PORTFOLIO PERFORMANCE ATTRIBUTION[1]

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Abstract

In this paper, we provide further insight into the performance attribution by development of statistical models based on minimizing ETL performance risk with additional constraints on Asset Allocation (AA), Selection Effect (SE), and Total Expected Value Added by the portfolio managers (S). We analyze daily returns of 30 stocks traded on the German Stock Exchange and included in the DAX30-index. The benchmark portfolio is the equally weighted portfolio of DAX30-stocks. The portfolio optimization is based on minimizing the downside movement of the DAX-portfolio from the benchmark subject to constraints on AA, SE and S. We investigate also the distributional properties of AA, SE and S sequences by testing the Gaussian distribution hypothesis versus stable Paretian hypothesis. Finally, we propose an empirical comparison among suggested portfolio choice models comparing the final wealth, expected total realized return of the optimal portfolio, and performance ratios for obtained sequences of excess returns.

Key words: Performance attribution, risk measures, tracking error, portfolio optimization.
JEL Classification: C61, C11, C32.

1. Introduction

We start with the definition of performance attribution. David Spaulding in his book (2003) wrote that "attribution is the act of determining the contributors or causes of a result or effect"; we refer to the extensive reference list in this book for complete review of performance attribution for financial portfolios. Tim Lord (1997) stated that "the purpose of performance attribution is to measure total return performance and to explain that performance in terms of investment strategy and changes in market conditions. Attribution models are designed to identify the relevant factors that impact performance and to assess the contribution of each factor to the final result".

In this paper we evaluate the performance of the portfolio relative to the benchmark applying the attribution technique. Our main goal is to determine the source of the portfolio's "excess return", defined as the difference between the portfolio's return and the benchmark's return. To evaluate the effects, causing the "excess return", and performance-attribution effects, we formulate several optimization problems based on minimizing Expected Tail Lost (ETL) (see Rachev S., Ortobelli S. et al. (2007) for a survey on risk measures) with constraints on Asset Allocation (AA) and Selection Effect (SE).

The main motivation of the paper is two-fold:

1. Having a portfolio of assets or, fund of hedge funds from the portfolio manager or, fund manager, would like

   (1a) to minimize tracking error over a given bench-mark portfolio or bench-mark index, and at the same time:

   (1b) guaranteeing excess mean returns at a given benchmark level; and

   (1c) keeping portfolio attributions of its assets (funds) in desirable bounds.

2. We perform statistical analysis on the optimal portfolios obtained in the back testing in order to better understand important statistical features of the optimal performance attribution con-

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straint portfolios. Having an optimal portfolio of type 1 (a, b, c), or a portfolio- or fund-manager will have much more assurances that this optimal portfolio will preserve the bounds on the performance attribution until the next re-allocation. In the classical performance attribution literature, the optimization component is missing. As advocated in Bertrand Ph. (2005), such a drawback could lead to non-desirable discrepancies between optimal (tracking) portfolios and portfolios with given attribution bounds. While in this paper we consider only relative optimization (based on tracking downside error), similar results will be put forward for absolute optimizations in another forthcoming work.

We now briefly describe our optimization problem: we solve the optimization problems daily, in the period from 07.10.2003 to 02.03.2007 (total 632 days), making forecast for the next day, based on observations of prior year (250 working days), and observed the realized excess return over the benchmark portfolio. We have chosen as a benchmark an equally weighted portfolio of 30 shares, included in DAX30-index. We assume that portfolio manager can outperform the benchmark making different allocation decisions across industry classes (AA-effect,) or picking different securities than are in the benchmark (called SE). To model the AA-effect, we divide the assets into industry classes according to their trading volumes. Recall that AA effect consists in adjusting the weights of the portfolio in order to outperform the benchmark, see for example Bertrand Ph. (2005) and the references there in. The portfolio manager might overweight the sector relative to the benchmark if she is bullish and underweight the sector if she is bearish. Selection effect consists in the picking different securities that, by the manager's opinion, outperform those in the benchmark. She can also pick the same securities as in the benchmark, but buy more or less of them than are in the benchmark. Spaulding D. (2003) formulated one of the Laws of Performance Attribution: "The sum of attribution effects must equal the excess returns. We need to account 100% of the excess return. Consequently, when we've calculated all the effects, their sum must equal our excess return". It means that 100% of the excess return should be explained by analyzed attribution effects.

In our setting, the AA and SE will define constraint sets in optimization problems of type tracking error minimization. We will address the following questions:

- Are our optimization models successful at determining the weights of sectors (or assets' classes) to help manager outperform the benchmark?
- Are our optimization models successful at selecting stocks within each sector to help manager outperform the benchmark?

To evaluate our optimal portfolio selection models, we perform the empirical analysis of final wealth and expected total realized return of the obtained optimal portfolios with respect to the benchmark. We compare the results over different suggested portfolio models to determine the most profitable model for portfolio managers which are best at outperforming the chosen benchmark.

