“Optimisation of Conditional-VaR in an Actuarial Model for Credit Risk Assuming a Student Copula Dependence Structure”

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OPTIMISATION OF CONDITIONAL-VAR IN AN ACTUARIAL MODEL FOR CREDIT RISK ASSUMING A STUDENT COPULA DEPENDENCE STRUCTURE

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Abstract

In this paper we present a model for the valuation of the risk of credit portfolios. It uses both traditional tools of credit risk valuations and more recent ones like copula functions and Conditional VaR theory. The model we propose is based on some key assumptions we here summarise: first of all, the risk of default is modelled using the time-until-default of an exposure; moreover the hazard rates are random variables whose values follow gamma distributions coherently with CreditRisk+ proposed by Credit Suisse and others; recovery rates themselves are supposed to be stochastic (following a Beta distribution).

The main aspect of our proposal is the introduction of credit migration in the context of an intensity-based model with copula function dependence structure (we use a Student copula to model correlations between the obligors). This permits to quantify the loss distribution of the portfolio and to calculate some useful indexes of risk for the probability distribution of the values of the portfolio: expectation, variance, $\alpha -$VaR, and, following Rockafellar & Uryasev, the $\alpha -$conditional VaR ($\alpha -$CVaR) of the portfolio itself.

The final aim of the model is to present a more flexible and realistic approach to valuation and management of the risk of credit portfolios. Infact, in comparison with the traditional approaches, we remove some restrictive assumptions and try to generalize the valuation scheme (i.e. CreditMetrics considers constant hazard rates while CreditRisk+ takes into account constant recovery rates with no credit migrations).

We conclude the article with a large numerical example in order to test the model.

Key words: credit risk; copula functions, copula modified Monte Carlo simulation, Conditional VaR.

JEL Classification: G11, G12, G13, G21, G33.

1. Introduction

The aim of the paper is the valuation of the risk of a credit portfolio following an approach with some elements that are usual in the insurance and actuarial field.

The model we propose is based on some key assumptions we here summarise; first of all, the risk of default, following Li (2000), Mashal & Naldi (2002), Meneguzzo & Vecchiato (2002) and Di Clemente & Romano (2003), is modelled using the time-until-default of an exposure; moreover the hazard rates are random variables whose values follow gamma distributions coherently with CreditRisk+ proposed by Credit Suisse (1997), Micocci (2000), Burgisser, Kurth & Wagner (2001) and Menzietti (2002); recovery rates themselves are supposed to be stochastic as in Gupton, Finger & Bathia (1997), and following a Beta distribution.

The central aspect of our proposal is the introduction of credit migration in the context of an intensity-based model with copula function dependence structure. This permits to quantify the loss distribution of the portfolio and to calculate some useful indexes of risk for the probability distribution of the values of the portfolio: expectation, variance, $\alpha -$VaR, and, following Rockafellar & Uryasev (2000), the $\alpha -$conditional VaR ($\alpha -$CVaR) of the portfolio itself.

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The paper is structured as follows: section 2 presents the model for default and credit migration; the section is divided in subsections facing the problems of time-until-default, the hazard rate function and the recovery rates, the credit migration, the exposure valuation and the loss distribution. Moreover we highlight the possibility of mixing our model with the $\alpha-$conditional VaR to build the efficient return/CVaR frontier of the credit portfolios.

Section 3 is devoted to present mathematical background of copula functions and to develop some useful algorithms of Monte Carlo simulation. Section 4 shows an example of application of the proposed model to a portfolio of credits with all the needed data. Section 5 concludes.

2. The model for default and credit migration

2.1. Time-until-default

The first aspect we treat is the risk of default, which is modelled in all the approaches to credit risk in different ways. In our model, following Li and other studies (Li (2000), Mashal & Naldi (2002), Meneguzzo & Vecchiato (2002), Di Clemente & Romano (2003)) we define a new random variable, the survival time from now to the time of default or the time-until-default for an exposure.

Time-until-default can be modelled as the survival time in life insurance[1]. We denote this random variable as T_0, and we assume some properties: it must take only positive values; it is continuous and has a density function $f_0(t)$ that we suppose continuous. We denote as $F_0(t)$ its distribution function:

$$F_0(t)=\Pr\{T_0\le t\}=\int_0^t f_0(u)\,du \quad \text{with} \qquad F_0(0)=0 \qquad F_0(\infty)=1.$$

Introducing the hazard rate function *h(t)* we also have

$$F_0(t)=1-e^{-\int_0^t h(u)\,du}.$$

The conditional survival probability until t_2 conditional on survival until t_1, with $t_1<t_2$, will be:

$$\Pr\{T_0>t_2 \mid T_0>t_1\}={}_{t_2-t_1}p_{t_1}=\frac{1-F_0(t_2)}{1-F_0(t_1)}=e^{-\int_{t_1}^{t_2} h(u)\,du}$$

from which we have the conditional default probability in the horizon $(t_1,\, t_2)$, given survival until t_1:

$$\Pr\{t_1<T_0\le t_2 \mid T_0>t_1\}={}_{t_2-t_1}q_{t_1}=1-e^{-\int_{t_1}^{t_2} h(u)\,du}.$$

We denote with q_{t_1} the one-year default probability given survival until t_1 ($q_{t_1}={}_1q_{t_1}$) and with ${}_sq$ the non conditional default probability in the horizon $(0,\, s)$ (${}_sq={}_sq_0$).

We have also:

$$f_0(t)=h(t)\cdot e^{-\int_0^t h(u)\,du}.$$

If τ is the time horizon and $h(t)=h$ for $t\in[0,\tau]$ we have $f_0(t)=h\cdot e^{-h\cdot t}$ and the time until default follows an exponential distribution with parameter h. In the hypothesis of con-

[1] See Bowers et al. (1997) and Pitacco (2000).

stancy, the hazard rate can be estimated from the one-year default probability (or for a different time horizon) as follows:

$$ {}_sq_t = 1 - e^{-h \cdot s} \Rightarrow h = -\frac{\ln\left(1 - {}_sq_t\right)}{s}, \tag{1} $$

$$ q_t = 1 - e^{-h} \Rightarrow h = -\ln\left(1 - q_t\right). \tag{1-bis} $$

2.2. Hazard rate function and recovery rate

The relationship between the distribution function of survival time and the hazard rate function allows representing the default process by modelling the hazard rate function.

So for our model we need the value of the hazard rate. This can be found in three different ways:

(1) using the expression (1) or (1-bis) and the default frequency from rating agencies we can calculate the hazard rate each year;

(2) from market data in the context of reduced-form models or intensity-based approach. Indeed in the market price of defaultable bonds, a credit spread curve is implied. If we assume a deterministic value for the recovery rate, a specific credit spread implies a value for default probability and so for the hazard rate;

(3) in the framework of the structural models using the Merton option theoretical approach (1974) and its following generalizations. In this case the default probability equals the probability that the firm asset value goes under the liabilities values.

