Noureddine Lahouel (Tunisia), Mokhtar Kouki (Tunisia)

GARCH option-pricing model with analytical solution when interest rate and risk premium change randomly

Abstract

We investigate the GARCH option-pricing model with random interest rate and random risk premium. The developed analytical solution generalizes the formula of Black & Scholes (1973) adjusted with skewness and kurtosis of standardized cumulative returns. In fact, the hypothesis of constant interest rate and risk premium considered in the GARCH option-pricing framework seems unrealistic. It becomes indispensable to solve this weakness of GARCH option-pricing models by lifting that hypothesis. The study of numerical and empirical effects of new state variables completes this work.

Keywords: GARCH, option-pricing, interest rate, risk premium, analytical approximation, performance, hedging.
JEL Classification: C22, C23, G12, G13.

Introduction

The family of GARCH models occupied an important place in the empirical asset pricing and the financial risk management. The success of GARCH models in fitting asset returns, and the failure of deterministic volatility models in fitting option prices, leaded the searchers to think about introducing the GARCH model in option-pricing framework. The first application in this domain was the one of Duan (1995) that established the foundation of option-pricing under GARCH. Duan, Ritchken & Sun (2005) derived an option-pricing theory extending the standard GARCH models to include jumps.

The success of GARCH models in option-pricing is due to the fact that the option-pricing theory is flexible, because it can be adapted to any GARCH specification. Also, the GARCH processes are linked up with stochastic volatility models[1]. Indeed, Nelson (1990) showed that univariate GARCH processes can be used to approximate stochastic volatility models. Duan (1996) generalized this established fact proving that the existing bivariate diffusion models are limits of the GARCH models[2].

In reality, the totality of studied GARCH specifications considered a constant risk free interest rate and constant risk premium. This hypothesis appears unrealistic, given the results of empirical studies elaborated to study the dynamics of interest rate and risk premium[3]. A natural extension can concern the violation of the two hypotheses. In this paper, we propose addressing a possible response to this question turning to the GARCH-M processes of Engle, Lilien & Robins (1987) to describe the new random variables (interest rate and risk premium). This new approach considers the GARCH option-pricing model when interest rate and risk premium are two stochastic state variables governed by a GARCH-M process, in which the relationship between the conditional mean and variance is nonlinear. The factors motivating the interest of this search are: on the one hand, the stochastic volatility models consider more than one state variable in the valuation process, while in the GARCH models framework we consider a single state variable (the underlying asset return). This can make up a limit of GARCH models comparatively to stochastic volatility models. On the other hand, the use of nonlinear processes to describe the new state variables is justified by several works like Lahouel (2007) for the description of interest rate, and Linton & Perron (1999) for the description of risk premium.

In this work, we estimate the dynamic relationships between the state variables, looking for a suitable formulation allowing studying the mechanisms of correlations between those variables in a multivariate framework. This procedure allows the use of VAR model with multivariate GARCH error (VAR-MGARCH). Indeed, the multivariate approach was expanded with the VAR models (Engle & Kroener, 1995). The multivariate approach of GARCH models becomes the more widespread, in financial econometric, to analyze the shock transmissions and to study the simultaneous variability of volatilities of different variables.

[1] Hull & White (1987), Stein & Stein (1991), Heston (1993), Bakshi, Cao & Chen (1997) and Schöbel & Zhu (1999) are examples of stochastic volatility option valuation models.

[2] Other searchers, like Heston & Nandi (2000), Duan, Ritchken & Sun (2005) and Christoffersen, Heston & Jacobs (2003), established the same observation.

[3] The behavior of the interest rate was examined by Kambhu & Mosser (2001), Jones (2003) and Lahouel (2007), among others. The risk premium was taken into account by Perron (1999) and Linton & Perron (1999), among others.

The calculation of option price in the GARCH model framework is carried out using numerical methods, which need an important time of application[1]. For this reason, the empirical studies on the GARCH option-pricing models are limited. In fact, the limitation can be caused by the difficulty of calculations or, perhaps, by the unique specification of the dynamic of underlying returns. Therefore, the search of alternative procedures becomes necessary. Hanke (1997) proposed an approximation of the GARCH option-pricing model by neural networks. A concurrent approach proposed by Heston & Nandi (2000)[2], consists in developing an analytical solution for the European options under GARCH. This method is based on the characteristic function of cumulative returns. It is an interesting approach, but it needs to resolve the characteristic function analytically, which not always possible with all GARCH specifications. More generally, Duan, Gauthier, Sasseville & Simonato (2006) developed an analytical approximation for the GARCH option-pricing model, using the Edgeworth expansion of the risk-neutral normal density function. We suggest, in the next, to adapt this approach to the proposed model. The obtained formula is similar to the one of Black & Scholes (1973), adjusted by skewness and kurtosis of standardized cumulative returns under GSIRSRP process.

The obtained analytic approximation allows getting hedging parameters favoring the effective neutralization of risks attached to option portfolios. Indeed, with the model of Black & Scholes (1973), we can talk only about a static signification, since this formula assumes that volatility and interest rate are constant. Such a hypothesis doesn't permit to have real hedging parameters but only some static comparisons on these variables. Also, with GARCH models, we can not obtain real measures of risk of optional portfolio. This fact is often allocated to the non-appropriation between the theoretical foundation of valuation model and the market reality.

We organize the rest of the paper as follows: in section one, we present the analytical framework of the GSIRSRP model that permits to result in establishing an approximated formula to valuate the European calls (section two). In the third section, we make the numerical study of the obtained model. The study of empirical performance will be treated in section four. In this section, we compare the GSIRSRP model and the GJR-GARCH of Glosten, Jagannathan & Runkle (1993) using data on S&P500 index observed on the CBOE. Finally, the last section concludes.

1. Specification of the GSIRSRP model

Since Black & Scholes (1973), the search on the modelling of the underlying asset return dynamic was very intensive. The more important developments in this sense concern especially continuous-time option-pricing models. Concerning the discrete-time option valuation, we always quote the GARCH models. Searches that investigated the empirical aspects of GARCH model performance are not numerous. The more known are Amin & Ng (1993), Engle & Mustafa (1992), Duan (1996), Hardle & Hafner (2000), Heston & Nandi (2000) and Christoffersen & Jacobs (2004). The proposed GARCH option-pricing models consider that the compound conditional returns, $R_t = Log(S_t \;/\; S_{t-1})$, where S_t is the underlying asset price at time t, can be modelled as:

$$R_t = r + \lambda + \varepsilon_t \tag{1}$$

with r is the risk-free interest rate, λ is the constant price of risk, $\varepsilon_t = \sqrt{h_t}\, z_t$ with $z_t \to N(0,1)$ and h_t is the conditional variance of returns, time varying and governed by a GARCH processes.

