}_tp_x\left\{\sum_{h=t+1}^{n}\left[\ddot{a}_{\overline{n-h+1}|_i}\ {}_{h-1-t}p_{x+t} - \left({}_{/m}\pi_{x,h+1} + \ddot{a}_{\overline{n-h+1}|_i}\right){}_{h-t}p_{x+t}\right]E\left[v(t,j)/T_t\right]\right\}\right]. \quad (12)$$

RT_t expresses the variability of the reserve at time t due to the randomness in the choice of the mortality table, the effects of the other two risk components (stochastic interest rates and random deviations of mortality) having been averaged out, as in Di Lorenzo et al. (2002).

4. The mortality risk

The portfolio of insured loan is not most likely a large portfolio able to absorb the uncertainty connected to the random deviations of deaths from the expected values increasing the number of policies, in other words exploiting the pooling nature of the mortality risk (cf. Coppola et al., 2002). It is opportune to investigate the impact of this risk component on the portfolio valuations.

Let's consider the stochastic value of the insured loan portfolio mathematical provisions in the case of the single premium paid in $t = 0$. On the basis of formula 7 we write the following mortality risk measure quantified at time t:

$$MR_t = E\left[E\left[Var\left[U_t / \Im'_t\right] / T_t\right]\right] = E\left[E\left[Var\left[\sum_{h=t+1}^{n}\left[\left(N_{h-1} - N_h\right)\left(\ddot{a}_{\overline{n-h+1|}} / \ddot{a}_{\overline{n|}}\right)v(t,h)\right] / \Im'_t\right] / T_t\right]\right]. \tag{13}$$

The variance in formula (13) can be calculated observing that : $N_h = \sum_{j=1}^{c} X_h^j$

having indicated with X_h^j the variable assuming value 1 if the j-th insured aged x at issue and belonging to the initial c is living at the age $x+h$ (j = 1,2,…,c; $h = t+1, t+2,..., t+n$), and 0 otherwise. The first two moments of X_h^j are the same and also noting that the random variables X_h^j's are independent and identically distributed, we can write:

$$E(N_h) = cE\left(X_h^j\right) = c\ {}_tp_x\ {}_hp_{x+t},$$
$$Var(N_h) = cVar\left(X_h^j\right) = c\ {}_{t+h}p_x\ (1 - {}_{t+h}p_x).$$

In order to calculate the covariance, we have that:

$$E(N_h N_k) = c\ {}_{t+h}p_x + 2\sum_{j=1}^{c-1}\sum_{i=j+1}^{c} E\left(X_h^j X_k^i\right)$$

with $t<h<k$, getting in particular:

$$E\left(X_h^j X_k^i\right) = {}_{t+h+k}p_x$$

finally obtaining

$$E[N_h N_k] = c\ {}_{t+h}p_x\ [1 + (c-1)\ {}_kp_{x+t+h}].$$

Proposition 1: If N_h is the random variable "number of survivors at time h" belonging to the c initial insured, and if the indicator variables X_h^j are independent and identically distributed, the autocovariance function of N_h is expressed by the following equation:

$$\mathrm{cov}(N_h, N_k) = c\ {}_{t+h}p_x\ [1 + (c-1)\ {}_kp_{x+t+h} - c\ {}_{t+k}p_x] \tag{14}$$

with $h<k$.

5. The application

The items proposed in the paper are here numerically applied; setting the analysis into a defined scenario for the financial and the demographic description, at first we calculate the current values of an insured loan portfolio reserve, graphically representing the numerical values and remarking the different behavior in case of single and periodic premiums. The numerical evidence shows in particular that the number of periodic premiums influences the trend and the sign of the mathematical reserve fair value during the contract life.

A further analysis deepens the role of the mortality table chosen for valuation in the same matter in the general insured loan risk map. The graphical illustrations put in evidence the impact of the different tables and make interesting the quantification of the *table risk* referred to the considered business. The trend of the table risk is reported as function of the number of policies in portfolio and the time of valuation. Finally, the mortality risk impact on the insured loan portfolio liabilities is calculated and several illustrations and tables show its peculiarities.