Furthermore, we investigate distributional properties of obtained (as results of our optimizations) values of AA, SE, and S, defined as the difference between the total expected return of the portfolio and the total expected return of the benchmark. In our study of those distributional properties we emphasize the skewness and kurtosis we observed in the values for AA, SE, and S showing a non-Gaussian (so called stable Paretian distributional behavior). Recall that the excess kurtosis, found in Mandelbrot's (1963) and Fama's (1963, 1965) investigations on the empirical distribution of financial assets, led them to reject the normal assumption (generally used to justify the mean variance approach) and to propose the stable Paretian distribution as a statistical model for asset returns. The behavior, generally stationary over time of returns, and the Central Limit Theorem and Central Pre-limit Theorem for normalized sums of i.i.d. random variables theoretically justify the stable Paretian approach proposed by Mandelbrot and Fama. Their conjecture was supported by numerous empirical investigations in the subsequent years (see Biglova A. et al. (2004 a, b); Rachev S. et al. (2003)) and the references therein. In our work we will provide additional empirical evidence testing normal and stable Paretian hypotheses for AA-, SE- and S-values.

The remainder of the paper is organized as follows. Section 2 provides a brief description of our data and methodology. Section 3 provides a description of the optimization problems. Section 4 provides a numerical analysis of suggested optimization models. Section 5 concludes the paper.

2. Data and Methodology

2.1. Description of the Data

Our sample comprises 30 stocks traded on the German Stock Exchange and included in the DAX30-index. We analyze the daily returns of these stocks for the period between 07.10.2003 and 02.03.2007. Daily returns were calculated as $r(t) = \log(S(t)/S(t-1))$, where *S(t)* is the stock daily closing at *t* (the stocks are adjusted for dividends). Everyday, we solve the optimization problem using the observations from the prior 250 working days and make a forecast for the next day. We analyze two portfolios: the benchmark portfolio and the portfolio of DAX30-stocks we want to optimize. Our benchmark portfolio is equally weighted portfolio of 30 shares, included in DAX30-index. We assume that portfolio manager would like to outperform the benchmark making different allocation decisions across industry classes. For that, we divide shares of DAX30 into 5 industry classes according to their trading volume:

$$Trading\_volume(share(i)) = Volume(T) * Stock\_value(T)),$$

where *i* is share's number, *T* is the time period, *Volume(T)* is the average volume over the entire period *T, and Stock value (T)* is the average stock value over the entire period *T.* Every class contains 6 shares (6 shares with the smallest trading volume refer to the first class, 6 shares with the largest trading volume refer to the 5th class). The structure of our benchmark portfolio is presented in Table 3 (see Section 4.2).

2.2. Calculation of the portfolios' parameters

Daily, we calculate the following parameters separately for each portfolio, based on observations of prior year (250 working days):

Portfolio of DAX-stocks	Benchmark portfolio
r_l is the random daily return of asset *l* in the DAX-index In the time series setting, $r_l(t)$ is the return of the asset l at time period (day) *t,* $l = 1,...,n; t = 1,...,T$, where n is a number of assets equal to 30, *T*=250. We will use the same notation for observed historical values of $r_l(t)$.	
R_l – mean return (expected value of r_l), $R_l = E(r_l)$	
z_{pl} – the weight of asset l in the portfolio of DAX30-stocks. $z_p = \begin{bmatrix} z_{p1} \\ z_{p2} \\ ... \\ z_{pn} \end{bmatrix}$ $\sum_{l=1}^{n} z_{pl} = 1$	z_{bl} – the weight of asset l in the benchmark portfolio. $z_b = \begin{bmatrix} z_{b1} \\ z_{b2} \\ ... \\ z_{bn} \end{bmatrix}$ $\sum_{l=1}^{n} z_{bl} = 1$ In our case, the benchmark portfolio is equally weighted, $z_{bl} = 1/30$.

