

# “Credit Risk, Credit Derivatives and Firm Value Based Models”

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## Credit risk, credit derivatives and firm value based models

### Abstract

Recently, the credit crisis, originating in the US, has affected many countries. Has the credit risk not been appropriately evaluated and anticipated? The financial market has now developed derivatives and structured financial products that have become extremely complex. This paper looks at the extent to which credit derivatives have been growing in the financial market and the way credit derivatives can be, from a practical point of view, evaluated. We favor, what has been called, the firm value-based model of evaluating credit risk. We present a method and an algorithm whereby the asset price of a firm and its debt capacity can be computed. This is done by relating the asset-value to the debt-capacity in an explicit and novel way. It has a sound theoretical foundation and suggests practical and new methods for evaluating credit risk.

**Keywords:** dynamic, decision, credit, risk, optimization, asset pricing, stochastic.

**JEL Classification:** C61, D81, G11, G12, G33.

### Introduction

In recent times there have been developed many instruments to transfer credit risk. These instruments are called credit derivatives. There have also been developed many models and methods to evaluate credit risk. They range from practical market methods to theory guided methods relying on firm value. In this paper, first, some well known instruments for transferring credit risk are discussed and then, second, firm value based models on evaluating credit risk are studied. Of course, there are other evaluation methods of credit risk, for example, intensity based models or credit rating models but here we want to focus on firm value based models. Those ones have a sound theoretical foundation and, they are based on the theoretical development of the 1970, put forward by Black and Scholes (1973) and Merton (1974). Further theoretical foundations of this approach can be found in Schönbucher (2003), Grüne and Semmler (2005) and Grüne, Semmler and Bernard (2006).

### 1. The relevance of credit derivatives

The market for credit derivatives was created in the early 1990s in London and New York and it is the fastest growing derivative market at the moment. Considering only the period between June 2001 and June 2004, the notional amounts outstanding in billions of US dollars were 695 and 4.477 respectively according to a recent survey of the Bank for International Settlements, Switzerland (see Table 3 in the Appendix). That is a growth of more than 500 per cent in only three years.

Participants in the market of credit derivatives can be divided into five major groups. Banks form the largest group with a fraction of about 47 per cent. The second largest group consists of insurances and re-insurances which cover about 23 per cent of the

market's notional outstanding. Other groups are hedge funds (8 per cent) and investment funds (5 per cent) as well as industrials (4 per cent) of different branches.

Table 1. Market share by instrument type (rounded numbers)

Instrument	Share (%)
Credit default swaps (including FtDs)	67
Synthetic balance sheet CDOs	12
Tranched portfolio default swaps	9
Credit-linked notes, asset repackaging, asset swaps	7
Credit spread options	2
Managed synthetic CDOs	2
Total return swaps	1
Hybrid credit derivatives	0,2

Source: Risk (Patel, 2002).

When one takes a look at the derivative market with respect to instrument types, one can see that credit default swaps (CDS) represent about 67 per cent of all transactions made in that field (see Table 1). A reason for this may be the standards for "plain vanilla" CDSs developed by the International Swaps and Derivatives Association (ISDA), leading to lower transaction costs and simplifying the whole business. Further types are discussed later in this paper.

Purposes for using credit derivatives are, as the types of instruments themselves, manifold. One can think of using credit derivatives as investments, for the credit risk management of bond portfolios, for hedging counterparty or country risk in isolated cases, as a funding opportunity for banks through the securitization of loan portfolios or for portfolio optimization for bond and loan portfolio managers. Referring to former times, a bank could only manage its credit risk at origination. During the whole lifetime of a loan the risk remained on the books until the loan was paid off or the obligor

defaulted. With the possibilities of these instruments, however, a bank and all the other previously mentioned institutions are able to conduct active risk management. Due to these features and the fact that credit is now a trading asset, the market of credit derivatives is growing and should keep growing in the future.

After giving a short introduction about the important role credit derivatives play in the financial world today, the terminology of the general credit derivative is described. Next we provide an overview of different types of credit derivatives, and give an example to show how they are used to conduct active risk management. In the second part of the paper, we talk about the idea of firm's value models and their connection to credit derivatives. In particular the Black/Scholes-Merton model and Moody's KMV are discussed.

As we have described previously<sup>1</sup>, the recent boom in US home mortgages has clearly demonstrated how the complex interplay of regulatory, tax, and other considerations can serve as catalysts, energizing an entire industry. Thus, the surge in US home prices can be seen as not only a function of housing needs and shifting demographics, but also as a consequence of the tax-deductibility, within certain limits, of mortgage interest payments for individuals, the lowering of default-risk standards on the part of issuing institutions, and the development of new financial instruments, thus enhancing the leverage potential available to consumers. The securitization of these loans and the resulting market for mortgage-backed products that has evolved to trade them, have taken on a life of their own.

Basel II and the regulatory structures thus spawned have significantly changed the environment in which, as Tavakoli (2001) has explained, banks and other large institutional players enter into agreements not only to gain economic advantage, but to manipulate the regulatory capital constraints under which they operate. This is similar to the position of a home buyer who purchases a home not because he either needs a place to live or believes that real estate is a good investment, but because home ownership will place him in a more favorable tax position. Since banks, at their core, are lending institutions that take credit risk, their goal is to seek an optimal return on regulatory capital.

Globalization, developments in the infrastructure of electronic trading and information systems, the rapid growth of several significant economies, and many other factors have created both the demand for and the deliverable capability of a wide variety of risk-related products. From sugar traders in Sao Paulo who are trying to preserve stability for their parent companies to currency desks attempting to maintain neutral positions in volatile times, the need for risk-related instruments is vast.

The breadth of this market notwithstanding, the whole spectrum of dangers that challenge companies ultimately filter down to what appears in their financial statements. On the one hand, and in the interest of having an efficient market, regulators cannot, to a large degree, require that banks have one set of lending rules for one industry and another set of rules for another. Thus, lending and investing regulations are largely based on the financial positions of companies. In other words, one measures the competency of a firm by how well its interaction with existing conditions boils down to cash flows, return on assets, etc., and keeps it solvent. The ability to securitize and trade default risk adds a powerful device to the financier's toolbox.

Credit derivatives are a relatively new class of financial contracts that allow market participants to separate risks associated with financial products from the actual ownership of those underlying products. For example, in the simplest case, a purchaser of a bond expects to receive an income stream from that bond, but understands that there is a risk that the issuer of the bond may default. In the case of default, there may be some recovery value.

A credit derivative allows the owner of the bond, or another party, to sell the default risk (to buy default protection) on the underlying security. Similarly, one can purchase the default risk (to sell default protection). In either case, there are several quantities that will critically determine the price of such a contract. First of all, there is the probability of default itself. Second of all, there is the expected recovery value, i.e., what amount may be realistically recovered in the event of default. Last, but not least, are technical marketplace considerations, e.g., liquidity issues, tax factors, etc.

Obviously, there is nothing new in the phenomena of financial default, and banks have been dealing with it for centuries. What is new is the emergence of an increasingly liquid market for contracts that are based on such phenomena. Thus, there is a corresponding demand for ways to assess these

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<sup>1</sup> See Semmler & Bernard (2007) for more material on this.

risks and value these contracts. However, in order to have a market, one needs counterparties, i.e., investors who are willing to pick up these products for their portfolios.