5.1. The current value of the insured loan reserve. Before approaching the first application, some considerations are opportune. The insured loan, from the strictly insurance point of view, is a temporary life insurance with decreasing sums at risk, the benefits payable by the insurer having a decreasing amount. The expected annual costs, the so called natural premiums, depend, in addition to the death probabilities increasing with time, on the decreasing insured capital amount; hence their trend can be decreasing. As known, the constant premium payable during the whole duration of the contract is the weighted average of the natural premiums and the insurer applying this premium payment could result, in the first period of the contract, not fully financed. In this circumstance, seeing the policy in a contractual perspective, the insurer becomes a creditor, a position not acceptable in order to avoid problems such as rescissions of the contracts (Pitacco, 2000).

In the light of what precedes, the fair values of the reserve will not be always positive for any number of periodic premiums.

The numerical application we present is referred to a portfolio of c = 1000 unitary insured loans with the duration 10 years each, issued on a male policyholder aged 40. On the basis of a loan fixed rate equal to 4% and of the Italian male mortality table SIM 2002, the single premium and the constant premiums payable at the beginning of each year if the insured is alive and at most, respectively, for 7, 8, 9, and 10 years, are reported in Table 1.

Table 1. Premium amounts per unit insured

Number of payments	Premium amount
Single premium	0.00931
7 years	0.00149
8 years	0.00133
9 years	0.00121
10 years	0.00111

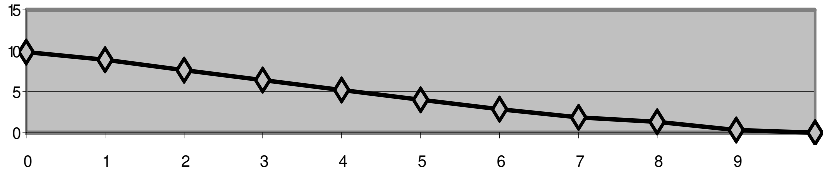
Note: The first column indicates the number of premium payments; unitary insured loan, x = 40, n = 10.

The background hypothesis on the stochastic scenario we choose for the description of the interest rate behavior in the general requirements of the fair value assessment, is the Cox-Ingersoll-Ross square root model, described by the SDE:

$$dr_t = -k(r_t - \gamma)dt + \sigma\sqrt{r_t}\,dB_t$$

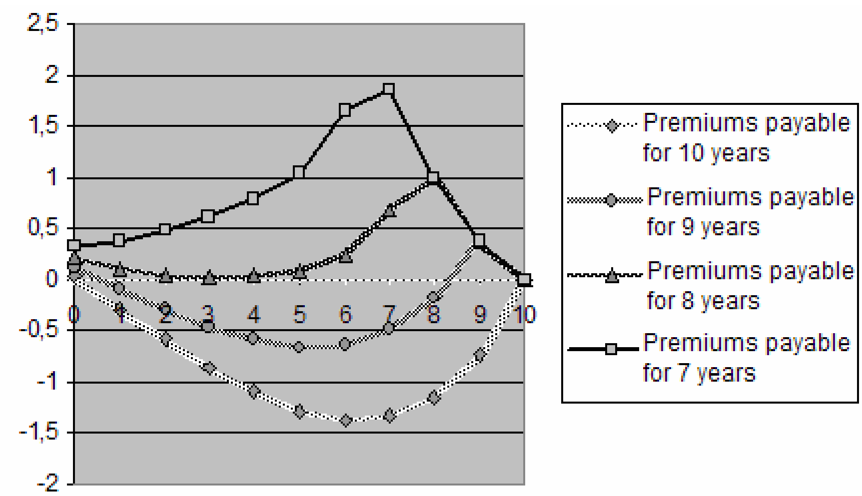
with k and σ positive constants, γ the long term mean and B_t a Brownian motion. According to Cocozza et al. (2007), we assign the following values to the parameters: $r_0 = 0.0172$, $\gamma = 0.0452$, $\sigma = 0.0052$.

The two graphs in Figures 1 and 2 show the reserves expressed in their current values, calculated at the beginning of each year in the case of a single premium (Figure 1) and of the periodic premiums listed in Table 1 (Figure 2).



Note: Portfolio of 1000 unitary insured loan policies, on male policyholders aged 40 at issue, $n = 10$.

Fig. 1. Current values of the portfolio reserve: single premium



Note: Portfolio of 1000 unitary insured loan policies, on male policyholders aged 40 at issue, $n = 10$.