$r^{(p)} = \sum_{l=1}^{n} z_{pl} r_l$ is the total random return from the DAX30-portfolio	$r^{(b)} = \sum_{l=1}^{n} z_{bl} r_l$ is the total random return from the benchmark portfolio
$w_{pi} = \sum_{l \in U_i} z_{pl}$ is the weight of asset class U_i in the portfolio of DAX30-stocks	$w_{bi} = \sum_{l \in U_i} z_{bl}$ is the weight of asset class U_i in the benchmark portfolio
$w_p = \begin{bmatrix} w_{p1} \\ w_{p2} \\ \ldots \\ w_{pm} \end{bmatrix}$, where *m* is a number of classes (in our case *m* is equal to 5) $\sum_{l=1}^{m} w_{pi} = 1$	$w_b = \begin{bmatrix} w_{b1} \\ w_{b2} \\ \ldots \\ w_{bm} \end{bmatrix}$. For our equally weighed benchmark portfolio, $w_{bi} = 1/5$ $\sum_{l=1}^{m} w_{bi} = 1$
$w_{pl}^{(i)} = \frac{z_{pl}}{\sum_{l \in U_i} z_{pl}} = \frac{z_{pl}}{w_{pi}}$ is the weight of asset *l* in its asset class U_i of the portfolio of DAX30-stocks	$w_{bl}^{(i)} = \frac{z_{bl}}{\sum_{l \in U_i} z_{bl}} = \frac{z_{bl}}{w_{bi}}$ is the weight of asset *l* in its asset class U_i of the benchmark portfolio For our equally weighed benchmark portfolio, $w_{bl}^{(i)} = \frac{1/30}{1/5} = \frac{1}{6}$
$r_{pi} = \sum_{l \in U_i} w_{pl}^{(i)} r_l$ is the random return from asset class *i* of the portfolio of DAX-stocks. $r^{(p)} = \sum_{l=1}^{n} z_{pl} r_l = \sum_{i=1}^{m} w_{pi} r_{pi}$, is the total DAX30-portfolio (daily) return $R_{pi} = E(r_{pi})$ is the expected return of asset class *i* of the portfolio of DAX-stocks $R^{(p)} = E(r^{(p)}) = \sum_{l=1}^{n} z_{pl} R_l = \sum_{i=1}^{m} w_{pi} R_{pi}$ is the total expected (daily) return of the portfolio of DAX-stocks over all classes	$r_{bi} = \sum_{l \in U_i} w_{bl}^{(i)} r_l$ is the random return from asset class *i* in the benchmark portfolio. $r^{(b)} = \sum_{l=1}^{n} z_{bl} r_l = \sum_{i=1}^{m} w_{bi} r_{bi}$ is the total benchmark-portfolio (daily) return $R_{bi} = E(r_{bi})$ is the expected return of asset class *i* of the benchmark $R^{(b)} = E(r^{(b)}) = \sum_{l=1}^{n} z_{bl} R_l = \sum_{i=1}^{m} w_{bi} R_{bi}$ is the total expected (daily) return of the benchmark portfolio

After these parameters are calculated, we calculate S, the total expected value added by the portfolio managers:

$$S = R^{(p)} - R^{(b)} = \sum_{i=1}^{m}(w_{pi}R_{pi} - w_{bi}R_{bi}) = \sum_{l=1}^{n}(z_{pl} - z_{bl})R_l .$$

The aim of portfolio attribution is to break down total value added into its main sources, namely: asset allocation (AA), security selection (SE), and interaction (I).

Asset Allocation (AA)

The contribution of the asset class *i* to the total value added measured by:

$$AA_i = (w_{pi} - w_{bi})(R_{bi} - R^{(b)}) .$$

The total asset allocation effect is measured by:

$$AA = \sum_{i=1}^{m}(w_{pi} - w_{bi})(R_{bi} - R^{(b)}) .$$

Selection Effect (SE)

The contribution of the total out performance of the choice of security within each asset class is given by:

$$SE_i = w_{bi}(R_{pi} - R_{bi})$$

The total selection effect is given by:

$$SE = \sum_{i=1}^{m} w_{bi}(R_{pi} - R_{bi})$$

Interaction

Bertrand Ph. (2005) stated that the sum of the asset allocation and selection effects is not equal to the total outperformance of asset class *i*, S_i. To ensure equality, it is necessary to add a term referred to as interaction that is defined by:

$$I_i = (w_{pi} - w_{bi})(R_{pi} - R_{bi}) .$$

It can be interpreted as the part of the excess return jointly explained by the asset allocation and selection effects. It can be defined as an extension of the effect of security selection: it is the security selection effect on the over- or under-weighted part of asset class i.

$$S = \sum_{i=1}^{m}(w_{pi}R_{pi} - w_{bi}R_{bi}) = \sum_{i=1}^{m}(AA_i + SE_i + I_i) .$$

3. Description of Optimization Problems

Our portfolio optimization models are based on minimizing a downside (tail) risk measure, called Expected Tail Loss (ETL), also known as Total Value-at-Risk (TVaR), Expected Shortfall, Conditional Value-at-Risk (CVaR), and defined as

$$ETL_\delta(X) = \frac{1}{\delta}\int_0^\delta VaR_q(X)dq ,$$

where $VaR_\delta(X) = -F_X^{-1}(\delta) = -\inf\{x / P(X \le x) \ge \delta\}$ is the Value-at-Risk (VaR) of the random return X. If we assume a continuous distribution for the probability law of X, then $ETL_\delta(X) = -E(X / X \le -VaR_\delta(X))$ and thus, *ETL* can be interpreted as the average loss beyond VaR (see Rachev S., Ortobelli S. et al. (2007)).

3.1. Optimization Tracking Error Problem with Constraints on AA, SE, S, and individual asset weights

Our goal is to find an optimal portfolio minimizing the tracking error measured by $ETL_{\delta}(r^{(p)} - r^{(b)})$. We shall examine various optimization problems choosing different δ =0.01, 0.05, 025 and 0.50, subject to constraints on the AA, SE and S in contrast to the standard tracking error given by the standard deviation $STD(r^{(p)} - r^{(b)})$, by using $ETL_{\delta}(r^{(p)} - r^{(b)})$. Thus, we do not penalize for positive deviations of our portfolio from the benchmark; we only minimize the downside movement of the optimal DAX30-portfolio from the benchmark (see Rachev S., Ortobelli S. et al. (2007)).