It is natural to wonder not only what price a purchaser of risk might demand, but also why he/she would be interested in the first place. Obviously, purchasers may wish to gain exposure to certain classes of investments to which they do not have access, receive high interest rates, etc. However, the two most important reasons are correlation between assets and leverage. Basic portfolio diversification requires that uncorrelated or negatively correlated products be included. Credit derivatives open up whole classes of investments which may meet these needs. Further, and perhaps most importantly, most credit derivatives are structured in ways that allow for significant leverage. This is particularly attractive to hedge funds, which constitute a growing industry themselves.

We can see how the growth of the market for credit derivatives shares many of the positive-feedback features with the mortgage-backed securities industry and the size of it is rapidly growing. Although the transactions are mostly off-balance sheet, making it difficult to obtain accurate data, the British Banker's Association estimated the credit derivatives market size reached \$5 trillion by the end of 2004, almost \$7 trillion by the end of 2005, and will be over \$8 trillion by the end of 2006. It is clear, therefore, that the sheer size of this market has serious implications for macroeconomic analysis. Not only because of the amount of money flowing through it, but because, as the market becomes more and more liquid, of the informational content implicit in its valuations.

## 2. Terminology

A useful definition of credit derivatives is formulated by Phillip Schönbucher (2003): "A credit derivative is a derivative security that has a payoff which is conditioned on the occurrence of a credit event. The credit event is defined with respect to a reference credit (or several reference credits), and the reference credit asset(s) issued by the reference credit. If the credit event has occurred, the default payment has to be made by one of the counterparties. Besides the default payment a credit derivative can have further payoffs that are not default contingent. This definition can be extended to include derivative securities whose payoffs are materially affected by credit events and derivatives on defaultable underlying securities."

For most derivatives, one can use the following definitions:

- ◆ **A** is the counterparty which receives a payment in the event of a default.
- ◆ **B** is the counterparty which has to make the payment in the event of a default.
- ◆ **C** is the reference credit.
- ◆ **Reference entity/reference credit** is the issuer of the reference obligation/reference credit asset whose default triggers the credit event.
- ◆ **Reference obligations/reference credit asset** is a set of assets issued by the reference entity.
- ◆ **Credit event/default event** occurs, e.g., for the following reasons:
  - bankruptcy;
  - failure to pay with certain requirements;
  - obligation default;
  - ratings downgrade below given thresholds (only for ratings-triggered credit derivatives).
- ◆ **Default payment** is the payment which has to be made by **B** if a credit event occurs.

## 3. Some types of credit derivatives

**3.1. Total Return Swaps (TRS).** In a total return swap (or total rate of return swap), **A** wants to change its entire payoff from a defaultable investment (e.g. a bond, denoted by **C** with the entire payoff **B** receives from its default-free Libor investment).

There are several effects appearing from this contract. First, **B** is long the **C**-bond without having paid for this investment. Therefore **B** normally has to put collateral (this can be the **C**-bond, which legally still belongs to **A**), depending on its creditworthiness. Second, **A** has hedged its exposure to the **C**-bond and bears a certain counterparty risk now, but which should be minimized because of the collateral.

Concerning the purpose of credit derivatives, **A** transmits the credit AND market risk of the **reference credit C** to **B** and ensures a risk-free Libor interest rate plus a certain spread, reflecting the creditworthiness of **B**.

In this agreement<sup>1</sup>, called a Total Rate of Return Swap (TRORS), the underlying asset is a bond with fixed interest rate and maturity. Here, one party, the payer,

<sup>1</sup> *Ibid.*

agrees to pay the coupon on the bond plus any increase in value in the price of the bond. The counterparty, the receiver, agrees to pay some floating rate, e.g.,

LIBOR, plus something extra, and any decrease in the price of the bond. The receiver also agrees to pay the par value of the bond in the case of default.

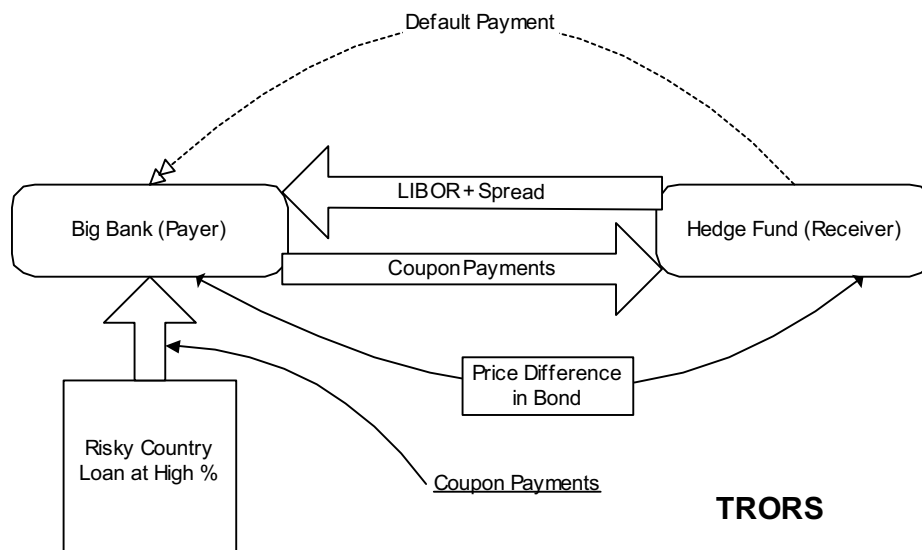


Fig. 1. Typical TRORS diagram

In the above illustration, it is presumed that the bank is interested, for business reasons, to make certain loans. However, it doesn't like the risk profile of the country. At the same time, the hedge fund does not have the cash to make such a loan, but is enticed by the interest rate. Since, and this is a key reason why these contracts are growing in popularity, the hedge fund may only need to put up 5% collateral, it is able to leverage its position. Realistically, the bank may not hope to collect much from the hedge fund in the case of default; however, it can keep, at least, the collateral and the recovery value (assuming the hedge fund reneges).

At the same time, we note that since the bank neither sells the bond, nor buys any asset, this is an off-the-balance-sheet transaction. Similarly, the hedge fund does normally have access to, and may not wish admit having exposure to, such asset classes.

**3.2. Credit Default Swaps (CDS).** The most important difference between a TRS and a CDS is the matter of isolating credit risk. While a TRS transfers both credit AND market risk (whereas a certain risk remains for counterparty A because only the risk of one of the reference credit is transferred, not the whole default risk), the default risk of this type of credit derivative is completely isolated.

In a credit default swap (or credit swap), B takes the default risk of A's defaultable asset and has to make a default payment of a credit event occurs. In exchange for this service, A pays a fee for the default protection.

With respect to the default payment, there are several possibilities. A physical delivery requires the delivery of the reference assets against a repayment at par. When a cash settlement is arranged, B has to pay the difference between the post-default market value and the face value of the asset. A default digital swap, in contrast, demands a fixed amount of money, agreed to at the time of the contract.

Since A and B can declare any asset of C they want, they are able to widen the range of assets so that the default risk of C is completely transferred.

According to the International Swaps and Derivatives Association (ISDA), the following information should be part of a CDS contract:

- ◆ the reference obligor and his reference assets;
- ◆ the definition of a credit event that is to be insured;
- ◆ the notional of the CDS;
- ◆ the start of the CDS;
- ◆ the maturity date;
- ◆ the credit default swap spread;
- ◆ the frequency and day count convention for the spread payments;
- ◆ the payment of the credit event and its settlement.