Fig. 2. Current values of the portfolio reserve: periodic premiums

We can observe that the fair values of the reserve, though very low, present a rather regular trend in the reported four cases of periodic premiums, remaining negative in the cases of payments going on at most for 10 years and becoming completely positive in the case of payments going on for 7 years. In our example the maximum value of the premium payment duration assuring the reserve being always positive is 7 years.

The influence of the model used as the best estimation of the mortality in the fair value calculation is considered introducing, as an example of choice, three different sets of survival probabilities. In particular, we consider the following tables:

- the survival probabilities in the table SIM 2002 (Istat);
- the survival probabilities deduced by the Lee Carter model. This law is considered as a good description of the survival phenomenon, being able to correct itself year by year capturing the changes in the trend. As shown in De Feo (2005), this model furnishes an acceptable representation of the survival phenomenon in intervals of 8-10 years making it particularly appropriate in the case of the considered portfolio. The probabilities have been obtained by means of the tables of the parameters as reported in Cocozza et al. (2005);
- the survival probabilities deduced by the Weibull model:

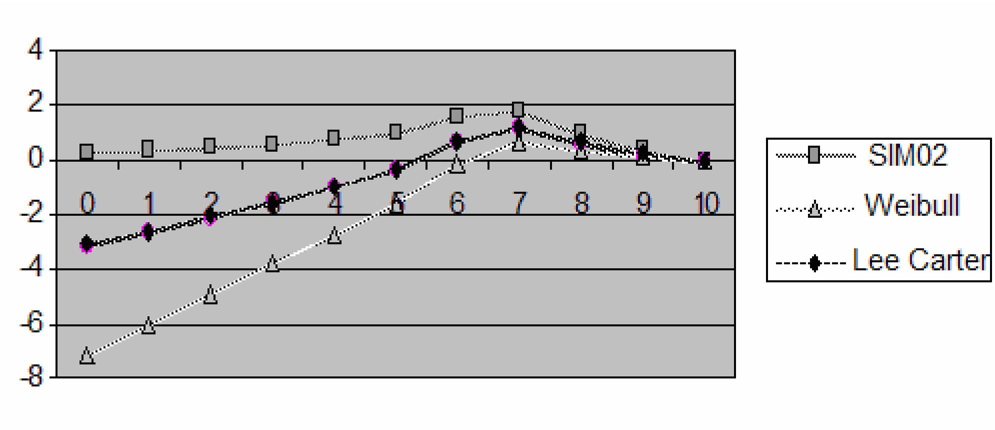
$$S(x) = exp\left[-(x/\alpha)^{\gamma}\right],\ x>0,$$

with $\alpha = 85.2$ and $\gamma = 9.15$, according to the realistic projection parameters proposed in Olivieri (1998).

The three mortality tables listed above, even though representing an example of application, are characterized by an increasing projection level, meaning increasing survival probabilities, taking into account that a fair description of the survival phenomenon evolution in time must contain the contribution of the betterment in the longevity.

Basing on the results above reported, we consider the insured loan with 7 annual premium payments. In Figure 3 the trends of the reserve current values in the case of premiums paid for years in case of life, evaluated on the basis of the three different mortality tables, are shown and the importance of a correct description of the future demographic scenario is evident.

Figure 3 shows that 7 premiums guarantee a positive reserve for the whole duration of the insured loans only in the case of the SIM 02; the use of tables characterized by an increasing projection leads towards lower fair reserves and, as a consequence, to shorter premium durations ensuring positive values for the reserve during the contract life. The problem of the choice of the "right" mortality description consists in its practical importance and the measure of its impact appears to be an interesting information from the business management point of view.



Note: Portfolio of 1000 unitary insured loan policies, on male policyholders aged 40 at issue with seven premiums, calculated on the basis of three different mortality models, $n = 10$.

Fig. 3. Current values in different mortality assumptions

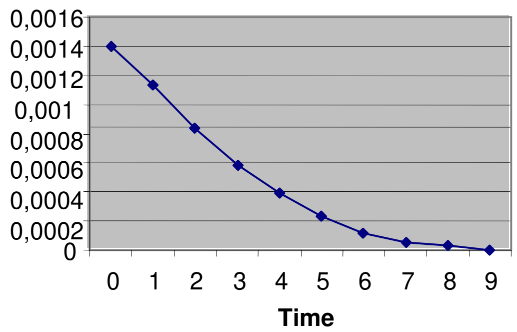
5.2. The table risk. We refer to two portfolios of $c = 1000$ and $c = 10$ unitary insured loan policies issued on individuals aged $x = 40$ with duration $n = 10$, in the case of a single premium payment calculated at the contractual interest rate $I = 0.04$ and on the basis of the SIM 2002 mortality table. As an example of application of formula 12 in Section 3, we describe the insurance scenario assigning the probabilities 0.2, 0.5 and 0.3 to the choice of respectively the SIM2002, the Lee Carter table and the Weibull table and calculate the table risk at time $t = 2$. The values of TR_2 are reported in Table 2.