Optimization Problem 3.1.1: Minimum ETL-Tracking Error with Constraints on asset weights, AA and SE:

$$\min_{z_{pl}} ETL_{\delta}(r^{(p)} - r^{(b)})$$

such that

(i) $z_{pl} > 0$, where z_{pl} is the weight of individual asset l in the portfolio of DAX30-stocks,

$$z_p = \sum_{l=1}^{n} z_{pl} = 1;$$

(ii) $a \le AA = \sum_{i=1}^{m} (w_{pi} - w_{bi})(R_{bi} - R^{(b)}) \le b$;

(iii) $c \le SE = \sum_{i=1}^{m} w_{bi}(R_{pi} - R_{bi}) \le d$.

The constants a,b,c,d can be pre-specified to meet particular needs of the portfolio manager. In our case they can take arbitrary values.

Optimization Problem 3.1.2: Minimum ETL-Tracking Error with Constraints on asset weights, AA , SE and S:

$$\min_{z_{pl}} ETL_{\delta}(r^{(p)} - r^{(b)})$$

such that (i),(ii),(iii) hold and

(iv) $S = R^{(p)} - R^{(b)} \ge s$,

where s>0 the excess total (benchmark) expected value added we want to achieve with minimum ETL-tracking error.

3.2. Optimization Tracking Error Problem with constraints AA, SE , S, and asset classes weights

Optimization Problem 3.2.1: Minimum ETL-Tracking Error with Constraints on classes weights, AA, and SE:

$$\min_{w_{pi}} ETL_{\delta}(r^{(p)} - r^{(b)})$$

such that

(I) $w_{pi} > 0$, where w_{pi} is the weight of asset class i in the portfolio of DAX-stocks,

$$\sum_{l=1}^{m} w_{pi} = 1,$$

and (ii) and (iii) are hold.

In this optimization problem, after the optimal portfolio of DAX30-stocks is found, we determine the weights of assets within the classes in correspondence with the structure of the benchmark portfolio. As our benchmark portfolio is equally weighted portfolio, we impose that the classes in the optimal portfolio of DAX-stocks are also equally weighted portfolios. Each class contains 6 shares. It means that the weight of asset in the class will be found according to this strategy as $w_{pi}/6$, where w_{pi} is the optimal weight of the class *i* in the portfolio of DAX30-stocks.

Optimization Problem 3.2.2: Minimum ETL-Tracking error with constraints on classes weights, AA, SE and S:

$$\min_{w_{pi}} ETL_{\delta}(r^{(p)} - r^{(b)}),$$

such that (I), (ii), (iii) and (iv) hold .

The weights of assets are found as described in Optimization Problem 3.2.1.

4. Empirical Analysis of Optimal Portfolio Performance

Suppose an investor has an initial wealth of $W_0 = 1$ on September 20, 2004. Every day she solves the optimization problem described above using daily observed returns from the prior year. Once she determines the optimal portfolio of DAX30-stocks z_{pl}, at time *t,* based on the historical return values until t (including), that is, $z_{pl} = z_{pl}(t)$, *l=1,...,n* the portfolio wealth at time *(t+1)* generated by the portfolio allocation at time *t* is evaluated according to

$$W(t+1) = W(t)(1 + r^{(p)}(t+1)),$$

where the portfolio's return $r^{(p)}$ *(t+1)* at time *t+1* is given by

$$r^{(p)}(t+1) = z_{p1}(t)r_1(t+1) + ... + z_{pn}(t)r_n(t+1).$$

The cumulative portfolio return CR *(t+1)* at time *t+1*, generated by the portfolio allocation made at time *t,* is defined iteratively by

$$CR(t+1) = CR(t) + r^{(p)}(t+1).$$

Values of the final wealth and cumulative return for the benchmark portfolios were calculated in the same way.

4.1. Results Summary

In Table 1 (Panels A, B, C, D), we first present results obtained by the four different optimizations with different constraints on AA, SE, and S.

Our results show that

- Optimization Problem 3.1.2, based on $ETL_{0.05}$-tracking error-minimization, and
- Optimization Problem 3.1.1, based on $ETL_{0.05}$-tracking error-minimization with constraints $-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, z_{pl} \ge 0$,

provide the largest realized wealth and total realized return at the end of the period T. These strategies were referred to the most profitable strategies. Furthermore, Optimization Problem 3.1.2 provides consistently most profitable strategies over a variety of constraints sets, while Optimization 3.1.1 is best only in the long short strategy where the Optimization Problem 3.1.2 is marginally second. Overall, the best performing portfolio, regardless whether we can consider long-only or long-short strategy, is given Optimization Problem 3.1.2.