Each approach has same drawbacks; to avoid these difficulties the solution proposed (see Li (2000), Schönbucher & Schubert (2001), Frey & McNeil (2001), Mashal & Naldi (2002)) is a copula function approach. The individual (marginal) survival probabilities of the obligors are taken from an intensity-based approach and the dependency is obtained with an appropriate copula function.

Such a solution is not feasible in the case we don't have market prices of defaultable financial instruments for the same obligor. For this kind of exposure the solution we propose is to use the rating agencies default probability for single exposure and to use a copula function for the dependence structure. In this case we can model the heterogeneity of default frequencies between obligors of the same rating class assuming that the default probability is not deterministic but stochastic.

In our model we assume that the hazard rate for an obligor in a given rating class is constant each year but the value is not the expectation of historical default frequencies for its class but a random variable following a Gamma distribution with parameters specific for the rating class[1]. We choose a Gamma distribution for many reasons:

- analysis of historical default frequencies shows that in each rating class the effective default propensity of the obligors is not homogeneous and that it follows a skewed distribution (see for example Kealhofer, Kwok & Weng (1998));
- the use of a Gamma distribution to represent the heterogeneity of claim frequencies for different risks is typical in risk models for general insurance (Daykin, Pentikäinen & Pesonen (1994));
- a Gamma distribution for default probability has yet been used with good results in many models for credit risk (CreditRisk+ developed by Credit Suisse (1997) and its numerous generalization as Micocci (2000), Bürgisser, Kurth & Wagner (2001), Menzietti (2002)).

With these hypotheses, denoting $H_k^{(i)}$ the random variable hazard rate for a given obligor i ranked in rating class k we have:

$$ H_k^{(i)} \sim \text{Gamma}\left(\alpha_k, \beta_k\right) \Rightarrow u_k\left(h^{(i)}\right) = \frac{\beta_k^{\alpha_k} \cdot e^{-\beta_k h^{(i)}} \cdot h^{(i)\alpha_i - 1}}{\Gamma\left(\alpha_k\right)}, $$

[1] The assumption that the hazard rate is a random variable is made to explain the risk heterogeneity of default frequency for rating classes published by rating agencies, but could be used as well if we use hazard rate structure inferred from market data to introduce random noise. Indeed, even in reduced-form models, the hypothesis of constant hazard rate for a time horizon of one year or more is not realistic.

where $u_k\left(h^{(i)}\right)$ is the density function and α_k and β_k are the parameters of the Gamma distribution for the rating class k.

We denote with μ_k and ${\sigma_k}^2$ mean and variance of the hazard rate for the rating class k in which is ranked the obligor i. The parameters for the Gamma distribution can be estimated with maximum likelihood or moments method. So the time-until-default for the obligor i, $T_0^{(i)}$ conditional on a value $h^{(i)}$ for hazard rate, is exponentially distributed with parameter $h^{(i)}$:

$$f_0^{(i)}(t)\Big|_{h^{(i)}} = h^{(i)} \cdot e^{-h^{(i)} \cdot t}.$$

The default will occur if $T_0^{(i)} < \tau$ with τ time horizon for the evaluation. So the probability of default for the obligor i, conditional on a value $h^{(i)}$ is:

$${}_{\tau}q^{(i)}\Big|_{h^{(i)}} = \Pr\left\{T_0^{(i)} < \tau\Big|_{h^{(i)}}\right\} = 1 - e^{-h^{(i)} \cdot \tau}.$$

In case of default we assume the immediate recovery of the exposure with a random rate $R^{(i)}$ on exposure face value, associated to obligor i. The exposure value $V^{(i)}$ after default is:

$$V^{(i)}\Big|_{T_0^{(i)} < \tau,\, R^{(i)} = r^{(i)}} = N^{(i)} \cdot r^{(i)},$$

where $N^{(i)}$ is the face value of the exposure and $r^{(i)}$ is the value assumed by the recovery rate.

To represent the recovery rate uncertainty we assume (as Gupton, Finger & Bhatia (1997) in the framework of CreditMetrics) that it follows a Beta distribution and we choose the parameters so that $0 \le R^{(i)} \le 1$ (at this purpose we put $\nu = 1$). If expectation and variance of recovery rate distributions are known, the parameters p and q can be estimated with the moments method.

The introduction of random recovery rate, and more in general the recovery risk, is an important feature of the model; actually only few models introduce such a risk aspect although it represents an important risk factor.

2.3. Credit migration

We assume that in t_0 the obligor lies in the k th rating class, and that at the end of the time horizon he could end in $K+1$ different states: in default or, in survival case, in one of the other K rating classes. If the arrival class is better (worse) than k we have an upgrading (downgrading). We note that eventual credit migrations influence the exposure value.

To model credit migration we assume that information on credit quality can be inferred from time-until-default: a high value means that the default is not likely and so that the obligor stays in a "good" rating class; a low value means the default is likely and the obligor can be ranked in a "bad" rating class.

As we saw in paragraph 2.1, conditional time-until-default in this model follows an exponential distribution and the value that it assumes is used to evaluate if default is incurred or not. After this, if the exposure survives, the same value could be used to estimate down or upgrading. In other words we represent default process and credit migrations with the same marginal distribution for time-until-default.

We obtain this result fixing K bounds over the time-until-default distribution[1]. If $T_0^{(i)}$ assumes a value within 0 and the first bound (which is equal to time horizon τ) the obligor de-

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[1] The idea to model the migrations with bounds on some specific distribution was first proposed by Gupton, Finger & Bhatia (1997) in CreditMetrics with bound on asset return distribution (normal distribution) then in Menzietti (2002), in CreditRisk+ framework, with bound on risk factor distribution (Gamma distribution).

faults. If $T_0^{(i)}$ assumes a value within the first and the second bound, the obligor ends in the "worst" rating class, and so on. If $T_0^{(i)}$ value crosses the K^{th} bound, we put the obligor in the "best" rating class.

In order to define the K bounds we need not only the default probability but also the probabilities of switching in other rating classes. These probabilities are included in transition matrices that rating agencies publish but can be produced even in a Merton-type model as KMV (Kealhofer (2003a)).

We denote with $p_{k,j}^{(i)}$ the probability of staying in class j at the end of the time horizon for counterpart i that initially stays in class k. The final state $K+1$ corresponds to the default state. Obviously this probability depends on the initial rating class, which is information known for each obligor so that $p_{k,j}^{(a)} = p_{k,j}^{(b)} = p_{k,j}$ if both a and b initially stay in class k.