In this paper, with the new form of R_t, we obtain a vector of stochastic processes $y_t = (R_t, r_t, \lambda_t)'$ whose specification can employ a multivariate GARCH model (MGARCH) in order to capture the dynamic evolution of the variance-covariance matrix. There exist a large number of MGARCH specifications but, in the next, we will use the one of Engle (2002). We choose this model because it can be estimated from a sequence of univariate GARCH models. It is also relatively easy to implement for the practitioner or the professional, because it doesn't impose heavy hypothesis of distributions as in the MGARCH model of Bollerslev (1990), for instance.

Hypothesis 1. Discrete-time perfect market

We consider a discrete-time economy with possible bear sale of assets and with null transaction costs and taxes. The uncertainty is characterized by a probability space $(\Re, \tau, P)$, where $\Re$ is the set of real numbers, τ is a tribe and P is an objective probability measure. This probability space equipped with a filtration $(F_t)_{t=0,1,2,\ldots}$ that follows a standard Brownian movement.

We try to specify the price of a European call option with strike K, maturity T and writing on an underlying asset that doesn't pay any dividend. The price to specify is so function of a set of state variables of

[1] Among the numerical studies in existence, we quote Duan & Simonato (1999) and Ritchken & Trevor (1999).
[2] Su, Chen & Huang (2007) applied this approach on FTSE 100 index options.

$\Re_+$: the price the underlying asset, S_t; the risk premium, λ_t and the interest rate r_t. Consequently, the equation (1), describing the conditional returns, becomes:

$$R_t = r_t + \lambda_t + \varepsilon_{Rt}. \quad (2)$$

If we notice C_t the premium of a European call option, the knowledge of C_t is closely bound to the one of the vector of state variables $x_t = (S_t, r_t, \lambda_t)'$, and C_t will be accordingly more plausible as x_t. However, to define the process that describes the time varying of each state variable, we move towards the discrete-time processes of Markov. Consequently, we can consider a class of GARCH models. These processes seem thus to be more adapted to describe the volatility influence on the asset returns.

1.1. The model of the interest rate. In this model, we postulate that the interest rate curve is function of an alone state variable which is the short-term interest rate[1]. However, the short-term interest rates are characterized by a high persistence and by the presence of a conditional heteroskedasticity. Consequently, a realistic modelling of interest rates must consider these two essential features.

Hypothesis 2. The interest rates

The interest rates on the periods corresponding to option maturities are governed by a volatility process of the form:

$$r_t = \mu_r + \varepsilon_{rt};\ t = 1,2,... \quad (3)$$

with μ_r is a constant positive which is the mean level of interest rate, $\varepsilon_{rt} = \sqrt{h_{rt}}\, z_{rt}$; where z_{rt} forms a sequence of independently and identically distributed variables (iid) with unit variance.

In this model, where to a single underlying asset we correspond several sources of hazard that are not perfectly correlated (the dimension of the vector of white noises is not equal to the number of risky assets), the market is thus incomplete. However, the interest rate considered as a random variable is not clearly the cause of this market incompleteness. In fact, the uncertainty generated by the randomness of this variable can be diversified by a zero-coupon bond[2]. In this way, having assumed that the interest rate curve is function of a single state variable (the short-term interest rate, r_t), we consider that prices of bonds of different maturity present a correlated evolution when the short-term interest rate fluctuates. Otherwise, these prices can be likened to the rate of a short-term zero-coupon bond B_t. Consequently, B_t is specified by the instantaneous short-term interest rate during all the maturity of the bond, $\{r_i\}_{i\in[t,T]}$. In fact, since the sensitivity of the option price to the variations of the interest rate is relatively weak, we suppose that the term structure of interest rate depends on a single state variable as follows:

$$B_t = B_t(\tau, r_t). \quad (4)$$

To determine the analytical expression of B_t, we suggest the following proposition:

Proposition 1. Price of a zero-coupon bond

Let a zero-coupon bond with final flow equal 1 at expiry date T. If the short-term interest rate r_t is modelled by equation (3), the price $B_t(\tau, r_t)$ of the zero-coupon bond, which is function of the maturity τ and interest rate r_t, is given by:

$$B_t(\tau, r_t) = \exp[\tau(0.5 - \mu_r) - r_t]. \quad (5)$$

Proof: see Appendix A.

1.2. The risk-premium bound to the volatility of returns. The hypothesis of constant risk-aversion coefficient (λ), considered in the equation (1) of underlying asset returns, should be violated. However, the question that should be resolved is to specify the dynamic of this new state variable. In this respect, we postulate to stay always in the autoregressive model framework with error conditionally heteroskedastic. In fact, Lahouel (2006) considered the following hypothesis:

Hypothesis 3. The risk-premium of returns

The risk-premium bound to the volatility of underlying asset prices is governed by a GARCH process of the form:

$$\lambda_t = \mu_\lambda + \varepsilon_{\lambda t};\quad \varepsilon_{\lambda t} = \sqrt{h_{\lambda t}}\, z_{\lambda t} \quad (6)$$

with μ_λ is a constant parameter and $z_{\lambda t}$ is a sequence of random variables having an iid distribution.

The vector of state variables x_t can be likened to a VAR specification with multivariate GARCH error.

[1] Concerning the long-term interest rates, we can build for them a term structure of interest rates by considering a particular dynamic for the short-term interest rates. This question was studied in the work of Lahouel (2007).

[2] Merton (1973) generalizes the continuous-time model of Black & Scholes (1973) by assuming that the interest rate is stochastic. He calls in a state variable which is the zero-coupon bond price, whose increments are correlated with those of underlying price, to diversify the risk bound to stochastic character of the interest rate.

1.3. Econometric specification. Consider a vector y_t of stochastic processes with N components. We can consider that the vector y_t depends on the past information constructed by its delayed values until the instant t-1. In the following, giving the vector of unknown parameters θ, we can write:

$$y_t = \mu_t(\theta) + \varepsilon_t \quad \text{with} \quad \varepsilon_t = H_t^{1/2} z_t \tag{7}$$

with $\mu_t(\theta)$ is a vector of conditional mean of y_t, $z_t = iid$ and $H_t^{1/2}$ is a positive definite matrix of dimension NxN such as H_t is the conditional variance matrix of y_t. We can, following Engle (2002), decompose the matrix H_t as follows:

$$H_t = D_t R_t D_t \quad \text{with} \quad D_t = diag\left(h_{11t}^{1/2}, \dots, h_{NNt}^{1/2}\right), \tag{8}$$

where R_t is the matrix of time-varying conditional correlations given by:

$$R_t = (diagQ_t)^{-1/2} Q_t (diagQ_t)^{-1/2}, \tag{9}$$

where Q_t is a symmetric positive definite matrix of dimension NxN, defined by:

$$Q_t = (1 - \theta_1 - \theta_2)\overline{Q} + \theta_1 \varepsilon_{t-1}\varepsilon_{t-1}^{'} + \theta_2 Q_{t-1}. \tag{10}$$

$diagQ_t$ is a diagonal matrix containing the elements on the diagonal of Q_t, $\overline{Q}$ is the unconditional variance-covariance matrix, and θ_1 and θ_2 are non-negative parameters satisfying $\theta_1 + \theta_2 < 1$.