Table 1

Summary statistics over different analyzed optimization models

Portfolios	ETL(0.01)		ETL(0.05)		ETL(0.25)		ETL(0.5)	
	Realized wealth	Annualized Total Realized Return (%)	Realized wealth	Annualized Total Realized Return (%)	Realized wealth	Annualized Total Realized Return (%)	Realized wealth	Annualized Total Realized Return (%)
Benchmark	1.47	16.26%	1.47	16.26%	1.47	16.26%	1.47	16.26%
Panel A: Long only constraints on AA and SE								
Optimization Problem 3.1.1 ($0 \le AA \le 1, 0 \le SE \le 1, z_{pl} \ge 0$)								
Optimal Portfolio	1.91	26.68%	2.04	29.33%	1.82	24.60%	1.88	26.01%
Optimization Problem 3.1.2 ($0 \le AA \le 1, 0 \le SE \le 1, S \ge 0, z_{pl} \ge 0$)								
Optimal Portfolio	1.94	27.41%	2.09	30.23%	1.83	24.88%	1.92	26.89%
Optimization Problem 3.2.1 ($0 \le AA \le 1, 0 \le SE \le 1, w_{pi} \ge 0$)								
Optimal Portfolio	NO feasible solution							
Optimization Problem 3.2.2 ($0 \le AA \le 1, 0 \le SE \le 1, S \ge 0, w_{pi} \ge 0$)								
Optimal Portfolio	NO feasible solution							
Panel B: Long-short constraints on AA and SE								
Optimization Problem 3.1.2 ($-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, S \ge 0, z_{pl} \ge 0$)								
Optimal Portfolio	1.88	26.03%	2.13	31.03%	1.85	25.38%	1.89	26.09%
Optimization Problem 3.2.1 ($-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, w_{pi} \ge 0$)								
Optimal Portfolio	0.82	-5.56%	1.19	8.70%	1.20	8.89%	0.92	0.08%
Optimization Problem 3.2.2 ($-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, S \ge 0, w_{pi} \ge 0$)								
Optimal Portfolio	NO feasible solution							
Panel C: No constraints on AA, long only constraints on SE								
Optimization Problem 3.1.1 ($-1 \le AA \le 1, 0 \le SE \le 1, z_{pl} \ge 0$)								
Optimal Portfolio	1.88	26.03%	2.10	30.53%	1.82	24.63%	1.89	26.16%
Optimization Problem 3.1.2 ($-1 \le AA \le 1, 0 \le SE \le 1, S \ge 0, z_{pl} \ge 0$)								
Optimal Portfolio	1.88	26.03%	2.12	30.74%	1.83	24.87%	1.90	26.38%
Optimization Problem 3.2.1 ($-1 \le AA \le 1, 0 \le SE \le 1, w_{pi} \ge 0$)								
Optimal Portfolio	NO feasible solution							
Optimization Problem 3.2.2 ($-1 \le AA \le 1, 0 \le SE \le 1, S \ge 0, w_{pi} \ge 0$)								
Optimal Portfolio	NO feasible solution							
Panel D: No constraints on SE, long only constraints on AA								
Optimization Problem 3.1.1 ($0 \le AA \le 1, -1 \le SE \le 1, z_{pl} \ge 0$)								
Optimal Portfolio	1.92	26.87%	2.06	29.66%	1.81	24.40%	1.89	26.08%
Optimization Problem 3.1.2 ($0 \le AA \le 1, -1 \le SE \le 1, S \ge 0, z_{pl} \ge 0$)								
Optimal Portfolio	1.94	27.41%	2.09	30.23%	1.83	24.92%	1.93	26.91%
Optimization Problem 3.2.1 ($0 \le AA \le 1, -1 \le SE \le 1, w_{pi} \ge 0$)								
Optimal Portfolio	1.23	8.62%	1.28	10.28%	1.33	11.90%	1.36	12.52%
Optimization Problem 3.2.2 ($0 \le AA \le 1, -1 \le SE \le 1, S \ge 0, w_{pi} \ge 0$)								
Optimal Portfolio	NO feasible solution							

Table 1 reports values of realized wealth and annualized total realized return obtained over different mathematical models with different restrictions on AA, SE, and S. The sample includes a total of 30 stocks traded on the German Stock Exchange during the period of October 2003 and March 2007.

As the investigation of the strategies based on comparison of realized wealth and total realized return doesn't take risk into account, we further analyze the sequences of realized excess returns, obtained over the most profitable optimization problems, consider their tail-risk profile and select the ones with best risk-return performance. Our next goal is to determine a model which achieves the Best Tracking Error Portfolio with Performance Attribution Constraints (we call shortly this portfolio BTEP) taking tail-risk (probability for large losses) into account. For that, we consider the sequences of realized excess returns:

$s(t) = r^{(p)}(t) - r^{(b)}(t)$, *t=1,...,T*,

where

$r^{(p)}(t)$ is the DAX30-portfolio return at time *t* generated by the portfolio allocation at time *(t-1)*, obtained by the optimal strategies obtained in solving the corresponding optimization problems 3.1.1, 3.1.2, 3.2.1, and 3.2.2;

$r^{(b)}(t)$ is the equally weighted benchmark portfolio return at time t.

The sequence of excess returns contains total T= 632 observations starting from the 251st day of the period examined as the "first-day" optimization problem is solved based on first 250 observations of the first year.