We need to find two bounds $s_{k,K+1-j}$ and $s_{k,K+2-j}$ such that the unconditional probability that $T_0^{(i)}$ assumes a value within the bounds is $p_{k,j}$ (obviously $s_{k,0} = 0$).

We calculate at first the density probability function $f_0^{(i)}(t)$ of time-until-default $T_0^{(i)}$. As we pointed out previously, $T_0^{(i)}$ is the composition of an exponential distribution with a Gamma distribution. By abuse of notation we omit the indexes representing the obligor i and the starting class k.

An elementary calculation gives the following result:

$$f_0(t) = \frac{\alpha \cdot \beta^{\alpha}}{(\beta + t)^{\alpha+1}}.$$

So the bound s_1 (the default bound) must satisfy the condition $\Pr\{0 < T_0 \le s_1\} = q$, which leads to

$$s_1 = \left(\frac{1-q}{\beta^{\alpha}}\right)^{-1/\alpha} - \beta \tag{2}$$

In general, using again the initial notation, we have

$$\Pr\left\{s_{k,K+1-j} < T_0^{(i)} \le s_{k,K+2-j}\right\} = p_{k,j}$$

so that the bounds can be determined recursively by the following:

$$s_{k,K+2-j} = \left(\frac{-p_k + \beta_k^{\alpha_k} \cdot \left(\beta_k + s_{k,K+1-j}\right)^{-\alpha_k}}{\beta_k^{\alpha_k}}\right)^{-1/\alpha} - \beta. \tag{3}$$

2.4. Exposure valuation and loss distribution

The exposure valuation at time t is obtained as present value of cash flows (the same solution is adopted in CreditMetrics, Gupton, Finger & Bhatia (1997)). We assume that the spot and forward zero curves for each rating class are known[1] as well as the vectors of cash flows $\mathbf{z}^{(i)} = \left(z_1^{(i)}, z_2^{(i)}, \ldots, z_T^{(i)}\right)$ and maturities $\mathbf{t}^{(i)} = \left(t_1^{(i)}, t_2^{(i)}, \ldots, t_T^{(i)}\right)$ for each exposure.

The exposure value in t_0 will be:

[1] We assume a given and deterministic forward zero curve for each rating category, so the market risk is not included.

$$V_{t_0}^{(i)} = \sum_{s=1}^{T} z_s^{(i)} e^{-\delta_k(t_0,t_s)\cdot(t_s-t_0)} \tag{4}$$

with $\delta_k(t_0,t_s)$ spot rate in the time horizon (t_0,t_s) for an exposure with rating k in t_0.

We assume one year time horizon ($t_0 = 0; t_1 = 1$). If the exposure survives, its value depends on forward rate term structure, cash flows and the arrival state j:

$$V_1^{(i)}\Big|_{s_{k,K+1-j} < T_0^{(i)} \le s_{k,K+2-j}} = \sum_{s=1}^{T} z_s^{(i)} \cdot e^{-\delta_j(1,t_s)\cdot(t_s-1)}. \tag{5}$$

If default occurs, as we have already said, the exposure value will be:

$$V_1^{(i)}\Big|_{T_0^{(i)} < \tau,\ R^{(i)} = r^{(i)}} = N^{(i)} \cdot r^{(i)}. \tag{5-bis}$$

Let denote with $y \in R^m$ the random vector which represents uncertainties which can affect the value (such as hazard rate, rating and (eventually) recovery rate). We assume besides that the distribution of $\mathbf{y}$ in R^m has density $p(\mathbf{y})$.

After finding the value of the exposure conditional to $\mathbf{y}$, $V^{(i)}(\mathbf{y})$ it is possible to calculate the loss (or gain):

$$L^{(i)}(\mathbf{y}) = V^{(i)}(\mathbf{y}^*) - V^{(i)}(\mathbf{y})$$

with $V^{(i)}(\mathbf{y}^*)$ exposure value if the credit characteristics (specifically the rating) don't change.

In many applications we are interested in a model for a portfolio of n exposures. In this case each scenario $\mathbf{y}$ is obtained from n Gamma distributions for the hazard rate (with specific characteristics for each rating class), then from a random vector of n time-until-default (with marginal exponentially distributed), finally we evaluate the single exposures and calculate portfolio value and consequent loss for the specific scenario. We denote with $\mathbf{x} = (x_1,...,x_i,...,x_n)^{\mathbf{T}}$ the vector of the quotas held for each exposure (it belongs to the set of available portfolios $X \subset R^n$), with $\mathbf{L}(\mathbf{y}) = \left(L^{(1)}(\mathbf{y}),...,L^{(i)}(\mathbf{y}),...,L^{(n)}(\mathbf{y})\right)$ the vector of the loss functions for single obligors and we assume that in t_0, $x_i = 1$ $(i = 1,...,n)$. The loss function will be:

$$L(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} L^{(i)}(\mathbf{y}) \cdot x_i = \mathbf{L}(\mathbf{y}) \cdot \mathbf{x}. \tag{6}$$

For each fixed value of $\mathbf{x}$, the loss function $L(\mathbf{x},\mathbf{y})$ is a random variable which distribution in R is induced by the distribution of $\mathbf{y}$. We can then define the probability that $L(\mathbf{x},\mathbf{y})$ does not exceed a threshold ς as:

$$\Psi(\mathbf{x},\varsigma) = \int_{L(\mathbf{x},\mathbf{y}) \le \varsigma} p(\mathbf{y})\, d\mathbf{y}. \tag{7}$$

For fixed $\mathbf{x}$, $\Psi(\mathbf{x},\varsigma)$ viewed as a function in ς represents the cumulative distribution function for the loss associated with $\mathbf{x}$. The function $\Psi(\mathbf{x},\varsigma)$ is no decreasing in ς and we assume for simplicity it is also continuous in ς.

We are interested in some features of the loss distribution: expectation, variance, $\alpha -$ VaR and $\alpha -$ Conditional VaR ($\alpha -$ CVaR).

The portfolio expected loss is:

$$\mu(\mathbf{x}) = \int_{Y} L(\mathbf{x},\mathbf{y}) \cdot p(\mathbf{y})\, d\mathbf{y}, \tag{8}$$

where Y represents the set of all the possible values for $\mathbf{y}$. The variance of portfolio loss is:

$$\sigma^2(\mathbf{x}) = \int_Y \left(L(\mathbf{x},\mathbf{y}) - \mu(\mathbf{x})\right)^2 \cdot p(\mathbf{y})\,d\mathbf{y} = \int_Y L(\mathbf{x},\mathbf{y})^2 \cdot p(\mathbf{y})\,d\mathbf{y} - \mu(\mathbf{x})^2 \cdot \tag{9}$$

The function (7) will turn out to be fundamental in defining VaR and CVaR, for which we use notations as in Rockafellar & Uryasev (2000).