Concerning the estimation of the parameters, Engle & Sheppard (2001) and Engle (2002) use a Pseudo Maximum Likelihood (or Quasi Maximum Likelihood: QML) as an estimation method which is a two-step estimation approach. In the first step, we find the value of $\hat{\phi}$ that maximizes the function $L_V(\phi)$, given as:

$$L_V(\phi) = -\frac{1}{2}\sum_{t=1}^{T}\left\{2Log|D_t| + \varepsilon_t^{'}\varepsilon_t\right\}. \tag{11}$$

In the first estimation, we apply a univariate GARCH model to the conditional variance of each variable[1]. In this way, we obtain the volatility coefficients of each variable taken individually.

In the second phase of estimation, the volatility coefficients obtained in the first step, $\hat{\phi}$, are maintained constants and serve to condition the likelihood function $L_C(\phi, \delta)$ used to estimate the parameters of the correlations, $\hat{\delta}$. This function is given by:

$$L_C(\phi, \delta) = -\frac{1}{2}\sum_{t=1}^{T}\left\{Log|R_t| + \varepsilon_t^{'} R_t^{-1} \varepsilon_t - \varepsilon_t^{'}\varepsilon_t\right\}. \tag{12}$$

2. The analytical approximation formula of the European call

Jarrow & Rudd (1982) elaborated a general theoretical framework to develop an option valuation analytical approximation formula. They proposed a technique to approximate the probability distribution, called the true distribution, to an alternative distribution, called the approximating distribution. In the statistical literature, this technique is called the generalized Edgeworth series expansion. Using a similar approach, we can derive an analytical approximating formula to valuate the European call option under the GSIRSRP model.

Proposition 2. Analytical approximation of a European call option

Let $\rho_\tau = Log(S_T / S_t)$, the cumulative return shaving a mean m_τ and a standard deviation σ_τ. Let $u_\tau = (\rho_\tau - m_\tau)/\sigma_\tau$ the standardized cumulative returns. In the proposed GSIRSRP framework, the premium of a European call option, with strike price K and maturity T, can be approximated with the following formula:

$$C_{approx} = C_1 + \gamma_3 C_2 + (\gamma_4 - 3)C_3, \tag{13}$$

where

$$C_1 = S_t B_t \exp\left(m_\tau + 0.5\sigma_\tau^2\right) N(U + \sigma_\tau) - KB_t N(U), \tag{14}$$

$$C_2 = (1/6) S_t \sigma_\tau B_t \exp\left(m_\tau + 0.5\sigma_\tau^2\right) \cdot \cdot \left|(\sigma_\tau - U) n(U + \sigma_\tau) + \sigma_\tau^2 N(U + \sigma_\tau)\right|, \tag{15}$$

$$C_3 = (1/24) S_t \sigma_\tau B_t \exp\left(m_\tau + 0.5\sigma_\tau^2\right) \cdot \cdot \left|\left(U^2 - \sigma_\tau U + \sigma_\tau^2 - 1\right) n(U + \sigma_\tau) + \sigma_\tau^3 N(U + \sigma_\tau)\right|, \tag{16}$$

with $\tau = T - t$ and $U = \left(Log(S_t / K) + m_\tau\right)/\sigma_\tau$,

n(.) and N(.) represent, respectively, the density and the cumulative functions of a standard normal random variable. The terms γ_3 and γ_4 are, respectively, the skewness and kurtosis of the standardized cumulative returns u_τ.

Proof: see Appendix B.

[1] We adopt the univariate GJR-GARCH specification of Glosten, Jagannathan & Runkle (1993) in the rest of the work. In this model, the conditional variance is given by:
$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1} z_{t-1}^2 + \beta_3 h_{t-1} \max(0, -z_{t-1})^2$..

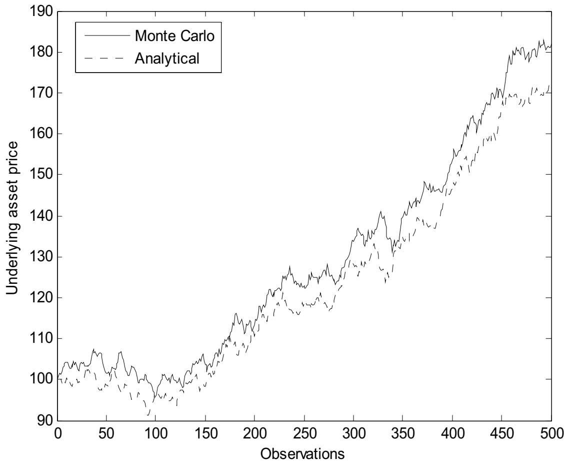
The application of this analytical approximation needs knowing the expressions of the four first moments of the cumulative return, ρ_τ, for all maturity T. The question is to obtain these moments as functions of the model parameters and the maturity. However, for all entire $m \in \{1, 2, 3, 4\}$, we have[1]:

$$E_t^P\left[\rho_\tau^m\right] = E_t^P\left[\left(Log(S_T / S_t)\right)^m\right]$$
$$= E_t^P\left[\left(\sum_{i=t+1}^{T} r_i + \sum_{i=t+1}^{T} \lambda_i + \sum_{i=t+1}^{T} \varepsilon_{Ri}\right)^m\right]$$
$$= E_t^P\left[\left(\tau\,\mu + \sum_{i=t+1}^{T} \delta\varepsilon_i\right)^m\right], \qquad (17)$$

with $\mu = \mu_r + \mu_\lambda$; $\delta = (1,1,1)$; $\varepsilon_i = H_i^{1/2} z_i$ and $\tau = T - t$.

3. Numerical performance of the GSIRSRP model

3.1. Simulation of underlying asset price. To simulate the process of returns and after that the underlying asset price, we need to fix values for the GSIRSRP model parameters[2]. We assume that μ_r = 1.37E-04 and μ_λ = -0.005, and we fix the same values for the conditional variance parameters with all state variables (return, interest rate and risk premium): β_{i0} = 1.0E-05, β_{i1} = 0.7, β_{i2} = 0.02 and β_{i3} = 0.005, for $i = R$ or r or λ. The parameters of shocks and correlations between state variables are fixed as: θ_1 = 0.1, θ_2 = 0.8, ρ_{Rr} = 0.01, $\rho_{R\lambda}$ = 0.7 and $\rho_{r\lambda}$ = -0.5. We obtain finally underlying asset prices fixing the initial value at S_0 = 100. The obtained results are reported in Figure1 below:



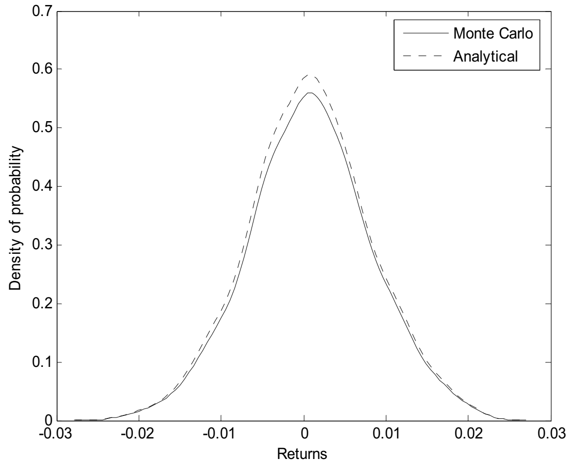


Fig. 1. Underlying asset price simulated with the GSIRSRP model

It is clear that the GSIRSRP model is a good approximation of the asset prices dynamic. Particularly, the analytical prices tend to confuse those of Monte Carlo.