Here, a holder of a risky asset may desire protection from default risk for a period of time. For this protection, a premium is paid.

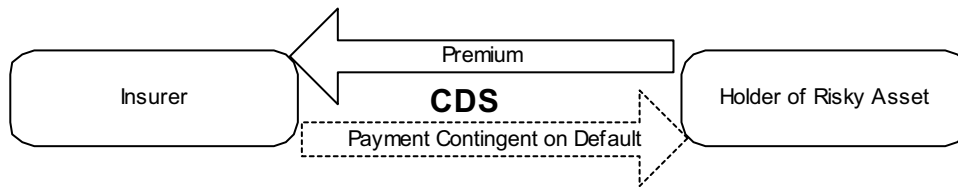


Fig. 2. Typical CDS diagram

With a CDS, the *credit event* that defines default is subject to negotiation. It can be any sort of event ranging from a change in a spread to even a political event in a foreign country.

The main point is that whatever the nature of the credit derivative is, the negotiable pricing criteria will be largely based on the probabilities of the defined credit event. As most of these contracts are written on financial transactions, modelers are particularly interested in those events that trigger traditional credit troubles, i.e., bankruptcy, change of rating, restructuring, etc. More specifically, in order to create effective strategies for the *control* of credit risk, understanding of the underlying dynamics, whether rooted in the firm's financial structure or in the statistical properties of the phenomena, is essential.

**3.3. Collateralized debt obligations (CDO).**

Collateralized debt obligations belong to the group of exotic credit derivatives as their construction is very special. The aim of a CDO is to securitize a complete portfolio of defaultable assets like a basket of bonds or loans in order to sell these securities and the credit risk of the assets with them.

The way a CDO is born looks like this: first, a portfolio of defaultable assets is set up and then sold to a company, exclusively created for this aim and denoted by special purpose vehicle (SPV). The second step is to divide the portfolio into several tranches in a way that every single tranche can be securitized and sold to investors with different risk aversions and different demands for the yield, respectively. The obligations sold by the SPV are collateralized by the underlying debt portfolio.

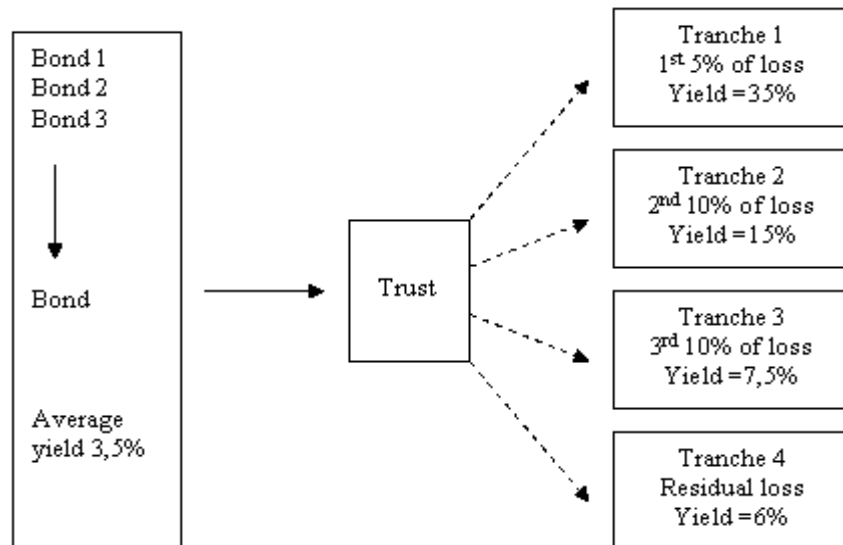


Fig. 3. Collateralized debt obligation

According to the tranche an investor owns, he or she is confronted with more or less risk. Assuming the investor has obligations of the first tranche, in the example given in Figure 1 he or she suffers already from the first 5 per cent of losses the portfolio gains. Since the risk of losing money is very high in this case, the yield one gets is correspondingly very high, too. Normally, it is a multiple of the average yield of the assets of the portfolio. An investor of the fourth tranche, in contrast, is only burdened with a loss when already more than 25 per cent of the assets of the portfolio defaulted. Of course, people

investing in this tranche have a lower expected yield than the average expected portfolio yield.

The Mortgage Backed Security (MBS) is a type of Collateralized Debt Obligations (CDO) in which the defaultable assets are mortgages instead of bonds or Credit Default Swaps. We note that the rise of this industry has exactly mirrored the housing boom as shown in the figure earlier. Let us go back to the MBSs and CDOs. MBSs operate by grouping together mortgages and using the interest income thus produced to compensate investors for taking

positions in which varying levels of default, called tranches, are guaranteed. The incentive to form such a structure is motivated primarily by the surplus cash generated – that not needed to compensate investors. The below diagram illustrates the basic mechanics and cash flows. A collection of mortgages with assumed default risk and assumed recovery value are grouped together. The monthly interest payments from those mortgages are income to the Special Purpose Vehicle (SPV) or “Trust.” Different tranches are assigned with attachment point in the following manner: If the number of defaults remains below the lower attachment point, the investors in that level simply collect the pre-arranged premium. However, once the percentage exceeds the lower attachment point, defaults are paid out of the capital posted by the investors of that tranche. Once the upper attachment point is reached, the next tranche takes over since the lower tranche is effectively exhausted. Investors in the MBS will demand compensatory interest commensurate with the assumed default risks and recovery values. These are paid from the interest income from the mortgages. The difference between the two cash flows is profit to the SPV investors. As long as it is profitable to construct these instruments, liquidity in the mortgage market will be limited only by the default probabilities, the recovery values, and the rates obtainable elsewhere.

**3.4. Example of a CDS with real quotes.** The following example should give an idea how a plain vanilla credit default swap looks in practice. Given the bid/offer quotes of a market maker in Table 2, one can think through several cases.

Table 2. Credit default swap quotes (basis points)

Company	Rating	Maturity			
		3 years	5 years	7 years	10 years
Toyota Motor Corp	Aa1/AAA	16/24	20/30	26/37	32/53
Merrill Lynch	Aa3/AA-	21/41	40/55	41/83	56/96
Ford Motor Company	A+/A	59/80	85/100	95/136	118/159
Enron	Baa1/BBB+	105/125	115/135	117/158	182/233
Nissan Motor Co.Ltd.	Ba1/BB+	115/145	125/155	200/230	244/274

Looking at Toyota, the market maker is prepared to buy three-year default protection for 16 basis points per year and sell three-year default protection for 24 basis points per year and so on (Hull, 2002).

Suppose that a bank had several hundred million dollars of loans outstanding to Enron and was concerned about its exposure. It could buy a \$100 million five-year CDS on Enron from the market maker for 135 basis points or \$1.35 million per year. This would shift part of the bank's Enron credit exposure to the market maker (Hull, 2002).

Another possibility could be an exchange in the bank's credit risk. If the bank is interested in shifting part of its credit risk to another industry, it could, for example, sell a five-year \$100 million CDS on Nissan for \$1.25 million per year while buying a similar CDS on Enron at the same time. The net cost of this strategy would be 10 basis points or \$100,000 per year. So the bank had changed part of its credit risk from Enron for a certain credit risk of Nissan. Due to the differences in these industries, one can say that the bank has diversified its credit exposure (Hull, 2002).