Let us fix a confidence level $\alpha \in (0,1)$. The $\alpha -$VaR and $\alpha -$CVaR values for the loss random variable associated with $\mathbf{x}$ at probability level α will be denoted $\varsigma_\alpha(\mathbf{x})$ and $\phi_\alpha(\mathbf{x})$ respectively and defined as:

$$\varsigma_\alpha(x) = \min\{\varsigma \in R : \psi(x,\varsigma) \geq \alpha\}, \tag{10}$$

$$\phi_\alpha(\mathbf{x}) = \frac{1}{1-\alpha} \cdot \int_{L(\mathbf{x},\mathbf{y}) \geq \varsigma_\alpha(\mathbf{x})} L(\mathbf{x},\mathbf{y}) \cdot p(\mathbf{y})\,d\mathbf{y} \cdot \tag{11}$$

In the first definition, $\varsigma_\alpha(\mathbf{x})$ turns out to be the left endpoint of the nonempty interval consisting of the values ς such that $\Psi(\mathbf{x},\varsigma) = \alpha$.

In the second definition we deduce that the probability that $L(\mathbf{x},\mathbf{y}) \geq \varsigma_\alpha(\mathbf{x})$ equals $1-\alpha$. Consequently, $\phi_\alpha(\mathbf{x})$ is seen as the conditional expectation of the loss associated with $\mathbf{x}$ relative to that loss being $\varsigma_\alpha(\mathbf{x})$ or greater.

Remark

It is a well known fact in financial literature (Pflug (2000)) that CVaR is a coherent risk measure in the sense of Artzner (Artzner et al. (1999)). Another important feature about CVaR, which was demonstrated by Rockafellar & Uryasev (2000), is the following: optimisation problems involving CVaR risk measure turn out to be a linear programming problem. Besides, equivalent optimisation problems can be set up by maximizing expected returns under CVaR constraints. This property can be thus very useful in determining the efficient frontier.

Unfortunately, the expression of the density function $p(\mathbf{y})$ is not generally known, besides closed forms for (10) and (11) are not available. So to determine the shape of the loss distribution and its features like variance, $\alpha -$VaR and $\alpha -$CVaR we need to resort to Monte Carlo simulation. The sampling we obtain generates a set of vectors $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_J$ with probabilities $\pi_j = \frac{1}{J}$ $(j = 1,2,...,J)$ and J simulations number[1].

The portfolio loss function in scenario $\mathbf{y}_j$ will be:

$$L(\mathbf{x},\mathbf{y}_j) = \sum_{i=1}^{n} L^{(i)} \cdot (\mathbf{y}_j) x_i \tag{6-bis}$$

with $L^{(i)}(\mathbf{y}_j)$ loss function in scenario $\mathbf{y}_j$ for the obligor i. The discretized version of the main characteristics of portfolio loss distribution is the following.

The portfolio expected loss is:

$$\mu(\mathbf{x}) = \frac{\sum_{j=1}^{J} L(\mathbf{x},\mathbf{y}_j)}{J} \cdot \tag{8-bis}$$

The variance of portfolio loss is:

[1] See section 3 for a description of the simulation procedure.

$$\sigma^2(\mathbf{x}) = \frac{\sum_{j=1}^{J}\left[L(\mathbf{x},\mathbf{y}_j)-\mu(\mathbf{x})\right]^2}{J}. \tag{9-bis}$$

The $\alpha-$VaR is defined as:

$$\varsigma_\alpha(\mathbf{x}) = QL_\alpha(\mathbf{x}) \tag{10-bis}$$

with $QL_\alpha(\mathbf{x})$ $\alpha-$quantile of the loss distribution.

The $\alpha-$CVaR is defined as:

$$\phi_\alpha(\mathbf{x}) = \frac{1}{J(1-\alpha)} \sum_{\substack{j=1 \\ L(\mathbf{x},\mathbf{y}_j)\geq\varsigma_\alpha(\mathbf{x})}}^{J} L(\mathbf{x},\mathbf{y}_j) = \varsigma_\alpha(\mathbf{x}) + \frac{1}{J(1-\alpha)}\sum_{j=1}^{J}\left[L(\mathbf{x},\mathbf{y}_j)-\varsigma_\alpha(\mathbf{x})\right]^+, \tag{11-bis}$$

where:

$$\left[L(\mathbf{x},\mathbf{y}_j)-\varsigma_\alpha(\mathbf{x})\right]^+ = L(\mathbf{x},\mathbf{y}_j)-\varsigma_\alpha(\mathbf{x}) \text{ if } L(\mathbf{x},\mathbf{y}_j)-\varsigma_\alpha(\mathbf{x})>0,$$

$$\left[L(\mathbf{x},\mathbf{y}_j)-\varsigma_\alpha(\mathbf{x})\right]^+ = 0 \text{ if } L(\mathbf{x},\mathbf{y}_j)-\varsigma_\alpha(\mathbf{x})\leq 0.$$

2.5. Correlation measures and copula function

The implication of such a model for a portfolio of n obligors is that we must generate scenarios for time-until-default from a multivariate distribution function[1]. So we have:

$$F_0(t_1,\dots,t_i,\dots,t_n) = \Pr\left\{T_0^{(1)}\leq t_1,\dots,T_0^{(i)}\leq t_i,\dots,T_0^{(n)}\leq t_n\right\}. \tag{12}$$

Useful tools for generating scenarios from this multivariate distribution are copula functions. We saw that many models use a copula functions approach to represent the dependence structure of a credit portfolio. We wish to ensure that the copula we use captures two features of dependence relationship in the joint distribution: correlation level and tail dependence.

Some specifications are needed for correlation level. If we have two obligors A and B, the individual default probability in a fixed time horizon (respectively q_A, q_B) and the joint default probability $q_{A,B}$ in the same time horizon, the linear default correlation coefficient is by definition (Lucas, 1995):

$$^{(D)}\rho_{A,B} = \frac{q_{A,B}-q_A\cdot q_B}{\sqrt{q_A\cdot(1-q_A)\cdot q_B\cdot(1-q_B)}}.$$

If we have more than two obligors, we can construct a correlation matrix $\mathbf{A}$ whose elements $\mathbf{a_{i,j}}$ are the linear default correlation between obligors i and j.[2]

Starting from this result Merton-Type models as CreditMetrics and KMV link the default correlation between each pair of obligors with the correlation of obligors' asset returns $^{(R)}\rho_{A,B}$ (included in the correlation matrix $\mathbf{R}$) (Gupton, Finger & Bhatia (1997), Kealhofer (2003a, 2003b)). Li (2000) proposed a more general definition of correlation: the survival time correlation. It can be calculated as follows:

[1] Instead the hazard rates are generated from n independent Gamma marginal distributions, because these express risk heterogeneity in each rating class for which we assume no reciprocal influence.