3.2. Approximation of the European call price with the GSIRSRP model. As shown by equation (13), the analytical approximation of the European call option price is composed of a term similar to the Black & Scholes formula and two adjustment terms for the skewness and kurtosis of standardized cumulative returns. However, it is necessary to examine the influence of the parameters on the European calls.

3.2.1. Influence of skewness and kurtosis on the call prices. It is a matter of study to analyze the influence of skewness and kurtosis on the European call prices through a comparison with term C_1 of formula (14). We consider the difference between the underlying asset price S_t and the strike price K. We represent the difference of prices (adjusted price – non adjusted price (C_1)) as function of the moneyness and maturity. We ask the following question: *which is the influence of parameters γ_3 and γ_4 on the call option prices?* Using the parameter values fixed at the top, we obtain the following simulations:

---

[1] The analytical expressions of the moments under the GSIRSRP model are available from authors upon request: noureddinelahouel@yahoo.fr.

[2] The first-order conditional variance h_1 is equal to the stationary variance, given by: $\bar{h} = \beta_0 / (1 - \text{volatility persistence})$. The quantities ψ_0 and R_0 are nulls.

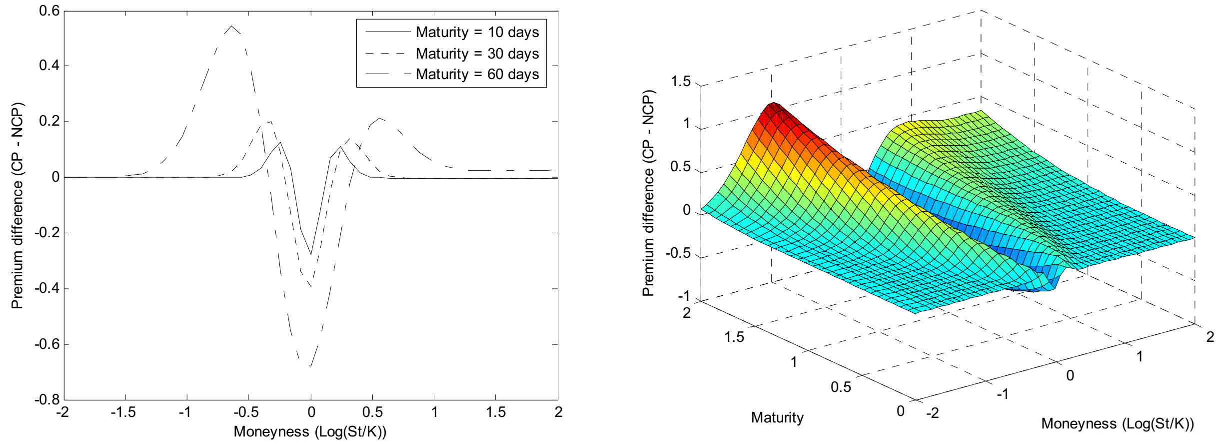


Fig. 2. Difference between corrected price, by skewness and kurtosis, and non corrected price

Figure 2 allows to come out again the following remarks:

- ♦ The difference between corrected and non-corrected prices is positive for the in-the-money and out-of-the-money European call options. The option prices are over-estimated, and the over-estimation is more important for the out-of-the-money options.
- ♦ The near-the-money call prices are under-estimated because the premium difference is negative.
- ♦ The difference, as function of the maturity level, is an increasing function for in-the-money and out-of-the-money options and a decreasing function for near-the money options.
- ♦ The premium difference becomes null for in-the-money and out-of-the-money options when the maturity decreases.

3.2.2. The typical European call price under the GSIRSRP model. It is possible now to provide an idea on the appearance of the option price given by GSIRSRP model as function of moneyness. Figure 3 draws the evolution of this price for different values of the maturity:

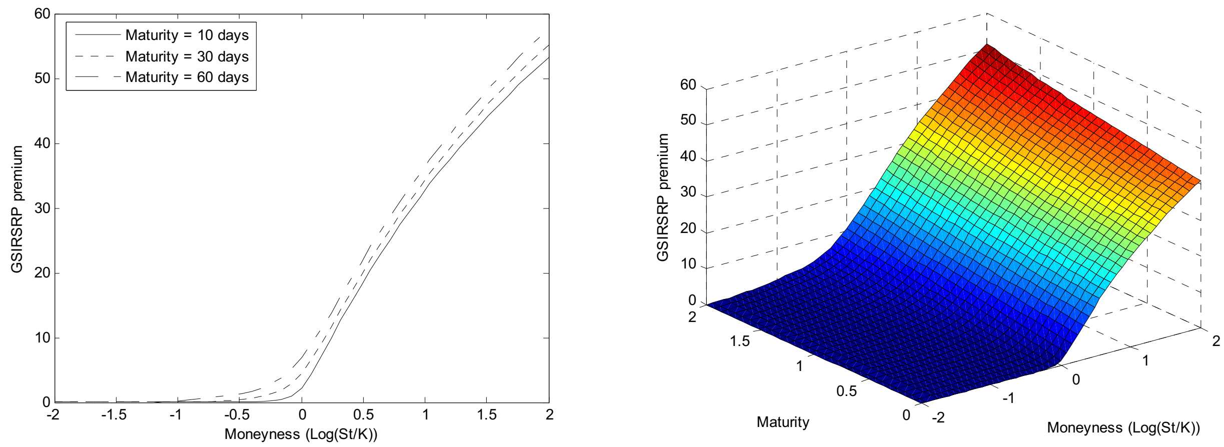


Fig. 3. Evolution of the European call option price under the GSIRSRP model

The examination of Figure 3 allows to deduce that the GSIRSRP European call price is an increasing function of moneyness and maturity.

4. Study of the GSIRSRP empirical performance on the CBOE

The objective of this section is to study the comparative empirical performance of the GSIRSRP valuation model and the GARCH model with constant interest rate and constant risk premium. The comparison of the valuation model performances allows improving the market dynamic knowledge because this analysis uses real market data. However, given that the performance of a valuation model depends on the correct specification of returns dynamic, the interest of this study consists in testing the internal accordance and coherence of the process parameters. It's a matter of establishing a relation between the conditional volatility parameters reflected implicitly in the option prices and the characteristics of the time-series of underlying asset returns. In this context, we can carry out the tests of model performance in valuation (static performance) and in hedging (dynamic performance). We focus the empirical study on the S&P 500 index options negotiated on the CBOE (Chicago Board Options Exchange).