#### 4. Firm value based models and Black and Scholes

So far we have talked about the characteristics of credit derivatives in general and how to use them as tools for active risk management. Now we will focus on the pricing of credit derivatives using a specific modeling approach: the approach of firm's value models.

To be able to price credit derivatives, we have to know something about the default risk (credit risk) of the underlying asset. Modeling the default risk is the aim of credit derivatives pricing models such as intensity and spread-based models. Compared to those, firm's value models use a much more fundamental approach to valuing defaultable debt and in addition try to provide a link between the values of equity and debt of the firm.

Firm's value models assume a fundamental process  $V$ , denoting the total value of the assets of the firm that has issued the bonds in question.  $V$  is described as a stochastic process, influenced by the prices of all securities issued by the firm. A very important point of this type of model is that all claims on the firm's value are modelled as derivative securities with the firm's value as underlying.

Black and Scholes (1973) and Merton (1974) were the first people modeling credit risk with what we know today as a firm's value model. Modeling credit risk means modeling default probability. In their consideration a default could only occur at maturity of the debt, i.e. if the difference *firm value  $V$  minus outstanding debt* at maturity is negative, a default happens, otherwise the firm continues to exist. Merton (1974) explicitly treated the corporate

liability from the perspective of derivative pricing. We will come to another and more realistic view later. For further theoretical development see Schönbucher (2003), Grüne and Semmler (2005) and Grüne, Semmler and Bernard (2007).

As already mentioned above, the value  $V$  of the firm's assets is described as a stochastic process. Fischer Black, Myron Scholes and Robert C. Merton set up for  $V$  the following geometric Brownian motion:

$$dV = \mu V dt + \sigma V dW, \tag{1}$$

$$\text{or } \frac{dV}{V} = \mu dt + \sigma dW, \tag{2}$$

where the variable  $\sigma$  is the volatility of firm value, the variable  $\mu$  is the expected rate of return and

$W$  as a Wiener process (for the derivation of this equation see Hull, 2002, 11.3).

From now on in this model, the prices of both debt  $\bar{B}(V, t)$  and shares  $S(V, t)$  are functions of the firm's value  $V$  and the time  $t$ . What Black and Scholes (1973) and Merton (1974) did was a breakthrough. They showed that both equity and debt of the firm can be seen as derivative securities on the value  $V$  of the firm's assets. The payoff structure of these derivative securities looks like this ( $\bar{D}$  is the exercise price):

$$\bar{B}(V, t) = \min(\bar{D}, V), \tag{3}$$

$$S(V, t) = \max(V - \bar{D}, 0). \tag{4}$$

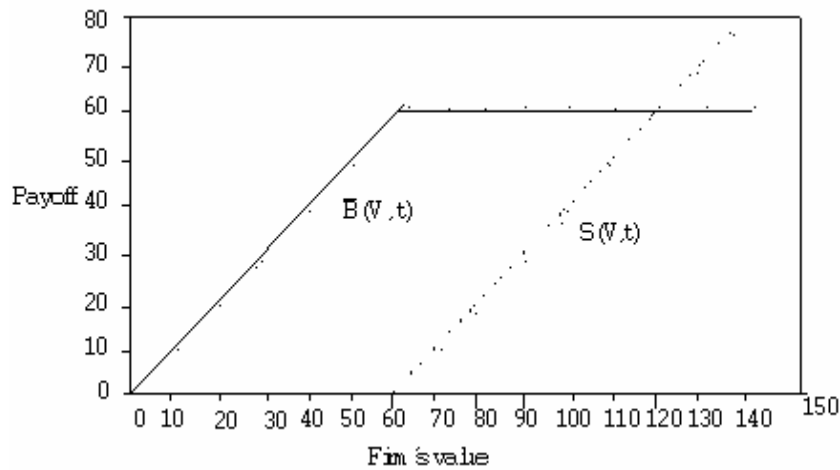


Fig. 4. Payoffs of shares and bonds at  $t = T$  for  $\bar{D} = 60$

As we are interested in pricing equity and debts of the firm and credit derivatives, respectively, we set up a risk-neutral portfolio by hedging one bond with  $\Delta$ -shares. The value of the portfolio is:

$$\Pi = \bar{B}(V, t) + \Delta S(V, t). \tag{5}$$

The change in value can be derived from Ito's lemma (see appendix 11A in Hull, 2002) and is:

$$d\Pi = d\bar{B}\Delta + dS = \left( \frac{\partial \bar{B}}{\partial t} + \frac{1}{2} \frac{\partial^2 \bar{B}}{\partial V^2} + \Delta \frac{\partial S'}{\partial t} + \frac{1}{2} \Delta \frac{\partial^2 S'}{\partial V^2} \right) dt + \left( \frac{\partial \bar{B}}{\partial V} + \Delta \frac{\partial S}{\partial V} \right) dV. \tag{6}$$

To be fully hedged and to have a predictable return, the number of shares must be:

$$S(V, t) = C^{BS}(V, t; \bar{D}, \sigma, r_f), \tag{9}$$

$$\Delta = - \frac{\partial \bar{B} / \partial V}{\partial S / \partial V}. \tag{7}$$

$$C^{BS}(V, t; \bar{D}, \sigma, r_f) = VN(d_1) - e^{-r_f(T-t)} \bar{D} N(d_2), \tag{10}$$

This leads to the well-known Black-Scholes partial differential equation:

$$\text{where } d_1 = \frac{\ln(V/\bar{D}) + \left( r_f - \frac{1}{2} \sigma^2 (T-t) \right)}{\sigma \sqrt{T-t}} \tag{11}$$

$$\frac{\partial}{\partial t} S + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2}{\partial V^2} S + rV \frac{\partial}{\partial V} S - \Gamma S = 0. \tag{8}$$

$$\text{and } d_2 = d_1 - \sigma \sqrt{T-t}. \tag{12}$$

Now we can compute the value of a share with the Black-Scholes formula  $C^{BS}$  for a European call option on  $V$ . The expiry date is denoted by  $T$ , the exercise price by  $\bar{D}$ , the underlying volatility by  $\sigma$  and the interest rate by  $r_f$ :

Note that in the risk neutral case the  $V$  in eq. (10) refers to the current value of the firm, but of course it is determined by the discounted future income stream of the firm. Yet in the risk free case we can have

$$S(V, t) = C^{BS}(V, t; \bar{D}, \sigma, r_f) = e^{-r_f(T-t)} (VN(d_1)e^{r_f(T-t)} - \bar{D}N(d_2)).$$

A Gauss computer program for the above evaluation of corporate debt from the perspective of derivative pricing is available<sup>1</sup>. Schönbucher (2003) extends the model by also taking into account a safety covenant acting as a default barrier. He also introduces bankruptcy cost, and a time varying interest rate, following a Brownian motion, for example, co-varying with stock market shocks. The firm value approach tries to model the whole obligor at once through linking the debt and equity with a hedge. A large and important disadvantage of the model is that one does not observe the process  $V$  with its driving factors.

### 5. Computing firm value and creditworthiness

In Grüne and Semmler (2005) the firm value is derived from an intertemporal behavior of firms. There, however, only for the deterministic case. Yet, in Grüne, Semmler and Bernard (2007) the stochastic case is also considered.