Table 2. Table risk

	c = 1000	c = 10
Table risk	8.365594428	0.000836559

Note: Portfolio of c unitary insured loans. $x = 40$, $t = 2$, $n = 10$. Table risk on the portfolio reserve.

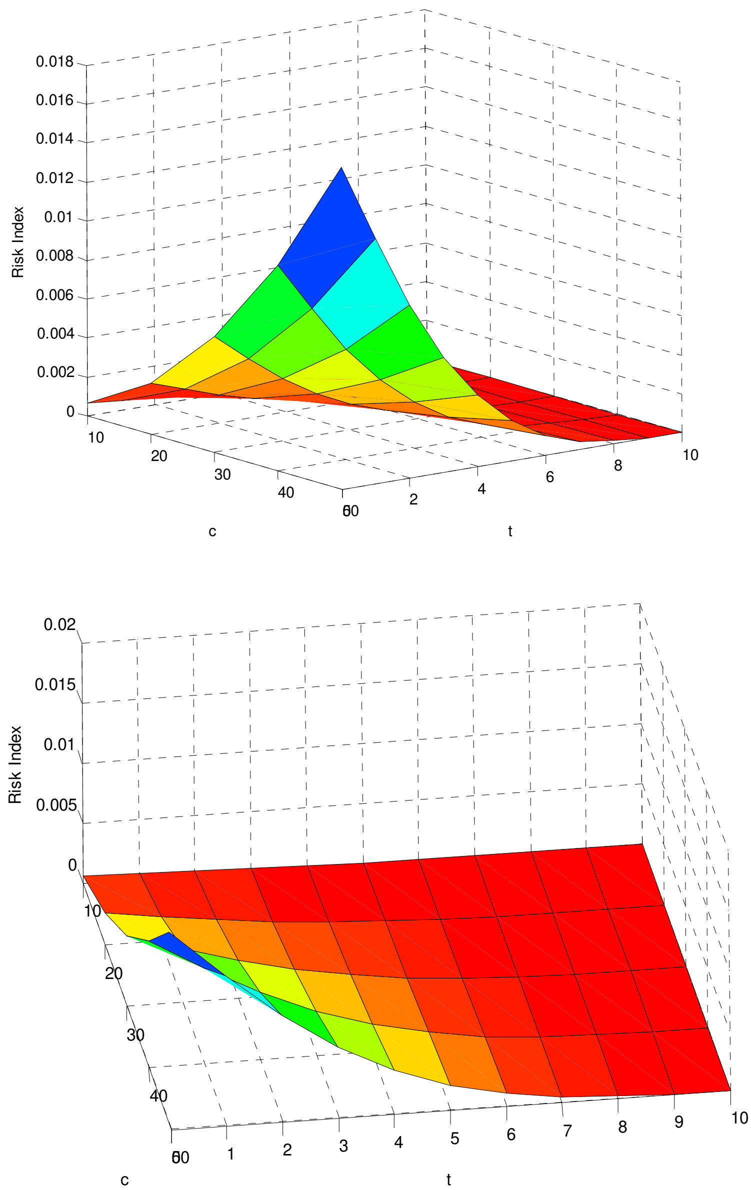
These values express the measures of the variability of TR_2 due to the randomness of the table used in the valuation, having averaged out the effects of the other two risk components (interest rates and mortality deviations) in the two portfolios at issue. In Figure 4 the decreasing trend of TR_2 is studied as function of the time of valuation in the case of a portfolio of $c = 10$ contracts.



Note: Portfolio of 10 unitary insured loans. $x = 40$, $n = 10$, $c = 10$. Table risk on the portfolio reserve.

Fig. 4. Table risk as function of time of valuation

Finally the behavior of the risk index is studied as function of the number of policies in portfolio ($c = 10, \ldots, 50$) and of the time of valuation, again in the case of the single premium payment. Figure 5 shows that the table risk increases with c and decreases with the time of valuation t.