4.1. Data. We use data on S&P500 index prices collected every 10 minutes covering the period from 06/19/2006 to 08/03/2006. The interest rate is approximated by the rate on the "Treasury-Bills" of the same maturity as the studied options (1 month, 2 months and 6 months). Table 1 summarizes the principal descriptive statistics of S&P 500 index returns during the period under study.

Table 1. Statistics of S&P500 index returns (06/18/2003 – 08/05/2003)

Mean	Standard deviation	Skewness	Kurtosis	Q(20)	Q²(20)	JB
0.0069	0.0173	-0.0123	6.6492	58.0940	523.2450	641.8673

$Q(20)$ and $Q^2(20)$ are statistics of autocorrelation Ljung-Box test of order 20 of returns and square returns, respectively. JB is the Jarque-Bera statistic testing the null hypothesis of normal distribution of returns. The descriptive analysis results don't allow to affirm that the empirical distribution of S&P500 index returns is assimilated to a normal distribution. This conclusion is justified by the values of skewness and kurtosis.

4.2. Estimation of the DCC-GARCH model. We estimate a DCC-GARCH model in which the variables R_t r_t and λ_t appear as endogenous variables. The estimated parameters are regrouped in Table 2.

Table 2. Estimation of structural parameters of GSIRSRP and GJR-GARCH models

Parameters		GSIRSRP			GJR-GARCH		
		τ<30	30≤τ≤60	60<τ≤180	τ<30	30≤τ≤60	60<τ≤180
μ_r		0.0412 (0.0005)	0.0406 (0.0003)	0.0408 (0.0002)	0.0412	0.0406	0.0408
μ_λ		-0.0401 (0.0005)	-0.0410 (0.0002)	-0.0420 (0.0001)	-0.0432 (0.0007)	-0.0431 (0.0007)	-0.0502 (0.0008)
β_0	Interest	1.34E-07 (1.27E-08)	1.49E-07 (2.66E-07)	1.77E-07 (2.08E-07)	- (-)	- (-)	- (-)
	Risk premium	1.55E-07 (3.12E-08)	1.84E-07 (2.62E-07)	1.44E-07 (2.20E-07)	- (-)	- (-)	- (-)
	Return	6.63E-08 (1.44E-10)	6.79E-08 (1.28E-09)	6.82E-08 (1.18E-09)	6.51E-08 (1.20E-08)	6.44E-08 (1.51E-08)	5.18E-08 (1.65E-08)
β_1	Interest	0.8055 (0.0273)	0.8122 (0.0396)	0.8110 (0.0378)	- (-)	- (-)	- (-)
	Risk premium	0.7992 (0.0304)	0.7984 (0.0466)	0.8074 (0.0276)	- (-)	- (-)	- (-)
	Return	0.7982 (0.0344)	0.8231 (0.0297)	0.8003 (0.0404)	0.7956 (0.0375)	0.8214 (0.0464)	0.8031 (0.0433)
β_2	Interest	0.1113 (0.0391)	0.1220 (0.0333)	0.1207 (0.0336)	- (-)	- (-)	- (-)
	Risk premium	0.1400 (0.0288)	0.1322 (0.0307)	0.1455 (0.0319)	- (-)	- (-)	- (-)
	Return	0.1398 (0.0225)	0.1388 (0.0258)	0.1421 (0.0588)	0.1202 (0.0228)	0.1143 (0.0300)	0.1185 (0.0202)
β_3	Interest	0.0100 (0.0828)	0.0101 (0.0888)	0.0102 (0.0876)	- (-)	- (-)	- (-)
	Risk premium	0.0140 (0.0622)	0.0139 (0.0568)	0.0137 (0.0600)	- (-)	- (-)	- (-)
	Return	0.0092 (0.0221)	0.0095 (0.0274)	0.0103 (0.0294)	0.0682 (0.0450)	0.0681 (0.0403)	0.0644 (0.0399)

Table 2 (cont.). Estimation of structural parameters of GSIRSRP and GJR-GARCH models

Parameters	GSIRSRP			GJR-GARCH		
	τ<30	30≤τ≤60	60<τ≤180	τ<30	30≤τ≤60	60<τ≤180
θ_1	0.1224 (0.0017)	0.1278 (0.0011)	0.1107 (0.0016)	- (-)	- (-)	- (-)
θ_2	0.8826 (0.0166)	0.8765 (0.0125)	0.8658 (0.0132)	- (-)	- (-)	- (-)
ρ_{Rr}	0.0207 (0.0019)	0.0192 (0.0022)	0.0199 (0.0011)	- (-)	- (-)	- (-)
$\rho_{R\lambda}$	0.7620 (0.0255)	0.7321 (0.0193)	0.7440 (0.0226)	- (-)	- (-)	- (-)
$\rho_{r\lambda}$	-0.5998 (0.0188)	-0.5983 (0.0176)	-0.6236 (0.0188)	- (-)	- (-)	- (-)
LogL	5346	5787	5776	6399	6200	6106

The GJR-GARCH model is estimated using the one-step maximum likelihood method. We fix for the interest rate the value obtained by the GSIRSRP model for the corresponding class of maturity.

The results obtained in Table 2 are comparable with the standard conclusions in the literature on GARCH processes. Estimations of the parameters β_1, β_2 and β_3 have approximately the same importance as in the existing literature. The volatility persistence deducted from the estimated parameters is also important in accordance with the literature. The standard errors (values in parentheses) indicate that estimated parameters are significant. The errors of the GSIRSRP model are smaller than those of the GJR-GARCH model except for the parameter β_1 for a maturity between 60 and 180 days. The LogL value (obtained at the optimum of the log-likelihood function) indicates that the GJR-GARCH model is more efficient than the GSIRSRP model. The risk premium is correlated negatively with the interest rate ($\rho_{r\lambda} < 0$) what is logical because the risk premium is decreasing with interest rate.

4.3. Study of GSIRSRP static performance in valuation. *4.3.1. In-sample approach.* This approach consists, given the parameters estimations, in testing the internal coherence of the models. However, we adopt the following method: we inject the structural parameter estimators, obtained at time *t*-1, in the volatility process. The repetition of this operation for all observations of the studied period gives us an implied volatility series. After, we calculate the variation of the obtained series, and the correlation between the reconstructed implied volatility variations and the underlying asset returns. The found value for the coefficient of variation is compared to the parameter β_1, and the second coefficient value is compared to zero because the correlation between volatility and returns is assumed to be null in the GARCH model framework. When we have a weak difference between these values, the considered specification for the model is correct.