We give a formal presentation of the deterministic model. We can say in the bilateral contract between a creditor and debtor there are two problems involved. The first pertains to the computation of debt and the second to the computation of the debt ceiling. The first problem is usually answered by employing an equation of the form

$$\dot{B}(t) = \theta B(t) - f(t), \quad B(0) = B_0,$$

where  $B(t)$  is the level of debt<sup>2</sup> at time  $t$ ,  $\theta$  is the interest rate determining the credit cost, and  $f(t)$  is the net income of the agent. The second problem can be settled by defining a debt ceiling such as

$$B(t) \leq C, \quad (t > 0)$$

or less restrictively by

$$\sup_{t \geq 0} B(t) < \infty$$

or even less restrictively by the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\theta t} B(t) = 0. \tag{13}$$

The ability of an obligator to service the debt, i.e. the feasibility of a contract, will depend on the obligator's source of income. Along the lines of

intertemporal models of borrowing and lending<sup>3</sup> we model this source of income as arising from a stock of capital  $k(t)$ , at time  $t$ , which changes with the investment rate  $j(t)$  at time  $t$  through

$$\dot{k}(t) = j(t) - \sigma(k(t)), \quad k(0) = k_0. \tag{14}$$

In our general model both the capital stock and the investment are allowed to be multivariate. As debt service we take the net income from the investment rate  $j(t)$  at capital stock level  $k(t)$  minus some minimal rate of consumption<sup>4</sup>. Hence

$$\dot{B}(t) = \theta B(t) - f(k(t), j(t)), \quad B(0) = B_0, \tag{15}$$

where  $\theta B(t)$  is the credit cost. Note that the credit cost is not necessarily a constant factor (a constant interest rate). We call  $B^*(k)$  the creditworthiness of the capital stock  $k$ . The problem to be solved is how to compute  $B^*$ .

If there is a constant credit cost factor (interest rate),  $\theta = \frac{H(B, k)}{B}$ , then, it is easy to see,  $B^*(k)$  is the present value of  $k$  or the asset price of  $k$ :

$$B^*(k) = \text{Max}_j \int_0^\infty e^{-\theta t} f(k(t), j(t)) dt - B(0) \tag{16}$$

s.t.

$$\dot{k}(t) = j(t) - \sigma(k(t)), \quad k(0) = k_s, \tag{17}$$

$$\dot{B}(t) = \theta B(t) - f(k(t), j(t)), \quad B(0) = B_0. \tag{18}$$

The more general case is, however, that  $\theta$  is not a constant. As in the theory of credit market imperfections we generically may let  $\theta$  depend on  $k$  and  $B$ , see below<sup>5</sup>. Employing a dynamic model of the firm<sup>6</sup> we can use the following net income

<sup>3</sup> Prototype models used as basis for our further presentation can be found in Blanchard (1983), Blanchard and Fischer (1989) or Turnovsky (1995).

<sup>4</sup> In the subsequent analysis of creditworthiness we can set consumption equal to zero. Any positive consumption will move down the creditworthiness curve. Note also that public debt for which the Ricardian equivalence theorem holds, i.e. where debt is serviced by a non-distortionary tax, would cause the creditworthiness curve to shift down. In computing the "present value" of the future net surpluses we do not have to assume a particular interest rate. Yet, in the following study we neither elaborate on the problem of the price level nor on the exchange rate and its effect on net debt and creditworthiness.

<sup>5</sup> The more general theory of creditworthiness with state dependent credit cost is provided in Grüne, Semmler and Sieveking (2004). Note that instead of relating the credit cost inversely to net worth, as in Bernanke, Gertler and Gilchrist (1998), one could use the two arguments,  $k$  and  $B$ , explicitly.

<sup>6</sup> The subsequent model can be viewed as a standard RBC model where the stochastic process for technology shocks is shut down and technical change is exogenously occurring at a constant rate.

<sup>1</sup> Available upon request.

<sup>2</sup> Note that all subsequent state variables are written in terms of efficiency labor along the line of Blanchard (1983).

function that takes account of adjustment investment and adjustment cost of capital.

$$f(k, j) = k^\alpha - j - j^\beta k^{-\gamma}, \quad (19)$$

where  $\sigma > 0, \alpha > 0, \gamma > 0$  are constants<sup>1</sup>. In the above model  $\sigma > 0$  captures both a constant growth rate of productivity as well as a capital depreciation rate. Blanchard (1983) used  $\beta = 2, \gamma = 1$  to analyze the optimal indebtedness of a firm (see also Blanchard and Fischer, 1989, Chap. 2).

The maximization problem (16)-(18) can be solved by using the necessary conditions of the Hamiltonian for (16)-(17). Thus we maximize

$$\text{Max}_j \int_0^\infty e^{-\theta t} f(k(t), j(t)) dt$$

s.t. (17).

The Hamiltonian for this problem is

$$H(k, x, j, \lambda) = \max_j H(k, x, j, \lambda)$$

$$H(k, x, j, \lambda) = \lambda f(k, j) + x(j - \sigma k)$$

$$\dot{x} = \frac{-\partial H}{\partial k} + \theta x = (\sigma + \theta)x - \lambda f_k(k, j).$$

We denote  $x$  as the co-state variable in the Hamiltonian equations and  $\lambda$  is equal to 1<sup>2</sup>. The function  $f(k, j)$  is strictly concave by assumption. Therefore, there is a function  $j(k, x)$  which satisfies the first order condition of the Hamiltonian

$$f_j(k, j) + x = 0, \quad (20)$$

$$j = j(k, x) = \left(\frac{x-1}{k^{-\gamma} \cdot \beta}\right)^{\frac{1}{\beta-1}} \quad (21)$$

and  $j$  is uniquely determined thereby. It follows that  $(k, x)$  satisfy

$$k = j(k, x) - \sigma k, \quad (22)$$

$$x = (\sigma + \theta)x - f_k(k, j(k, x)). \quad (23)$$

The isoclines can be obtained by the points in the  $(k, x)$  space for  $\beta = 2$ , where  $k = 0$  satisfies

$$x = 1 + 2\sigma k^{1-\gamma} \quad (24)$$

and where  $x = 0$  satisfies

$$x_{\pm} = 1 + \vartheta k^{1-\gamma} \pm \sqrt{\vartheta^2 k^{2-2\gamma} + 2\vartheta k^{1-\gamma} - 4\alpha\gamma^{-1}k^{\alpha-\gamma}}, \quad (25)$$

where  $\vartheta = 2\gamma^{-1}(\sigma + \theta)$ . Note that the latter isocline has two branches.

If the parameters are given, the steady state – or steady states, if there are multiple ones – can be computed and then the local and global dynamics studied. We scale the production function by  $\alpha$ <sup>3</sup>.

There is another solution technique which allows one to solve for firm value by using a dynamic programming approach. The alternative solution method uses the Hamilton-Jacobi-Bellman (HJB) equation.

In this appendix we present the solution technique of how to find the solution of the HJB-equation. We describe an algorithm which enables us to compute the asset price of the firm for the HJB equation of a type such as (1) which will give us the present value borrowing constraint. We show how one can explicitly compute firm value using modern dynamic decision theory.