[2] In the case of two obligors, we can reach the probabilities of all elementary events by using the linear correlation coefficient. If we had more than two obligors this would not be possible. With n obligors we have 2^n joint default events and only $n\cdot(n-1)/2$ correlations plus n individual default probabilities and the constrain that these probabilities must sum up to one. So the correlation matrix gives us the bivariate marginal distributions but not the full distribution (Schönbucher, 2003 chap. 10).

$$^{(T)}\rho_{A,B}=\frac{Cov\left(T_0^{(A)},T_0^{(B)}\right)}{\sqrt{Var\left(T_0^{(A)}\right)\cdot Var\left(T_0^{(B)}\right)}}=\frac{E\left(T_0^{(A)}\cdot T_0^{(B)}\right)-E\left(T_0^{(A)}\right)\cdot E\left(T_0^{(B)}\right)}{\sqrt{Var\left(T_0^{(A)}\right)\cdot Var\left(T_0^{(B)}\right)}}.$$

He demonstrated that if we use this concept of correlation and a bivariate normal copula function for dependence structure the correlation parameter $^{(T)}\rho_{A,B}$ is equal to the asset correlation between the two obligors $^{(R)}\rho_{A,B}$.

This result has been extended to t-copula by Mashal & Naldi (2002) and Meneguzzo & Vecchiato (2002).

In the next section we show basic concepts and principal results of copula theory.

3. Copula functions: main definitions and properties

Definition

An n-dimensional copula is a multivariate distribution function C, with margins uniformly distributed in [0,1] that satisfies the following properties:

(i) $C:[0,1]^n \to [0,1]$;

(ii) C is grounded and n-increasing;

(iii) C have margins C_i satisfying $C_i(u)=C(1,\cdots,1,u,1,\cdots,1)=u \quad \forall u\in[0,1] \quad (i=1,\cdots,n)$.

The most important result is the following, due to Sklar:

Theorem

Let F be an n-dimensional distribution, with marginals F_i. Then there exists an n-copula C such that $F(x_1,\cdots,x_i)=C(F_1(x_1),\cdots,F_i(x_i))$.

If the marginals F_i are continuous, then the copula C is unique.

The previous representation is called canonical representation of the distribution. Sklar's theorem is then a powerful tool to build n-dimensional distributions by using one-dimensional ones, which represent the marginals of the given distribution. Dependence between marginals is then characterized by the copula C.

An important multivariate copula is the Student's t-copula, the copula of the multivariate Student's t-distribution. Its parameters are the correlation matrix R and the degree of freedom υ. Besides, a well known algorithm permits to generate pseudo casual numbers from the Student copula.

The results in this section can be found in Embrechts et al. (2001, 2002), Frees & Valdez (1998), Genest et al. (1986, 1993), Kimberling (1974), Meneguzzo & Vecchiato (2002), Nelsen (1998), Roncalli (2000), Sklar (1959).

The copula approach for modelling the portfolio loss distribution follows these steps:

- at first, we have to determine the marginal distributions;
- secondly, we generate pseudo random $n-tuples$ from a Student copula with correlation matrix R. Each random $n-tuple$ represents a simulated time-until-default for each obligor;
- for each simulation and for each obligor, we examine the simulated time-until-default.
  - If it is less than one this obligor has defaulted. In this case, we extract a random recovery rate and then hence we determine the value of this credit at the end of our time horizon.
  - Otherwise, no default has occurred. The simulated value is compared with migration bounds so that we can determine the new rating class. We then evaluate the value of this credit at the end of our time horizon:

- for each simulation, we calculate the portfolio value at the end of the time horizon by summing the values of each credit;
- at last, we deduce the portfolio loss for each simulation and we estimate finally VaR and CVaR.

4. Application of the model to a credit portfolio

In this section we apply the model presented in the previous sections to a portfolio of twenty exposures ($n = 20$). We assume that for each exposure we know rating on Standard & Poor's scale[1], face value, coupon rate, time to maturity. The value in t_0 is calculated by expression (4) using a common term structure of spot rates.

The exposure characteristics are reported in Table 1, all the amounts are expressed in Euro.

Table 1

Exposure characteristics

Obligor	Rating	Face Value	Coupon rate	Maturity	$V_0(i)$
1	AAA	7 000 000	6,75%	3	7 587 181
2	AA	1 000 000	8,25%	4	1 144 159
3	A	1 000 000	7,25%	3	1 093 892
4	BBB	1 000 000	9,00%	4	1 152 020
5	BB	1 000 000	9,25%	3	1 097 229
6	B	1 000 000	13,00%	4	1 196 561
7	CCC	1 000 000	13,75%	2	977 605
8	A	10 000 000	10,75%	8	13 796 527
9	BB	5 000 000	6,75%	2	5 137 472
10	A	3 000 000	5,25%	2	3 105 183
11	A	1 000 000	8,50%	4	1 148 480
12	A	2 000 000	9,50%	5	2 416 986
13	B	600 000	11,00%	3	672 728
14	B	1 000 000	8,00%	2	1 041 126
15	B	3 000 000	7,75%	2	3 109 564
16	B	2 000 000	13,00%	4	2 393 122
17	BBB	1 000 000	10,75%	6	1 283 421
18	BBB	8 000 000	10,00%	5	9 704 110
19	BBB	1 000 000	7,25%	3	1 082 121
20	AA	5 000 000	9,25%	5	6 018 039

The portfolio value in t_0 is 65,157,524 Euro with a face value of 55,600,000 Euro. The time horizon is one year ($t_1 = 1$). The term structure of forward rates used for exposure valuation in $t = 1$ is extrapolated from the term structure of spot rate.

The recovery rates are extrapolated from a Beta distribution with $p = 1.4612$ and $q = 1.3966$, these values ensure a recovery rate expectation equal to 0.5113 and a standard deviation equal to 0.2545[2]. Obviously is it possible to assume that each exposure has different beta distributions. For simplicity we assume here the same distribution for each one.

[1] The basic rating scale of Standard & Poor's has 7 rating classes decreasing from AAA to CCC.
[2] These values are reported in a statistics for senior unsecured bond by Carty & Lieberman (1996).

The previous data for rates and recoveries allow us to calculate, by expression (5), the single exposure values conditional on rating state in $t = 1$. If default occurs, we calculate the expected value for expression (5-bis).

The portfolio value in $t = 1$, if there are no rating migration, $V_1^{\mathbf{x}}\left(\mathbf{y}^*\right)$ amounts to 67,673,167 Euro.