Table 3. Internal coherence of correlation and variance parameters

	Parameters estimated from returns				Parameters obtained from implied volatility of returns			
	GSIRSRP		GJR-GARCH		GSIRSRP		GJR-GARCH	
Maturity	β_1	ρ	β_1	ρ	β_1	ρ	β_1	ρ
τ< 30	0.7982	0	0.7956	0	0.8663	0.0267	0.7654	-0.0263
30-60	0.8231	0	0.8214	0	0.9230	-0.0322	0.8213	-0.0239
60-180	0.8003	0	0.8031	0	0.8205	-0.0120	0.8472	-0.0270

Table 3 shows clearly the internal incoherence of parameters in the studied models. This incoherence appears evident for the implied correlation between shocks of variance and S&P 500 returns. But, comparing the values of the coefficient of the implied volatility in observed prices to the values of reconstructed variances lightens the incoherence. However, the GSIRSRP model proves more coherent especially when it is a matter of estimation of the implied variances.

4.3.2. Out-of-sample approach. In this approach, the important question is: given the structural parameter estimations and those of the variance, what is the degree of the model performance in the option price forecasts? To answer this question, we use parameters estimated at *t*-1 to calculate the price, at *t*, of the option belonging to the same category of maturity from which we estimated the parameters values. In fact, the market operators are not capable to know the parameter values instantaneously, for that they use the estimations calculated at the previous date. The underlying asset price used to calculate the option premium is that of the current instant. The calculated price will be compared to the one observed at the same time and having the same maturity. The difference between the two prices is the forecasting error of the adopted model. By repetition of the procedure for each observation during the studied period, for three types of maturity, we calculate the root of the mean-squared errors (RMSE), the mean of the standardized absolute errors (MSAE) and mean of the percentage of valuation errors (MPE). The obtained results are:

Table 4. Out-of-sample forecasting errors

	GSIRSRP			GJR-GARCH		
	τ<30	30≤τ≤60	60<τ≤180	τ<30	30≤τ≤ 60	60<τ≤180
RMSE	3.1687	2.7706	2.4726	6.9657	5.6743	5.4466
MSAE	2.243	2.5121	2.1108	6.6784	5.0562	4.7689
MPE	0.0273	0.0238	0.0224	0.0843	0.0851	-0.0805

The forecasting performance of the models improves with maturity, given that the errors turn down when maturity increases. The absolute valuation error (MSAE) is stable for all categories of maturity, which indicates the absence of valuation bias tied to the maturity. The mean of the percentage of valuation errors (MPE) shows that the two considered models tend to over-estimate the options for all maturity levels, except for the GJR-GARCH model for a maturity superior to 60 days. Concerning the comparison of the two models, it is clear that the GSIRSRP model is more efficient for option valuation. In fact, the introduction of a random interest rate and a random risk premium in the GARCH model improves its performance for return adjustment and option valuation. This result is expected because of the unrealistic hypothesis of constant interest rate and constant risk premium[1].

4.4. Dynamical performance in optional portfolio hedging. *4.4.1. The hedging strategy.* To test the capacity of the GSIRSRP model to hedge option positions, we adopt a delta-neutral hedging strategy. We study the case of a trader trying to hedge a short position on a call option (target option, C_t) of maturity τ and strike price *K*. The logic of hedging strategy forces the trader to take position on the underlying asset in a proportion $P_{t,S}$ at the date *t* to wrap up against the risk of the price. The delta-neutral hedging principle and the randomness of the volatility lead the trader to take position, at time *t*, on a call option $C_{t,1}$ in a proportion P_{t,C_1}. This call option has a maturity identical to the target option C_t, but with strike price $K_1 \neq K$. Furthermore, the randomness of the interest rate leads the trader to take position on a zero-coupon bond B_t, in a proportion $P_{t,B}$ at time *t*, to wrap up against the interest rate risk. The replication portfolio value of the target option at the date *t* is given by:

$$\Pi_t = P_{t,0} + P_{t,S}S_t + P_{t,C_1}C_{t,1} + P_{t,B}B_t \tag{18}$$

with $P_{t,0}$ is the remainder of liquidities (cash) to the trader at the date *t*. Different proportions can be obtained using the following formulae:

$$P_{t,0} = C_t - P_{t,S}S_t - P_{t,C1}C_{t,1} - P_{t,B}B_t, \tag{19}$$

$$P_{t,S} = \Delta_{t,S}(\tau, x_t, K) - P_{t,C1}\Delta_{t,S}(\tau, x_t, K_1), \tag{20}$$

$$P_{t,C_1} = \Delta_{t,V}(\tau, x_t, K) / \Delta_{t,V}(\tau, x_t, K_1), \tag{21}$$

$$P_{t,B} = \left(P_{t,C_1}\Delta_{t,r}(\tau, x_t, K_1) - \Delta_{t,r}(\tau, x_t, K)\right) / B_t. \tag{22}$$

4.4.2. Variance minimum of hedging portfolio. The variance minimum criterion of the hedging portfolio returns is borrowed from Nandi (1996). According to this logic, the hedging portfolio is periodically adjusted in order to neutralize the sensitivity to the first order variations (delta). In Table 5, we regroup the mean variances of hedging portfolio value changes on the studied period according to BS, GJR-GARCH and GSIRSRP models:

[1] Figures drawing the time-variation of calculated and observed call prices, as well as figures drawing the time-variation of the valuation relative errors, are available from author upon request: noureddinelahouel@yahoo.fr

Table 5. Mean variances of hedged portfolio value changes under the GSISRP, GJR-GARCH and BS models

	GSIRSRP			GJR-GARCH			BS		
Maturity	OTM	ATM	ITM	OTM	ATM	ITM	OTM	ATM	ITM
τ < 30	0.2156	0.3402	0.0673	0.2614	0.3698	0.0775	0.6596	0.8347	0.2865
30-60	0.0110	0.0012	0.0009	0.0127	0.0022	0.0019	0.0204	0.0061	0.0052
30-180	0.0092	0.0011	0.0008	0.0121	0.0012	0.0010	0.0176	0.0059	0.0029

Comparing the models, we note that the GSIRSRP model has the more little variance for all categories of maturity and moneyness[1]. The mean variance of the portfolio value changes decreases with the maturity. It also decreases with the moneyness, except when maturity is less than 30 days it is more important for ATM options. For the BS model and with maturity between 30 and 60 days, the ATM option portfolio variance is weaker. The reduction percentage of variance is more important when we compare GSIRSRP with BS, than when we compare GSIRSRP with GJR-GARCH.

Conclusion

In this paper, we proposed a European option-pricing model (GARCH-Stochastic Interest Rate, Stochastic Risk Premium: GSIRSRP) which generalizes the GJR-GARCH model of Glosten, Jagannathan & Runkle (1993). This model allows capturing, as well as negative skewness and excess of kurtosis of return distribution, the non stationary character of interest rate and risk premium. In fact, all existing GARCH option-pricing models consider constant interest rate and constant risk premium hypothesis. Several authors, such as Christoffersen & Jacobs (2004), suggest the violation of this unrealistic hypothesis. In this way, to reply to those prospects, the GSIRSRP model considers that all state variables are governed by GARCH processes. To take correlations between different state variables into account, we considered the multivariate GARCH model of Engle (2002) (DCC-GARCH) to describe the conditional variance-covariance matrix.