The HJB-equation for our problem reads

$$\theta V = \max_j [k^\alpha - j - j^2 k^{-\gamma} + V'(k)(j - \sigma k)]. \quad (26)$$

Using the HJB equation we also can compute the steady state equilibria.

For the steady state, for which  $0 = j - \sigma k$  holds, we obtain:

$$V(k) = \frac{f(k, j)}{\theta}, \quad (27)$$

$$V'(k) = \frac{f'(k, j)}{\theta} = \frac{\frac{\partial}{\partial k}(k^\alpha - \sigma k - \sigma^2 k^{2-\gamma})}{\theta}. \quad (28)$$

Using the information of (27)-(28) in (26) gives, after taking the derivatives of (26) with respect to  $j$ , the steady states for the stationary HJB equation:

$$-1 - 2jk^{-\gamma} + \frac{\alpha k^{\alpha-1} - \sigma - \sigma^2(2-\gamma)k^{1-\gamma}}{\theta} = 0. \quad (29)$$

Note that hereby  $j = \sigma k$ <sup>4</sup>. Given our parameters the equation may admit multiple steady states.

<sup>1</sup> Note that the production function  $k^\alpha$  may have to be multiplied by a scaling factor. For the analytics we leave it aside here.

<sup>2</sup> For details of the computation of the equilibria in the case when one can apply the Hamiltonian, see Semmler and Sieveking (1998), appendix.

<sup>3</sup> We have multiplied the production function by  $\alpha = 0.30$  in order to obtain sufficiently separated equilibria, and take  $c = 0$ . We employ the following parameters:  $\alpha = 1.1, \gamma = 0.3, \sigma = 0.15, \theta = 0.1$ .

<sup>4</sup> Note that this gives us the same equilibria as using the Hamiltonian approach.

We specify the company's technology parameters to be  $\sigma = 0.15, A = 0.29, \alpha_i = 0.7, \beta_i = 2, \gamma = 0.3$  and  $\theta = 0.1$ . The remaining parameters are specified below.

As for the numerical procedure an example was computed for different  $k$ 's in the compact interval  $[0,2]$ , using dynamic programming with control range  $j \in [0,0.25]$ . The dynamic programming algorithm (DP) used here is built on the HJB equation and is explained, see appendix, in Grüne and Semmler (2004). From this algorithm we obtain the figure below which approximates the present value curve  $V(k)$  representing firm value.

We have considered our deterministic formulation above. In this case, debt is issued, but with no default premium. Thus, the credit cost is given by  $H(k, B) = \theta B$ . We have used the above mentioned DP algorithm in order to solve the discounted infinite horizon problem (16)-(18). Figure 3 shows the corresponding value function representing the present value curve,  $V(k)$ . The present value curve represents the asset value of the company for initial conditions  $k(0)$  and thus its creditworthiness.

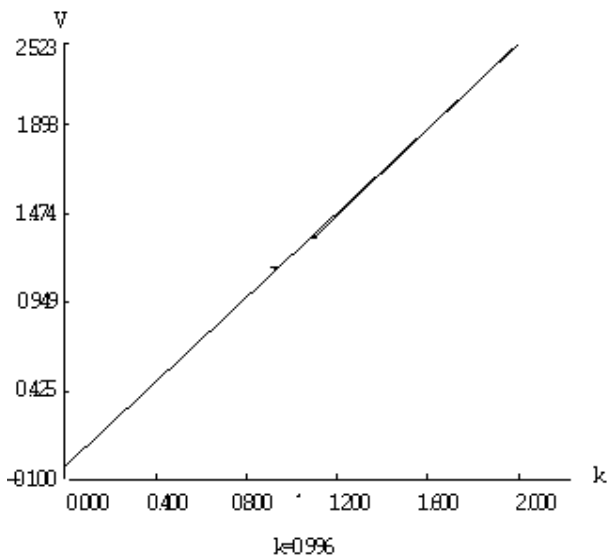


Fig. 5. Present value of company's capital assets

The debt control problem is solved whenever debt is below the firm's asset value, so that we have  $V - B \geq 0$ . The optimal investment strategy is not constrained and thus the asset value which represents the maximum debt capacity  $V$ , is obtained by a solution for an unconstrained optimal investment strategy, represented by the present value curve in Figure 3. For initial values of the capital assets above or below  $k^*$ , the optimal trajectories tend to the domain of attraction  $k^* = 0.996$ . For all initial conditions, the debt dynamics remain bounded as long  $V - B \geq 0$ , thus

allowing the company's equity holders to exercise the option of retiring the debt. Any initial debt above the present value curve will be explosive and the company will lose its creditworthiness, since it will not be able to pay its obligations.

For the more general case where a default premium is to be paid we can use the following function to represent risk premia:

$$H(k(t), B(t)) = \frac{\alpha_1}{\left(\alpha_2 + \frac{N(t)}{k(t)}\right)^\mu} \theta B(t).$$

For the models (16)-(18) with a risk premium included in the company's borrowing cost, it is not possible to transform the model into a standard infinite horizon optimal control problem. This results because debt is now an additional constraint on the optimization problem. Hence, we need to use another method firm value and one can undertake experiments for different shapes of the credit cost function representing different alternative functions for the risk premium. An important class of functions for risk premia is defined by the steepness of the slope defined by the parameter  $\alpha_2$ , for details, see Grüne, Semmler and Bernard (2007). There are also results reported as to what extent the value of this company is affected by a default premium. Overall, when investments are undertaken (by firms, households, or investors in real estate), this is often accompanied by extensive borrowing and, as it often turns out, by overleveraging. Our approach can determine the exact debt capacity, i.e., the degree of leveraging, the economic agent can afford and the lender should pay attention to.

## 6. Moody's KMV

Due to the difficulties in computing the present value for firm value models<sup>1</sup> in practice, a practical implementation has been developed which comes with solutions to this problem. The KMV model, named after the founders Kealhover, McQuown and Vasicek (2001), models credit risk and the default probability of a firm as follows. This structure is useful in seeing how our approach can be implemented.

**6.1. The distance-to-default.** The model states that there are three main elements determining the default probability of a firm:

- ◆ **Value of assets** is the market value of the firm's assets.

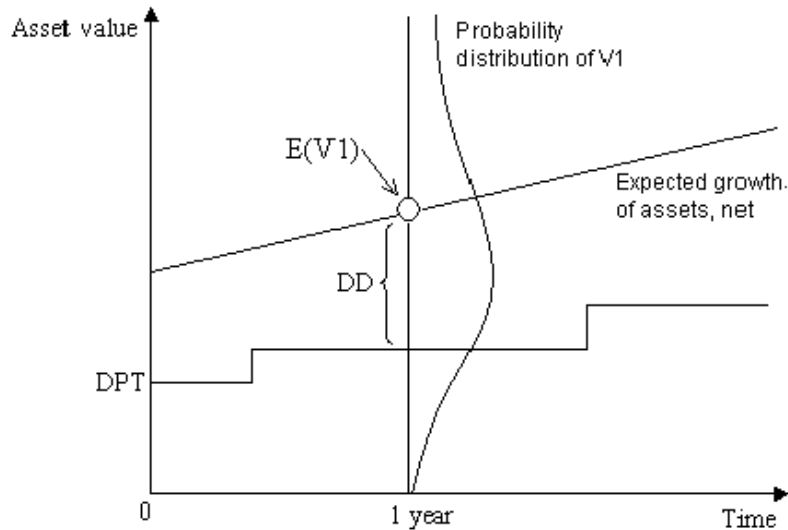
<sup>1</sup> A more practical method of computing firm value is proposed in Benninga (1998, chs. 2-3).