The expected hazard rates are extrapolated from one-year default probabilities included in an S&P-style transition matrix by through formula (1-bis). Such a matrix is used also for migration probabilities over one year.

Despite the criticisms on rating agencies transition matrices, we use these data from rating agencies for the following reasons:

- we assume that market data are not complete;
- it is very difficult to use market data to find implicit migration frequency, so in the reduced form approach it is difficult to implement a multinomial[1] model for credit risk[2];
- in the context of Merton-Type model it is possible to model the credit migration but with the hypothesis of a normal copula dependence structure;
- we model the heterogeneity of default frequency between obligors of the same rating classes assuming that the default probability is not deterministic but stochastic.

On the other hand it is possible to use default probabilities inferred from intensity-based model but in this case the transition matrix should be specially constructed. This could be a future model implementation.

The transition matrix $\mathbf{M}$ we use and the expected hazard rates can be easily recovered from specialized web pages. We made same settlement to the original S&P matrix to guarantee some coherence features[3].

To simulate hazard rate values we have found the gamma parameters with moments method. The standard deviation for each rating class has been assumed to be equal to a quota of expected value with different quotas for each class. The value of such quotas is coherent with statistical Gordy analysis (Gordy (2000)) but the trend from a class to the next one has been a little smoothed to have more regular shape. The expectation, standard deviation and parameters for the hazard rate of each rating class are reported in Table 2.

Table 2

Hazard rate data

Rating	μ_k	σ_k	α_k	β_k	σ_k/μ_k
AAA	0,0001	0,00014	0,51020	5101,79	1,40
AA	0,0002	0,00026	0,59172	2958,28	1,30
A	0,0006	0,00072	0,69444	1157,06	1,20
BBB	0,0018	0,00180	1,00000	555,06	1,00
BB	0,0107	0,00853	1,56250	146,62	0,80
B	0,0534	0,03204	2,77778	52,02	0,60
CCC	0,2205	0,11026	4,00000	18,14	0,50

The bounds which determine state transitions have been calculated by expression (3) with the probabilities included in the transition matrix $\mathbf{M}$ and the vector of expected hazard rates for each rating class $\overline{\boldsymbol{\mu}} = \left(\mu_1, \ldots, \mu_7\right)$.

Finally we need data about the correlation between each pair of exposures. We remind that the linear correlation between the time-until-default of two different obligors is equal to the

[1] We call multinomial model the model that includes rating migrations and binomial model the model with only default risk.
[2] See Jarrow, Lando & Turnbull (1997) and Bielecki & Rutkowski (2003) as example of intensity-based multinomial models.
[3] For more details see Gupton, Finger & Bhatia (1997) and Gordy (2000).

linear correlation between the asset return of the two counterparts[1]. This information is usually not easy to extrapolate, so the solution that has been proposed in literature[2] is to use linear correlation between the equity of each obligor as proxy variable. We do not exhibit the correlation matrix $\mathbf{R}$ for sake of brevity.

The first analysis we perform is a simulation with 20,000 draws assuming 10 degrees of freedom. We find a vector of portfolio values whose mean, $\mu\left(V_1^{\mathbf{x}}\right)$ is 67,332,482 Euro, so the expected return is:

$$\mu(r) = \ln\left[\frac{\mu\left(V_1^{\mathbf{x}}\right)}{V_0^{\mathbf{x}}}\right] = \ln\left[\frac{67,332,482}{65,157,524}\right] = 0.0328 \cdot$$

Considering that $V_1^{\mathbf{x}}\left(\mathbf{y}^*\right)$ is equal to 67,673,167, the expected loss is:

$$\mu(\mathbf{x}) = V_1^{\mathbf{x}}\left(\mathbf{y}^*\right) - \mu\left(V_1^{\mathbf{x}}\right) = 67,673,167 - 67,332,482 = 340,684 \cdot$$

We calculate also standard deviation of loss distribution, $\alpha -$ VaR and $\alpha -$ CVaR with different values of α which are reported in Table 3.

Table 3

Loss distribution characteristics

		mean (μ)	340 685		
		stand. dev. (σ)	1 002 911		
Simulated loss distribution (10,000 draw)			**Normal distribution (μ,σ)**		
α	**α-VaR**	**α-CVaR**	α	**α-VaR**	**α-CVaR**
50 (median)	32 537	900 568	**50 (median)**	340 685	1 140 475
10	1 516 926	2 680 712	**10**	1 625 967	2 099 336
5	2 193 306	3 520 083	**5**	1 990 327	2 406 885
2,5	2 930 212	4 516 589	**2,5**	2 306 355	2 680 891
1	4 182 184	6 127 243	**1**	2 673 805	3 004 488
0,5	5 398 931	7 605 907	**0,5**	2 924 013	3 225 210

The strong asymmetry of loss distribution is clear if we compare simulation results with the values of a normal distribution with the same mean and standard deviation. It's interesting to note that with this number of degrees of freedom the results we obtained are quite similar to those that we can obtain applying CreditMetrics model to the same data.

We know that with a high level of degrees of freedom the t-Student copula gives results similar to normal copula and that Li (Li 2000) demonstrates that CreditMetrics model uses an implicit normal copula dependence structure.

To study the effects of varying the number of degrees of freedom on tail dependence, we repeat the simulation with the same data but assuming different levels for degrees of freedom (3, 10, and 30). $\alpha - VaR$ and $\alpha - CVaR$ for α value of 0.95 and 0.99 are shown in Table 4.

[1] See § 2 and Li (2000).
[2] First of all Merton (1974), but this is a standard solution, adopted i.e. in CreditMetrics model (see Gupton, Finger & Bhatia (1997)) and in KMV model (see Kealhofer (2003a, 2003b)).

Table 4

$\alpha -$ VaR and $\alpha -$ CVaR with different degrees of freedom
(Binomial and Multinomial model)

Degree of freedom (v)	Multinomial model		Binomial model	
	VaR 99	VaR95	VaR 99	VaR 95
	CVaR 99	CVaR95	CVaR 99	CVaR 95
3	4 779 466	2 407 446	4 467 971	2 251 271
	7 116 326	3 997 526	6 540 321	3 719 451
10	4 182 185	2 193 307	3 763 611	2 110 941
	6 127 244	3 520 084	5 417 561	3 206 981
30	3 823 266	2 173 686	3 494 461	2 058 231
	5 555 126	3 314 516	5 228 641	3 081 401

We can see that the effect of tail dependency is clear[1].

In Table 4 we also reported the values for $\alpha - VaR$ and $\alpha - CVaR$ if we suppose that rating migrations are not possible (binomial model). These results show that the introduction of such migrations is a key feature for a credit risk model.