To have an analytical option-pricing formula, we used the general theoretical framework elaborated by Jarrow & Rudd (1982). This technique allows to approximate an unknown probability distribution (true distribution) using an alternative distribution (approximating distribution). However, doing the approximation of the standardized cumulative return distribution by the normal distribution, we managed to establish the analytical approximation of the European call option premium.

In the following, the path of the process describing the underlying asset price dynamic was confronted with the Monte Carlo simulation. Results show that the process calling in random interest rate and random risk premium seems well adapted to describe the financial return distribution, having a leptokurtocity. We also studied the numerical performance of the analytical approximation formula valuing the European call option, elaborated under GSIRSRP model. However, since this formula is corrected by skewness and kurtosis of cumulative returns, we tested the impact of the adjustment parameters on the European call prices. The two adjustment terms influence significantly European call values even if we vary the moneyness and the maturity.

The confrontation of the proposed GSIRSRP model with the one considering constant interest rate and constant risk premium showed the pertinence of these state variables in the underlying asset return process. In fact, the study of empirical and comparative performance of models, in option valuation and hedging, favors GSIRSRP model compared with the GJR-GARCH model of Glosten, Jagannathan & Runkle (1993).

Looking forward, the effort made in this article can be followed. We can quote some subsequent research prospects:

- The first development concerns applying the proposed GSIRSRP model on other data (for instance, data on the MONEP).
- The second direction can concern comparing the GSIRSRP model with continuous-time models like Bakshi, Cao & Chen (1997).
- Other alternative suggests including the transaction costs. However, the market reality shows the existence of transaction costs joined to operations between the traders. The considered difficulty concerns resolving the proposed models. It is therefore recommended to turn to numerical methods of resolution.
- Finally, it is important to consider applying the GSIRSRP model to other options. We can study the American options, the exotic options, the exchange options or barrier options. However, the majority of options exchanged on the markets are American options.

[1] The moneyness value is given by: $Log(S_t/K)$. For the ATM options, we considered moneyness between -0.05 and 0.05.

References

1. Amin K., and V. Ng (1993), "ARCH processes and option valuation", Manuscript, University of Michigan.
2. Bakshi G., C. Cao and Z. Chen (1997), "Empirical performance of alternative option pricing models", Journal of Finance, 52, 2003-2049.
3. Black F., and M. Scholes (1973), "The pricing of options and corporate liabilities", Journal of Political Economy, 81, 637-654.
4. Bollerslev T. (1990), "Modelling the coherence in short-run nominal exchange rates: a multivariate generalized ARCH model", Review of Economics and Statistics 72, 498-505.
5. Christoffersen P., Heston S. and Jacobs K. (2003), "Option valuation with conditional skewness", EFA 2004 Maastricht Meetings Paper No. 2964. Available at SSRN: http://ssrn.com/abstract=557079
6. Christoffersen P, and Jacobs K. (2004), "Which GARCH models for option valuation?", Management Science, 50, 1204-1221.
7. Duan J.C. (1995), "The GARCH option pricing models", Mathematical Finance, 5, 13-32.
8. Duan J.C. (1996), "Cracking the smile", Risk, 9, December, 55-59.
9. Duan J.C., Gauthier G., Sasseville C. & Simonato J.G. (2006), "Approximating the JGR-GARCH and EGARCH option pricing models analytically", Journal of Computational Finance, 9 (3).
10. Duan J.C, Ritchken P., and Sun Z. (2006), "Jump starting GARCH: Pricing and hedging options with jumps in returns and volatilities", FRB of Cleveland Working Paper No. 06-19 Available at SSRN: http://ssrn.com/abstract=479483.
11. Duan J.C., and Simonato J.G. (1999), "American option pricing under GARCH by a Markov chain approximation", working paper, Hong Kong University of Science and Technology.
12. Engle R.F. (2002), "Dynamic conditional correlation: a simple class of multivariate GARCH models", Journal of Business and Economic Statistics, 20, 339-350.
13. Engle R. and Kroner F.K. (1995), "Multivariate simultaneous generalized ARCH", Econometric Theory, 11, 122-150.
14. Engle R., Lilien D.M., and Robins R.P. (1987), "Estimating time varying risk premia in the term structure: the ARCH-M model", Econometrica, vol. 55, 391-407.
15. Engle R., and Mustafa C. (1992), "Implied ARCH models from options prices", Journal of Econometrics, 52, 289-311.
16. Engle R.F. and Sheppard K. (2001), "Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH", Mimeo, UCSD.
17. Glosten L.R., R. Jagannathan and D.E. Runkle (1993), "On the relation between the expected value and volatility of the nominal excess return on stocks", Journal of Finance, 48, 1779-1801.
18. Hanke M. (1997), "Neutral network approximation of option pricing formulas for analytically intractable option pricing models", Journal of Computational Intelligence in Finance, 5, 20-27.
19. Hardle W., and Hafner C. (2000), "Discrete time option pricing with flexible volatility estimation", Finance Stochastics, 4, 189-207.
20. Heston S. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", Review of Financial Studies, 6, 327-343.
21. Heston S. and S. Nandi (2000), "A closed-form GARCH option pricing model", Review of Financial Studies, 13, 585-626.
22. Hull J. and White A. (1987), "The pricing of options with stochastic volatilities", Journal of Finance, 42, 281-300.
23. Jarrow R.A. and Rudd A. (1982), "Approximate option valuation for arbitrary stochastic processes", Journal of Financial Economics, 10, 347-369.
24. Jones C. (2003), "The dynamics of stochastic volatility: Evidence from underlying and options markets", Journal of Econometrics, 116, 181-224.
25. Kambhu J., and Mosser P.C. (2001), "The effect of interest rate options hedging on term structure dynamics", FRBNY Economic Policy Review, December.
26. Lahouel N. (2006), "Evaluation des options avec prime de risque variable", www.tn.refer.org/CEAFE/Versions_definitives/CEAFE06_Lahouel2.pdf
27. Lahouel N. (2007), "La construction d'une structure par terme des taux d'intérêt à l'aide d'un modèle markovien à deux régimes avec erreurs GARCH", IFC4, Hammamet Tunisia.
28. Linton O., and Perron B. (1999), "The shape of the risk premium: Evidence from a semiparametric GARCH model", Cowles Foundation for Research in Economics, Yale University.
29. Merton R.C. (1973), "The theory of rational option pricing", Bell Journal of Economic and Management Science, 4, 141-183.
30. Nandi S. (1996), "Pricing and hedging index options under stochastic volatility: an empirical examination", Working paper, Federal Reserve of Atlanta.
31. Perron B. (1999), "Semi-parametric weak instrument regressions with an application to the risk-return trade-off", CRDE working paper 0199, University de Montréal.
32. Ritchken P., and Trevor R. (1999), "Pricing options under generalized GARCH and stochastic volatility processes", Journal of Finance, 54, 377-402.