We start our risk-analysis of the optimal portfolios by computing the most commonly accepted risk-reward measure, the Sharpe Ratio (Sharpe W. (1994)) (see (1) below), using the sequences of realized excess returns *s(t)* for *t=1,...,T.* However, in order to include the observed non-normality distribution of the realized excess returns *s(t)* in the risk-return analysis, we also calculate the STAR Ratio (STARR) and R-Ratio (see (2) and (3) below) as alternatives to the Sharpe ratio replacing the standard deviation in the Sharpe ratio with the tail-risk measured by ETL. We analyze and compare STARR Ratio (0.05), and R-Ratio (0.05, 0.05) using the 5% of the excess highest and lowest returns. The choice of those quantiles is based on the performance-evidence we have collected in our previous papers on portfolio optimization (see for example Biglova A. et al. (2004 a, b).

We now give a summary of the three performance ratios:

1. The Sharpe Ratio (see Sharpe W. (1994)) is the ratio between the expected excess return and its standard deviation of the realized excess returns *s* with stable distribution determined by the sample T *s(t) , t=1,...,T* :

$$\rho(s) = \frac{E(s)}{STD(s)} \ , \qquad (1)$$

where $E(s)$ and $STD(s)$ is the mean and sample standard deviation *s*. For this Ratio it is assumed that the second moment of the excess return exists, thus the stable distribution we use for modeling the probability distribution of *s* is, in fact, Gaussian. (We give the definition and discuss the basic properties of stable distributions for modeling asset returns in the next section.)

2. STARR (0.05) (see Rachev S. et al. (2007 a)) is the ratio between the expected excess return and its Expected Tail Loss:

$$\rho(s) = \frac{E(s)}{ETL_\delta(s)}, \qquad (2)$$

where $ETL_\delta(s)$ is defined in Section 3.

3. R-Ratio (0.05, 0.05) is the ratio between the Expected Tail Return ETR(s) = ETL(-s) at a given confidence level and the ETL of the excess return at another confidence level.

$$\rho(s) = \frac{ETL_{\gamma 1}(-s)}{ETL_{\gamma 2}(s)}. \qquad (3)$$

We analyze the R-Ratio for parameters $\gamma_1 = \gamma_2 = 0.05$.

For (2) and (3) to exist (to be well defined) we only need that the index of stability of *s* is greater than 1, which is the mean of *s* exists. All empirical analyses on the distribution of asset returns show that, without restriction, one can assume that the mean of asset returns is finite (see Rachev S. (2007 a) and the references there in).

Table 2 reports values of stable distribution parameters, performance ratios: Sharpe Ratio, STARR-Ratio (0.05), and R-Ratio (0.05, 0.05) for realized excess returns over analyzed strategies. Results, presented in Table 2, show that Sharpe Ratio is not suitable to be applied as the coefficients of stable fit confirm that the realized excess returns are non-Gaussian, heavy-tailed and skewed, hence STARR and R-Ratio are more reliable. This table shows that Optimization Problem 3.1.2 based on ETL (0.05) with long-short constraints on AA and SE: $-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, S \ge 0, z_{pl} \ge 0$ provides the best values of STARR equal to 0.0622 and R-Ratio equal to 1.8813, therefore it provides the best portfolio when we take into account the tail-risk of the realized excess returns.

We call this optimal portfolio the Best Tracking Error Portfolio with Performance Attribution Constraints (**BTEP**) and we shall analyze it now in more detail.

Table 2

Summary statistics of excess realized returns over the most profitable optimization problems

alpha	beta	sigma	mu	Sharpe Ratio	STARR Ratio (0.05)	R-Ratio(0.05,0.05)
Optimization Problem 3.1.2 based on ETL(0.05) long only constraints on AA and SE: $0 \le AA \le 1, 0 \le SE \le 1, S \ge 0, z_{pl} \ge 0$						
1.5557	0.3066	0.0031	7.021e-004	0.0879	0.0397	1.4005
Optimization Problem 3.1.2 based on ETL(0.05) long-short constraints on AA and SE: $-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, S \ge 0, z_{pl} \ge 0$						
1.5571	0.2927	0.0031	7.059e-004	0.0907	0.0622	1.8813
Optimization Problem 3.1.2 based on ETL(0.05) no constraints on AA, long only constraints on SE: $-1 \le AA \le 1, 0 \le SE \le 1, S \ge 0, z_{pl} \ge 0$						
1.5571	0.2927	0.0031	7.096e-004	0.0911	0.0324	1.3483
Optimization Problem 3.1.1 based on ETL (0.05) no constraints on SE, long only constraints on AA: $-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, z_{pl} \ge 0$						
1.5325	0.3120	0.0031	7.912e-004	0.0928	0.0278	1.1937

Table 2 reports values of stable distribution's parameters for the sequences of realized excess returns and values of performance Ratios for those sequences.

4.2. Analysis of the BTEP

Having solved the optimization problem daily, a total of 632 times, in the period from 07.10.2003 to 02.03.2007, we present mean statistics of the obtained daily optimal portfolios of DAX30-stocks in Table 3.