- ◆ **Asset risk** is the uncertainty or risk of the asset value. This is a measure of the firm's business and industry risk.
- ◆ **Leverage** is the extent of the firm's contractual liabilities. It is the book value of liabilities relative to the market value of assets.

As in equations (3) and (10) the default risk of the firm increases when the value of the assets

approaches the book value of the liabilities. The firm defaults when the market value of the assets is smaller than the book value of the liabilities.

The model specifies the financial structure of the firm in terms of assets, current debt, long-term debt, and preferred shares. Next, the default point (*DPT*), the asset value where the firm defaults, is computed. It is assumed that this point is above the size of its short-term debt.



Source: Journal of Banking & Finance (2000).

Fig. 6. Classic KMV model

The distance-to-default (*DD*) is the number of standard deviations between the mean of the distribution of the assets value and the default point (*DPT*).

$$DD = \frac{E[V1] - DPT}{\sigma}$$

where  $E(V1)$  = expected asset value in 1 year,  $DPT$  = (short-term debt) + 1/2 (long-term debt), and  $\sigma$  = volatility of asset returns (in dollars).

The last stage in this procedure is to construct a large list of firms and calculate their respective *DDs* and note the *EDF* as a function of *DD*. Thus an estimate of *EDF* based on valuation, capital structure, and the market as a whole is achieved. Once the *EDF* for a particular firm is determined, one can price, for example, a CDS using the same discounted expected value methodology as described in the beginning of this paper.

According to Peter Crosbie and Jeff Bohn (2003) who wrote the paper *Modelling Default Risk* for Moody's, their studies do not confirm this thesis in general. Not all the firms which reach the point where the asset value goes below the book value of their liabilities default. There are many which continue to serve their debt. The reason for this can be found in the long-term liabilities which enable the firms to continue their business until the debt becomes due. The firms may also have credit lines at their disposal.

Crosbie and Bohn draw the conclusion that the default point, the asset value at which the firm will default, generally lies somewhere between total liabilities and short-term liabilities. The relevant net worth of the firm is therefore defined as:

$$[\text{Market Net Worth}] = [\text{Market Value of Assets}] - [\text{Default Point}] \tag{30}$$

If the market net worth of a firm is zero, the firm is assumed to default. To measure the default risk, one can combine all three elements determining the

default probability in a single measure of default risk, the distance-to-default:

$$[\text{Distance - to - Default}] = \frac{[\text{Market Net Worth}]}{[\text{Size of One Standard Deviation of the Asset Value}]}, \tag{31}$$

$$\frac{[\text{Market Net Worth}]}{[\text{Size of One Standard Deviation of the Asset Value}]} = \frac{[\text{Market Value of Assets}] - [\text{Default Point}]}{[\text{Asset Volatility}]} \tag{32}$$

The distance-to-default is the number of standard deviations the asset value is way from default. The default probability can then be computed directly from the distance-to-default if the probability distribution of the asset value is known.

**6.2. The probability of default.** Crosbie and Bohn (2003) give 6 variables that determine the default probability of a firm over some horizon, from now until time H (see Figure 6):

- 1) the current asset value;
- 2) the distribution of the asset value at time H;
- 3) the volatility of the future assets value at time H;
- 4) the level of the default point, the book value of the liabilities;
- 5) the expected rate of growth in the asset value over the horizon;
- 6) the length of the horizon, H.

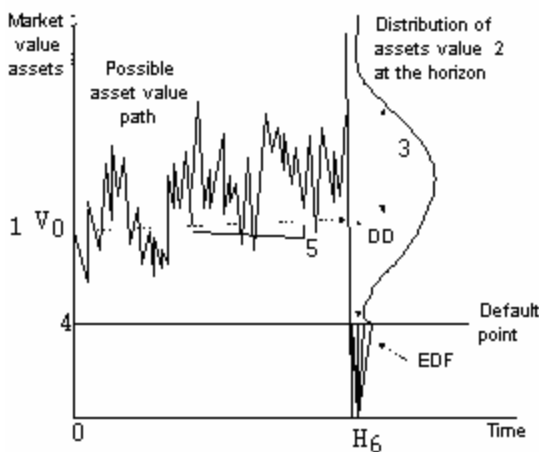


Fig. 6. Crosbie-Bohn model

The probability of default (expected default frequency or EDF value) can be computed with the aid of the measure we calculated above and data on historical default and bankruptcy frequencies. The database that Mody's uses consists of more than 400,000 company-years of data and more than 4,900 incidents of default or bankruptcy (see Figure 4). From these data, a frequency table can be generated which relates the likelihood of default to the distance-to-default measure.

For example, a firm that is 7 standard deviations away from default has an expected default frequency (EDF value) of 5 per cent which leads to a rating of AA. In this case, Moody's analyzes the default history of the fraction of firms which were 7 standard deviations away from the default point and

defaulted over the next year. According to Crosbie and Bohn (2003), Moody's tested the relationship between distance-to-default and default frequency for industry, size, time and other effects and has found that the relationship is constant across all of these variables.

Those relationships can be developed in mathematical terms. According to the Black-Scholes model and as above in equation (1) presumed, the market value of the firm's underlying assets is described by the following stochastic process:

$$dV_A = \mu V_A dt + \sigma_A V_A dz, \tag{33}$$

where  $V_A, dV_A$  are the firm's asset value and change in asset value;  $\mu, \sigma_A$  are the firm's asset value drift rate and volatility;  $dz$  is a Wiener process.

The probability of default is that the market value of the firm's assets will be less than the book value of the firm's liabilities by the time the debt matures:

$$p_t = Pr[V_A^t \leq X_t | V_A^0 = V_A] = Pr[\ln V_A^t \leq \ln X_t | V_A^0 = V_A], \tag{34}$$

where  $p_t$  is the probability of default by time  $t$ ;  $V_A^t$  is the market value of the firm's assets at time  $t$ ;  $X_t$  is the book value of the firm's liabilities due at time  $t$ .

The change in the value of the firm's assets is described by (16), so the value at time  $t, V_A^t$ , given that the value at time 0 is  $V_A$ , is:

$$\ln V_A^t = \ln V_A + \left( \mu - \frac{\sigma_A^2}{2} \right) t + \sigma_A \sqrt{t} \varepsilon, \tag{35}$$

where  $\mu$  is the expected return on the firm's asset;  $\varepsilon$  is the random component of the firm's return.

Equation (18) describes the asset value path shown in Figure 3. Combining (17) and (18), one can write the probability of default as:

$$p_t = Pr \left[ \ln V_A + \left( \mu - \frac{\sigma_A^2}{2} \right) t + \sigma_A \sqrt{t} \varepsilon \leq \ln X_t \right], \tag{36}$$

$$\text{or } p_t = Pr \left[ - \frac{\ln \frac{V_A}{X_t} + \left( \mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \geq \varepsilon \right]. \tag{37}$$

Since the Black-Scholes model assumes that  $\varepsilon$  is normally distributed, one can write the default probability as:

$$p_t = N \left[ - \frac{\ln \frac{V_A}{X_t} + \left( \mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \right]. \quad (38)$$

Since the distance-to-default measure is nothing else than the number of standard deviations that the firm is away from default, one can write this measure with the Black-Scholes notation as:

$$[\text{Distance - to - Default}] = \frac{\ln \frac{V_A}{X_t} + \left( \mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}}. \quad (39)$$

Given an example that we compute a distance-to-default from equation (22) that equals 3.0, the probability of default using equation (21) will then be 13 basis points or 13 per cent. In practice, this distance-to-default measure is adjusted to include several other factors which play a role in measuring the default probability.