33. Schöbel R. and Zhu J. (1999), "Stochastic volatility with an Ornstein-Uhlenbeck process: An extension", European Finance Review, 3, 23-46.
34. Stein E. and Stein J. (1991), "Stock price distributions with stochastic volatilities", Review of Financial Studies, 4, 727-752.
35. Su Y., Chen M. and Huang C. (2007), "An application of closed-form GARCH option pricing model on FTSE 100 option and volatility", National Taiwan University.

Appendix A. Proof of the proposition 1

In general, the price of zero-coupon bond is calculated by updating its expected value under a physical probability measure P as follows:

$$B_t(\tau, r_t) = E_t^P\left[\exp\left(-\sum_{i=t}^{T} r_i\right)\right] = E_t^P\left[\exp(-r_t)\exp\left(-\sum_{i=t+1}^{T} r_i\right)\right] = \exp(-r_t)E_t^P\left[\exp\left(-\sum_{i=t+1}^{T} r_i\right)\right].$$

If we put down $x = \sum_{i=t+1}^{T} r_i$, then x is a normal distribution with mean $E_t^P[x] = \tau\mu_r$ and variance $V_t^P[x] = \tau$, and after that:

$$E_t^P[\exp(-x)] = \int_{-\infty}^{+\infty} \exp(-x) f(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{\tau}} \exp(-x) \exp\left(-\frac{(x-\tau\mu_r)^2}{2\tau}\right) dx$$

$$= \exp[\tau(0.5-\mu_r)] \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{\tau}} \exp\left(-\frac{(x-\tau(\mu_r-1))^2}{2\tau}\right) dx.$$

Putting down $y = \dfrac{x-\tau(\mu_r-1)}{\sqrt{\tau}}$, we obtain: $E_t^P[\exp(-x)] = \exp[\tau(0.5-\mu_r)]$.

Appendix B. Proof of the proposition 2

From the equation $u_\tau = (\rho_\tau - m_\tau)/\sigma_\tau$, we can have:

$$S_T = S_t \exp(m_\tau + \sigma_\tau u_\tau).$$

It follows that:

$$S_T \geq K \iff u_\tau \geq -(Log(S_t / K) + m_\tau)/\sigma_\tau = -U .$$

Under the physical probability measure P, the call price is a martingale:

$$C(\tau, x_t, K) = B_t E_t^P[\max(S_T - K, 0)] = B_t E_t^P\left[(S_T - K)^+\right].$$

It follows that:

$$C(\tau, x_t, K) = B_t E_t^P\left[(S_t \exp(m_\tau + \sigma_\tau u_\tau) - K)^+\right] = B_t \int_{-U}^{+\infty} (S_t \exp(m_\tau + \sigma_\tau u_\tau) - K) f(u_\tau) du_\tau .$$

Following Jarrow & Rudd (1982), the true density of probability of u_τ can be approximated by:

$$f(u_\tau) \approx n(u_\tau) - \left(\frac{\gamma_3(F)}{3!}\right)\frac{d^3 n(u_\tau)}{du_\tau^3} + \left(\frac{\gamma_4(F)-3}{4!}\right)\frac{d^4 n(u_\tau)}{du_\tau^4},$$

where $F(.)$ is the cumulative function of the random variable u_τ , and since

$$\frac{d^3 n(u_\tau)}{du_\tau^3} = \frac{d^3}{du_\tau^3}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u_\tau^2}{2}\right)\right) = -(u_\tau^3 - 3u_\tau)n(u_\tau)$$

and $$\frac{d^4 n(u_\tau)}{du_\tau^4} = \frac{d^4}{du_\tau^4}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u_\tau^2}{2}\right)\right) = (u_\tau^4 - 6u_\tau^2 + 3)n(u_\tau)$$

we obtain finally:

$$f(u_\tau) \approx n(u_\tau)\left[1 + \frac{\gamma_3}{3!}\left(u_\tau^3 - 3u_\tau\right) + \frac{\gamma_4 - 3}{4!}\left(u_\tau^4 - 6u_\tau^2 + 3\right)\right].$$

It comes that:

$$C(\tau, x_t, K) = B_t \underbrace{\int_{-U}^{+\infty} (S_t \exp(m_\tau + \sigma_\tau u_\tau) - K) n(u_\tau) du_\tau}_{I_1} + \frac{\gamma_3}{3!} B_t \underbrace{\int_{-U}^{+\infty} (S_t \exp(m_\tau + \sigma_\tau u_\tau) - K)\left(u_\tau^3 - 3u_\tau\right) n(u_\tau) du_\tau}_{I_2}$$

$$+ \left(\frac{\gamma_4 - 3}{4!}\right) B_t \underbrace{\int_{-U}^{+\infty} (S_t \exp(m_\tau + \sigma_\tau u_\tau) - K)\left(u_\tau^4 - 6u_\tau^2 + 3\right) n(u_\tau) du_\tau}_{I_3} \quad .$$

To calculate the three terms I_1, I_2 and I_3, we must use the properties concerning the standard normal random variable. We obtain:

$$I_1 = \int_{-U}^{+\infty} S_t \exp(m_\tau + \sigma_\tau u_\tau) n(u_\tau) du_\tau - \int_{-U}^{+\infty} K n(u_\tau) du_\tau = S_t \exp(m_\tau) \int_{-U}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\sigma_\tau u_\tau - 0.5u_\tau^2\right) du_\tau - K \int_{-U}^{+\infty} n(u_\tau) du_\tau =$$

$$= S_t \exp\left(m_\tau + 0.5\sigma_\tau^2\right) \int_{-U}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-0.5(u_\tau - \sigma_\tau)^2\right) du_\tau - K \int_{-U}^{+\infty} n(u_\tau) du_\tau =$$

$$= S_t \exp\left(m_\tau + 0.5\sigma_\tau^2\right) \underbrace{\int_{-U}^{+\infty} n(u_\tau - \sigma_\tau) du_\tau}_{I_{11}} - K \underbrace{\int_{-U}^{+\infty} n(u_\tau) du_\tau}_{I_{12}} \quad .$$

Making the change of variable, $Y = u_\tau - \sigma_\tau$ for the first term I_{11}, and exploiting the standard normal random variable, we obtain:

$I_{11} = N(U + \sigma_\tau)$ and $I_{12} = N(U)$.

In the following:

$$I_1 = S_t \exp\left(m_\tau + 0.5\sigma_\tau^2\right) N(U + \sigma_\tau) - KN(U).$$

Using results of appendixes A and B, the same type of calculation allows writing I_2 and I_3 as:

$$I_2 = S_t \sigma_\tau \exp\left(m_\tau + 0.5\sigma_\tau^2\right)\left[(U - \sigma_\tau) n(U + \sigma_\tau) - \sigma_\tau^2 N(U + \sigma_\tau)\right],$$

$$I_3 = S_t \sigma_\tau \exp\left(m_\tau + 0.5\sigma_\tau^2\right)\left[\left(U^2 - \sigma_\tau U + \sigma_\tau^2 - 1\right) n(U + \sigma_\tau) + \sigma_\tau^3 N(U + \sigma_\tau)\right].$$