Table 3

Mean statistics of the Best Tracking Error Portfolio with Performance Attribution Constraints (BTEP)

Optimization Problem BTEP: Constraints: $-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, S \ge 0, z_{pl} \ge 0$, $ETL_{0.05}$					
Classes	Shares	Trading volume (Euro, Millions)	Mean of the asset weights in the optimal portfolio	Mean of the asset class weights in the optimal portfolio	Mean of the asset weight in the class
1st class	hrx	24.55	0.0214	0.1047	0.2044
	fme	27.39	0.1666		0.1591
	henkel	30.37	0.0166		0.1591
	alt	33.05	0.0166		0.1591
	lin	39.67	0.0166		0.1591
	tui-n	41.26	0.0166		0.1591
2nd class	lha	49.1	0.0166	0.3856	0.0432
	man	49.84	0.0166		0.0432
	meo	51.43	0.0166		0.0432
	con	65.73	0.1234		0.3202
	db1	67.11	0.1788		0.4638
	tka	69.61	0.0332		0.0863
3rd class	sch	69.84	0.1764	0.2598	0.6792
	dpw	75.19	0.0166		0.0641
	cbk	93.43	0.0166		0.0641
	ifx	93.87	0.0166		0.0641
	bmw	96.64	0.0166		0.0641
	bayer	135.37	0.0166		0.0641
4th class	vow	151.45	0.0166	0.1498	0.1112
	rwe-a	158.18	0.0451		0.3013
	bas	162.37	0.0166		0.1112
	ads	183.6	0.0380		0.2537
	muv2	195.43	0.0166		0.1112
	sap	227.13	0.0166		0.1112
5th class	dcx	238.95	0.0166	0.1000	0.1666
	eoa	262.08	0.0166		0.1666
	dbk	305.46	0.0166		0.1666
	siemens	330.33	0.0166		0.1666
	alv	333.44	0.0166		0.1666
	dte	347.28	0.0166		0.1666

Table 3 reports shares, divided into 5 classes according to their trading volumes, values of trading volumes of appropriate shares, presented in millions of Euros, means of the asset weights, asset class weights and asset weights in the class over 632 optimal portfolios of DAX-stocks, obtained solving the Optimization Problem 3.1.2 daily, a total 632 times, and based on minimizing of $ETL_{0.05}$ with constraints $-0.03 \le AA \le 0.3, -0.03 \le SE \le 0.3, S \ge 0, z_{pl} \ge 0$ during the period of October 2003 and March 2007.

Table 3 shows that for most cases, the main parts of optimal portfolios of DAX30-stocks were the shares of the second and the third classes (39% and 26%). It means that in most cases the portfolio of DAX-stocks consisting of shares with "average" trading volumes outperforms the portfolio of benchmark.

Figure 1 presents graphs of the realized final wealth of the portfolio of DAX-stocks and benchmark portfolio. This exhibit shows that the portfolio of DAX-stocks sample paths dominate the benchmark sample paths and they yield the maximum wealth of 2.11 at the end of the period examined, the maximum wealth of the benchmark portfolio is equal to 1.47 at the end of the period examined.

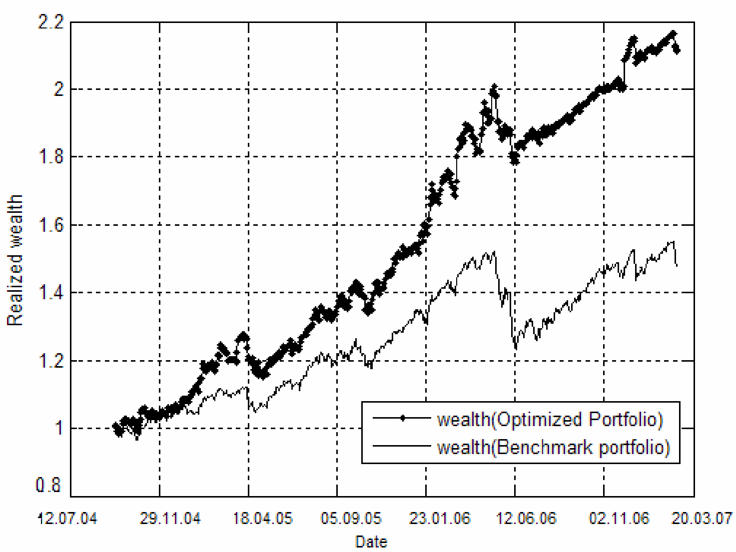

Fig. 1. Realized Wealth of the Optimized Portfolio (BTEP) and the Benchmark Portfolio

Figure 2 presents sample paths of cumulative returns for the portfolio of DAX-stocks and the benchmark portfolio. The plots also show that the portfolios of DAX-stocks always perform better than the benchmark portfolio and it yields the maximum total realized annualized return equal to 77.53% at the end of the period examined (and thus the annualized value is 30.67%, the total realized return of the benchmark portfolio is equal to 41.11% at the end of the period examined (with annualized value of 16.26%).