The third analysis we perform is the valuation of marginal risk contribution of each exposure and the definition of risk reduction actions. We use three different measures of risk: standard deviation, $\alpha -$ VaR and $\alpha -$ CVaR (with $\alpha = 0.99$). We measure each risk measure assuming that an exposure (each time a different one) is not in the portfolio (only 19 exposures). The variations in each measure are divided by the value in $t = 1$ of the excluded exposure in case of no migrations. Results are reported in Table 5.

Table 5

Marginal risk contribution

Obligor excluded	Marginal stand. dev. %	Marginal 0.99-VaR %	Marginal 0.99-CVaR %
1	0,35%	0,89%	4,11%
2	0,06%	0,00%	0,30%
3	0,08%	0,00%	0,29%
4	0,29%	-0,01%	1,54%
5	3,74%	14,88%	23,70%
6	5,28%	25,55%	24,25%
7	7,37%	33,61%	27,73%
8	0,45%	0,50%	3,80%
9	1,14%	6,05%	11,08%
10	0,03%	0,21%	0,16%
11	0,65%	4,59%	4,84%
12	0,32%	1,42%	3,01%
13	2,31%	8,77%	15,54%
14	1,62%	6,07%	8,80%
15	2,55%	16,27%	16,24%
16	7,04%	25,30%	28,99%
17	0,50%	3,74%	2,54%
18	0,37%	1,67%	2,51%
19	0,32%	0,00%	1,74%
20	0,03%	0,80%	0,36%

[1] We remind that in real applications the right number of degrees of freedom should be estimated with maximum likelihood methods as explained in section 3.

A risk reduction action can be realized excluding from portfolio exposures with high marginal risk contribution and high market value. Such a choice can be graphically represented as in figures below adding a curve in the marginal risk-market value space. The bounding curve is a hyperbola branch, this means that it is the geometric set of points such that exposure market value multiplied by the marginal risk value is an appropriate constant. The choice of the constant value is subjective.

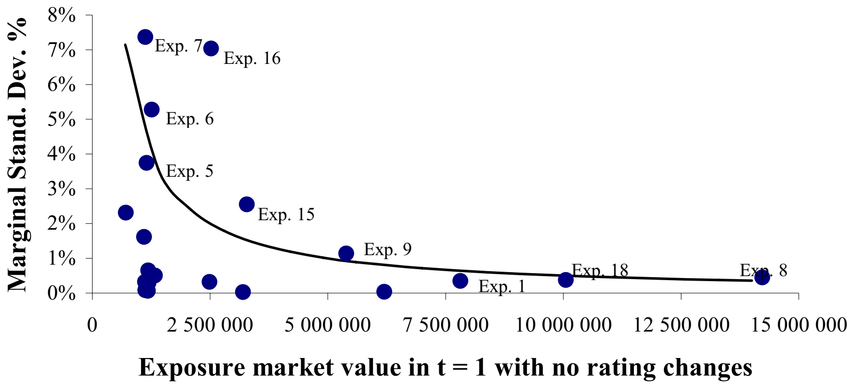

Fig. 1. Marginal standard deviation % – Market value

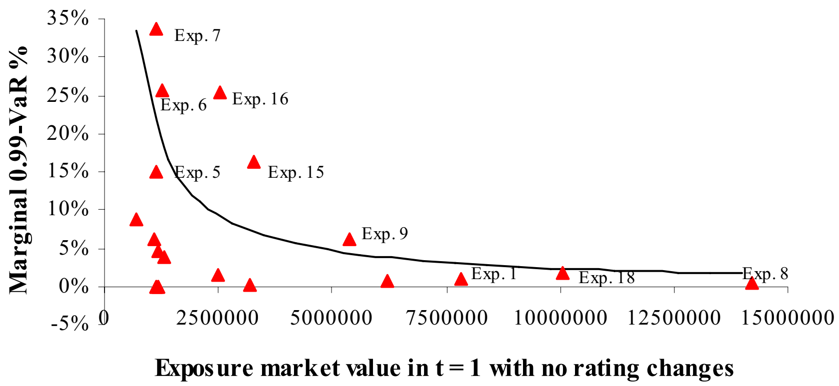

Fig. 2. Marginal 0.99-VaR % – Market value

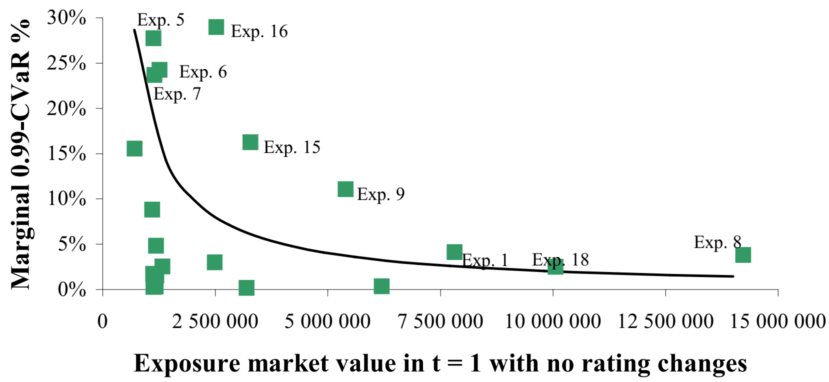

Fig. 3. Marginal 0.99-CVaR % – Market value

We can see that the risk reduction actions suggested under the three different measures are not always the same. So the exposures 15 and 16 are always over the line, but the choice about the exposures 1, 5, 6 and 8 depends on risk measure.

The last analysis we perform is the portfolio optimisation under value-at-risk and conditional-value-at-risk constraints.

We represent in Figure 4 the shape of the efficient frontier with $\alpha = 0.99$ ($\alpha -$ VaR is represented too). The results are reported in Table 6.

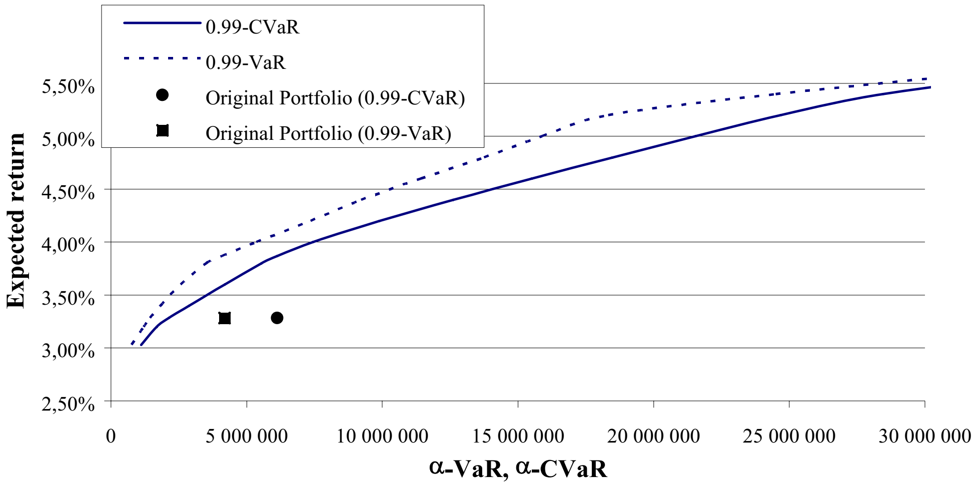

Fig. 4. Efficient frontier with *0.99* – CVaR optimisation

From the figure and the data it's evident the effect of the optimisation which allows a high risk reduction (or a high expected return improvement) with respect to initial portfolio.