## 7. Empirical evidence for firm value based models

There are several advantages and disadvantages that firm value based models have in practice. The predictions of firms value based models on the dynamics of share and debt prices of firms, are discussed briefly in this section. After a few empirical papers are discussed the general importance of these models will be evaluated.

While the majority of firm value based models predict a hilly shape for the term structure of credit spreads, Litterman and Iben (1991) showed that this is only true for rating classes of firms with bad rating. For other classes, like investment-grade rating classes, they observed increasing credit spreads rather than hilly ones.

The aim of another empirical work, the one by Lardic and Rouzeau (1999), was to reproduce the risk ranking of obligors using firm value models. The test was designed not to study the real market value of the firms but to derive the risk level of firms in such a way that allowed to differentiate between riskier and less risky assets. The results however showed that the models were not able to reproduce the risk ranking of obligors. Instead, they were only able to recognize changes in the credit quality of the same obligor.

Longstaff and Schwartz (1995) investigated credit spread movements. With their tests using Moody's

corporate bond yield averages, they found that there is a negative correlation between spreads and rates, meaning that firm value based models cannot be used for hedging purposes.

Concerning the pricing accuracy of firm value based models, Eom et al. (2000) run a test where they priced corporate bonds using the current share prices and balance sheet data of firms that issued the bonds. According to this test where the dynamics of the spreads were not included, it was found that there are pricing errors in all models.

Approximating data on fundamental is an essential strength of firm value based models, but defining the actual firm value can be a complex issue. The problems can quickly become too complex to be handled by empirical tests. Despite all the complications one has to deal with when using firm value models, a more practical approach like Moody's KMV shows that one can obtain acceptable results and a better pricing performance with some pragmatic approach (see section 7).

## Conclusion

Recent events in the credit market, that has experienced an extensive credit crisis, raises the question of whether credit risk has not been appropriately evaluated and anticipated. Among financial economists it is still debated whether the current credit crisis is a liquidity crisis or arising from an insolvency problem, Semmler (2006, ch. 4). Yet, in either case, proper evaluation of credit risk is needed.

The financial market had developed derivatives and structured financial products that have become extremely complex. We have looked at the extent to which credit derivatives have been growing in the financial market and the way credit derivatives can be, from a practical point of view, evaluated. We favor what has been called the firm value based model of evaluating credit risk. It has a sound theoretical foundation and suggests practical methods to evaluate credit risk. There have also been developed many models and methods to evaluate credit risk. They range from practical market methods to theory guided ones relying on firm value. In this paper, first some well-known instruments for transferring credit risk are discussed and then, second, firm value based models on evaluating credit risk are studied. Of course, there are other evaluation methods of credit risk, for example, intensity based models or credit rating models. They may also give us great insight of how credit risk is evolving. Yet, our approach has a sound theoretical foundation and is based on the theoretical development of the 1970, put forward

by Black and Scholes (1973) and Merton (1974). Further theoretical foundations of this approach can be found in Schönbucher (2003), Grüne and Semmler (2005) and Grüne, Semmler and Bernard (2007). This approach can also be applied to the recent US real estate and subprime mortgage

problems, see Bernard and Semmler 2008. Generally, we suggest that Brownian motion not be the sole dynamic model, but, instead, compute asset value directly. The practical dimension of our approach makes it suitable for real-world implementation.

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## Appendix

### The numerical solution of the model

We here briefly describe the dynamic programming algorithm as applied in Grüne and Semmler (2004) that enables us to numerically solve the dynamic model as proposed in section 3. The feature of the dynamic programming algorithm is an adaptive discretization of the state space which leads to high numerical accuracy with moderate use of memory.

Such algorithm is applied to discounted infinite horizon optimal control problems of the type introduced in section 3. In our model variants we have to numerically compute  $V(x)$  for

$$V(x) = \max_u \int_0^{\infty} e^{-rt} f(x, u) dt$$

$$\text{s.t. } \dot{x} = g(x, u),$$

where  $u$  represents the control variable and  $x$  is a vector of state variables.

In the first step, the continuous time optimal control problem has to be replaced by a first order discrete time approximation given by

$$V_h(x) = \max_j J_h(x, u), \quad J_h(x, u) = h \sum_{i=0}^{\infty} (1-\theta h)^i Uf(x_h(i), u_i),$$

where  $x_u$  is defined by the discrete dynamics

$$x_h(0) = x, \quad x_h(i+1) = x_h(i) + hg(x_i, u_i)$$

and  $h > 0$  is the discretization time step. Note that  $j = (j_i)_{i \in \mathbb{N}_0}$  here denotes a discrete control sequence.

The optimal value function is the unique solution of a discrete Hamilton-Jacobi-Bellman equation such as

$$V_h(x) = \max_j \{hf(x, u_o) + (1 + \theta h)V_h(x_h(1))\},$$

where  $x_h(1)$  denotes the discrete solution corresponding to the control and initial value  $x$  after one time step  $h$ .  
Abbreviating

$$T_h(V_h)(x) = \max_j \{hf(x, u_o) + (1 - \theta h)V_h(x_h(1))\}$$

the second step of the algorithm now approximates the solution on grid  $\Gamma$  covering a compact subset of the state space, i.e. a compact interval  $[0, K]$  in our setup. Denoting the nodes of  $\Gamma$  by  $x^i, i = 1, \dots, P$ , we are now looking for an approximation  $V_h^\Gamma$  satisfying

$$V_h^\Gamma(X^i) = T_h(V_h^\Gamma)(X^i)$$

for each node  $x^i$  of the grid, where the value of  $V_h^\Gamma$  for points  $x$  which are not grid points (these are needed for the evaluation of  $T_h$ ) is determined by linear interpolation. We refer to the paper cited above for the description of iterative methods for the solution of (A5). Note that an approximately optimal control law (in feedback form for the discrete dynamics) can be obtained from this approximation by taking the value  $j^*(x) = j$  for  $j$  realizing the maximum in (A3), where  $V_h$  is replaced by  $V_h^\Gamma$ . This procedure in particular allows for the numerical computation of approximately optimal trajectories.

In order to distribute the nodes of the grid efficiently, we make use of a posteriori error estimation. For each cell  $C_l$  of the grid  $\Gamma$  we compute

$$\eta_l := \max_{k \in C_l} |T_h(V_h^\Gamma)(k) - V_h^\Gamma(k)|.$$

More precisely we approximate this value by evaluating the right hand side in a number of test points. It can be shown that the error estimators  $\eta_l$  give upper and lower bounds for the real error (i.e., the difference between  $V_j$  and  $V_h^\Gamma$ ) and hence serve as an indicator for a possible local refinement of the grid  $\Gamma$ . It should be noted that this adaptive refinement of the grid is very effective for computing steep value functions and models with multiple equilibria, see Grüne and Semmler (2004).