Note: The risk index in a portfolio of c (c = 10,…,60) valued at time t (t = 0,…10), x = 40, n = 10.

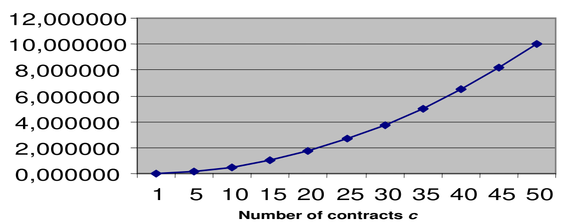
Fig. 5. Table risk as function of contracts and time of valuation

5.3. The mortality risk. The measure of the mortality risk in the case of a portfolio of c = 10 unitary insured loan policies with single premium, issued on insured aged 40 at issue and calculated in t = 2 are reported in Table 3, together with the values previously obtained for the table risk in order to compare them. The weight of the mortality risk, in particular compared to the correspondent table risk, is remarkable. Figure 6 shows the increasing trend of the mortality risk measure on the portfolio value as function of the number of contracts issued at time 0, quantified at time t = 2.

Table 3. Table and mortality risks

	c = 1000	c = 10
Mortality risk	3673.089	0.508368264
Table risk	8.365594428	0.000836559

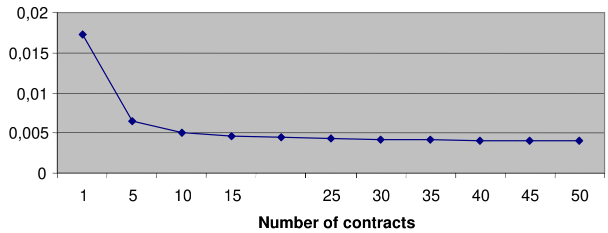
Note: Portfolio of c unitary insured loans. x = 40, t = 2, n = 10. Table risk and mortality risk on the portfolio reserve.



Note: Portfolio of c unitary insured loans. x = 40, n = 10, t = 2. Mortality risk on the portfolio reserve.

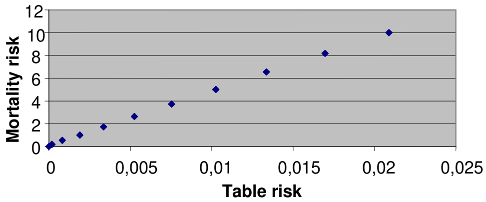
Fig. 6. Mortality risk as function of the number of contracts

The impact of the pooling effect on the mortality risk measure is visible in Figure 7, in which we report the mortality risk of the average mathematical provision per policy, as function of the number of contracts in portfolio. To very high initial values referred to low values of c, low and slowly decreasing values are in contrast for high values of c. In Figure 8 the 11 dots refer respectively to the 11 values of c and each of them gives contextually the two risk measures, mortality and table risks, calculated at time 2.



Note: Average reserve per policy. x = 40, n = 10, t = 2. Mortality risk.

Fig. 7. Mortality risk impact on the average reserve per policy



Note: Portfolio of c unitary insured loans, c=1, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50. x = 40, n = 10, t = 2. Mortality risk and table risk on the portfolio reserve for each value of c.

Fig. 8. Mortality and table risk for given values of c

Conclusions and further perspectives

The paper is dedicated to the insured loan considered in a fair valuation assessment. From the strictly actuarial point of view, the interest in this insurance contract is due to the behavior of the expected annual costs, the so called natural premiums, depending, in addition to the death probabilities increasing with time, on the insured capital amount, decreasing with time.

A relevant aspect of the analysis reveals to be the connection between the number of periodic premiums acceptable to the aim of positive reserve fair values and the mortality rate set used. The application shows that an increasing projection level in the survival probabilities produces lower reserve fair values, meaning the chance of negative values. The impact of the randomness in the choice of the table used to describe the evolution in time of the mortality phenomenon arises and the measurement tool proposed in the paper is applied in the numerical example. The mortality risk measure in the specific case of the portfolio at issue is proposed and the relevance of the mortality risk on a portfolio like this, not well diversifiable, is pointed out.

The item proposed and studied in the paper seems to have interesting fields of applications as, for example, the risk connected to the debtor default in a bank system. This will be the perspective in which the considerations developed in this paper will be addressed.

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