We conclude this theoretical overview by confronting with Markowitz mean-variance portfolio optimisation.

It is a well known fact that for normally distributed loss functions these two approaches are equivalent and lead so to the same efficient frontier. However, in the case of non-normal and especially non symmetric distributions (as often occur in credit risk framework) the two approaches may lead to significant differences.

We recall the original optimisation problem set up by Markowitz:

$$\min_x \sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ik} \cdot x_i \cdot x_k$$

subject to the constraints:
$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \text{budget constraint} \\ \sum_{i=1}^{n} E[r_i] \cdot x_i = r_P & \text{expected return constraint} \\ 0 \le x_i \le \upsilon_i, \quad i = 1,\cdots,n & \text{bounds on weights} \end{cases}$$

where r_i is the rate of return of instrument i, $\sigma_{ik} = \mathrm{cov}(r_i, r_k)$.

In our application, we found the following two efficient frontiers:

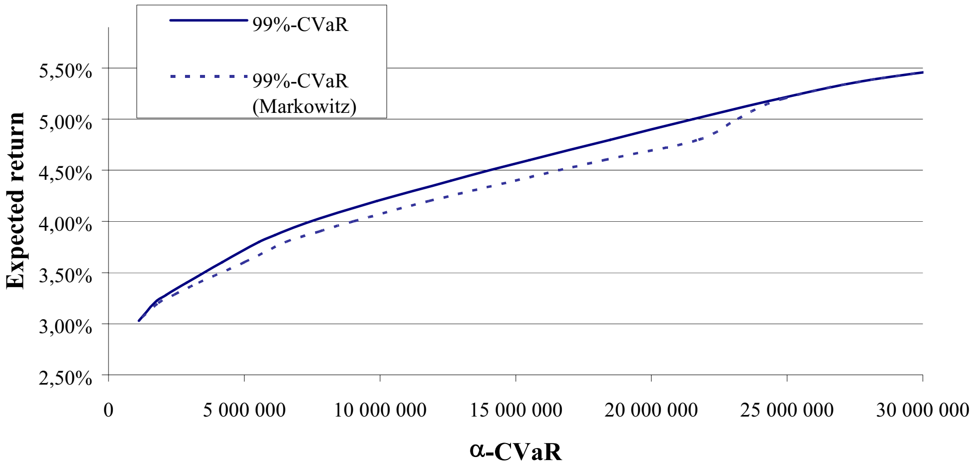

Fig. 5. Efficient frontiers

From the two figures we observe that for a given return, the Markowitz optimal portfolio has a higher CVaR risk level than the efficient return/CVaR portfolio. The difference reduces with high risk level. Nevertheless the two efficient frontiers are substantially different, as we expected from the previous theoretical considerations.

Numerical results are contained in the following table.

Table 6

Optimisation results [1]

Mean - 0.99-CVaR optimisation				Mean-Variance Optimisation		
Std. Dev.	99%-VaR	99%-CVaR	Exp. Return	Std. Dev.	99%-CVaR	Exp. Return
1 002 911	4 182 184	6 127 243	3,28%	1 002 911	6 127 243	3,28%
236 702	758 727	1 115 341	3,03%	103 407	1 246 532	3,06%
225 662	1 206 880	1 680 170	3,20%	193 903	1 829 451	3,20%
332 099	1 441 174	2 128 275	3,28%	279 055	2 412 636	3,28%
481 093	1 813 827	2 869 723	3,40%	406 724	3 318 911	3,40%
781 290	2 573 326	4 193 968	3,60%	646 520	5 010 463	3,60%
1 130 058	3 513 499	5 538 568	3,80%	862 259	6 547 801	3,80%
1 314 882	4 093 889	6 127 304	3,87%	1 002 919	7 764 569	3,90%
1 714 195	5 360 287	7 449 152	4,00%	1 170 244	9 018 529	4,00%
2 335 246	7 361 701	9 895 611	4,20%	1 579 508	11 795 311	4,20%
2 960 823	9 218 887	12 615 263	4,40%	2 030 301	14 973 267	4,40%
3 661 476	11 421 073	15 504 267	4,60%	2 509 950	18 249 165	4,60%
4 393 926	13 756 173	18 480 481	4,80%	3 008 233	21 734 893	4,80%
5 091 946	15 898 348	21 542 825	5,00%	3 772 916	23 172 830	5,00%
5 858 833	18 335 026	24 682 446	5,20%	5 440 317	24 840 751	5,20%
7 491 714	24 499 628	28 455 353	5,40%	7 491 714	28 455 353	5,40%
9 689 757	32 353 840	34 657 391	5,60%	9 689 757	34 657 391	5,60%
11 956 634	40 173 122	42 510 573	5,80%	11 956 634	42 510 573	5,80%
14 261 826	48 362 748	50 689 786	6,00%	14 261 826	50 689 786	6,00%
16 591 500	56 443 219	59 008 027	6,20%	16 591 500	59 008 027	6,20%
17 842 090	60 789 815	63 558 975	6,31%	17 842 090	63 558 975	6,31%

[1] In grey the values of the original portfolio are reported.

6. Conclusions

In the previous sections we presented our proposal for a new model for portfolio credit risk. In the framework of intensity-based model with copula function dependence structure we propose a solution which takes into consideration credit migration risk.

We saw with the application in section 4 that such a risk increases risk measures like VaR and conditional-VaR respectively of about 5% and 10% in case of strong tail dependence.

The examples in section 4 show the flexibility of the model as risk management tool in problems like measuring marginal risk contribution, defining risk reduction actions, or assessing portfolio selection in the context of risk-return efficient frontier.

As future improvements, we can use our model to evaluate derivatives on credit portfolio with spread sensitivity. This kind of pricing is not possible with traditional binomial models.

Another possible development is the introduction of systematic recovery risk which represents an important risk source not yet well represented in intensity based models[1].

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[1] Examples of introduction of such a risk in credit risk models are Bürgisser, Kurth & Wagner (2001) and Menzietti (2002).

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