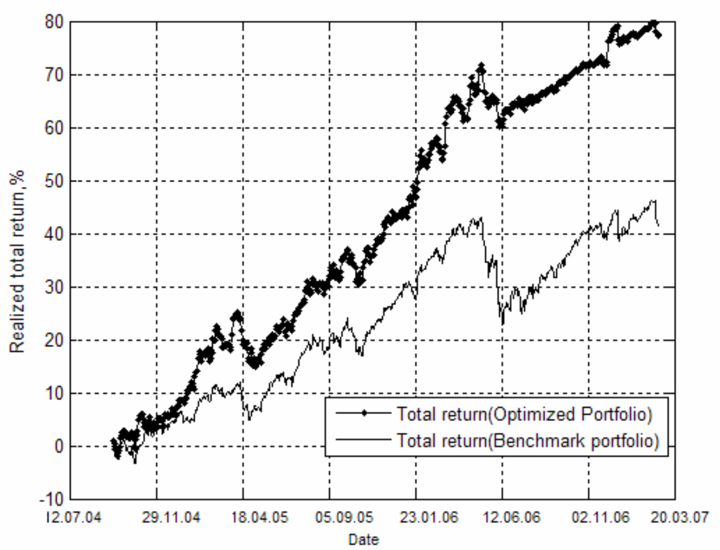

Fig. 2. Total Realized Return of the Optimized Portfolio (BTEP) and the Benchmark Portfolio

We now focus on the statistical analysis of the time series of AA, SE and S-values in the optimal DAX30-portfolio and the benchmark portfolio.

We view the observations of Asset Allocation (AA), Selection Effect (SE), and Total expected value, added by portfolio managers (S), calculated in solving the optimization problems 632 times, as three samples of size 632 each, and we would like to study the distributional properties of the AA, SE, and S, and in particular mean-values, dispersion, skewness and kurtosis.

The fist observation we made concerns the non-normality of the distribution of the samples for AA, SE, and S. We observe that by testing the hypotheses about normal (Gaussian) versus stable (non-Gausian, Paretian) distributions for the AA, SE and S values.

Let us first recall some basic facts on stable distributions. The α -stable distributions describe a general class of distribution functions which include leptokurtic and asymmetric distributions. A random variable X is *stable* distributed if there exists a sequence of i.i.d. random variables $\{Y_i\}_{i\in N}$, a sequence of positive real values $\{d_i\}_{i\in N}$ and a sequence of real values $\{a_i\}_{i\in N}$ such that, as $n \to +\infty$:

$$\frac{1}{d_n}\sum_{i=1}^{n} Y_i + a_n \xrightarrow{d} X,$$

where "$\xrightarrow{d}$" points out the convergence in distribution. The characteristic function which identifies a stable distribution is given by:

$$\Phi_X(u) = E(\exp(iuX)) = \begin{cases} \exp\left\{-\gamma^{\alpha}|u|^{\alpha}\left(1 - i\beta\,\mathrm{sgn}(u)\tan(\pi\alpha/2)\right) + i\mu u\right\} & \text{if } \alpha \neq 1 \\ \exp\left\{-\gamma|u|\left(1 + i\beta\frac{2}{\pi}\mathrm{sgn}(u)\log(u)\right) + i\mu u\right\} & \text{if } \alpha = 1 \end{cases}.$$

Thus, an α -stable distribution is identified by four parameters: the index of stability $\alpha \in (0,2]$ which is a coefficient of kurtosis, the skewness parameter $\beta \in [-1,1]$, $\mu \in \Re$ and $\gamma \in \Re^{+}$, which are respectively, the location and the dispersion parameter. If *X* is a random variable whose distribution is α -stable, we use the following notation to underline the parameter dependence (see Samorodnitsky G., Taqqu M. (1994)): $X \sim S_{\alpha}(\gamma, \beta, \mu)$.

When $\alpha = 2$ and $\beta = 0$ the α -stable distribution has a Gaussian density. The α -stable distributions with $\alpha < 2$ are leptokurtotic and present fat tails. While a positive skewness parameter ($\beta > 0$) identifies distributions whose tails are more extended towards right, the negative skewness parameter ($\beta < 0$) typically characterizes distributions whose tails are extended towards the negative values of the distribution. If $\alpha < 2$, then *X* is called stable (non-Gaussian, or Paretian) random variable.

We estimate the stable distribution parameters of the sequences AA, SE and S by maximizing the likelihood function (see McCulloch J. (1998), Stoyanov S. and Racheva-Iotova B. (2004 a, b)). It is possible to obtain optimal approximations of the stable parameters with STABLE program, developed and described in Stoyanov S., Racheva B. (2004 a, b)).

We compute the main parameters of the stable law: the index of stability α, skewness parameter β, which will characterize the heavy-tailedness and asymmetry of the observations' distributions respectively. We also compute μ and σ in the Gaussian fit. The normality tests employed are based on the Kolmogorov distance (KD) and computed according to

$$KS = \sup_{x\in R} | F_S(x) - \hat{F}(x) |,$$

where $F_S(x)$ is the empirical sample distribution and $\hat{F}(x)$ is the standard normal cumulative distribution function evaluated at *x* for the Gaussian or stable fit, respectively.

Our results show that we can reject the normality using the standard Kolmogorov-Smirnov test for observations of AA, SE and S values at the extremely high confidence level of 99%. In contrast, the stable-Paretian hypothesis is not rejected for these sequences at the same confidence level. Figure 3 presents the graphs of distribution densities of AA, SE, and S sequences. Figure 4 presents the histograms of AA, SE, and S values with respect to normal distributed values.

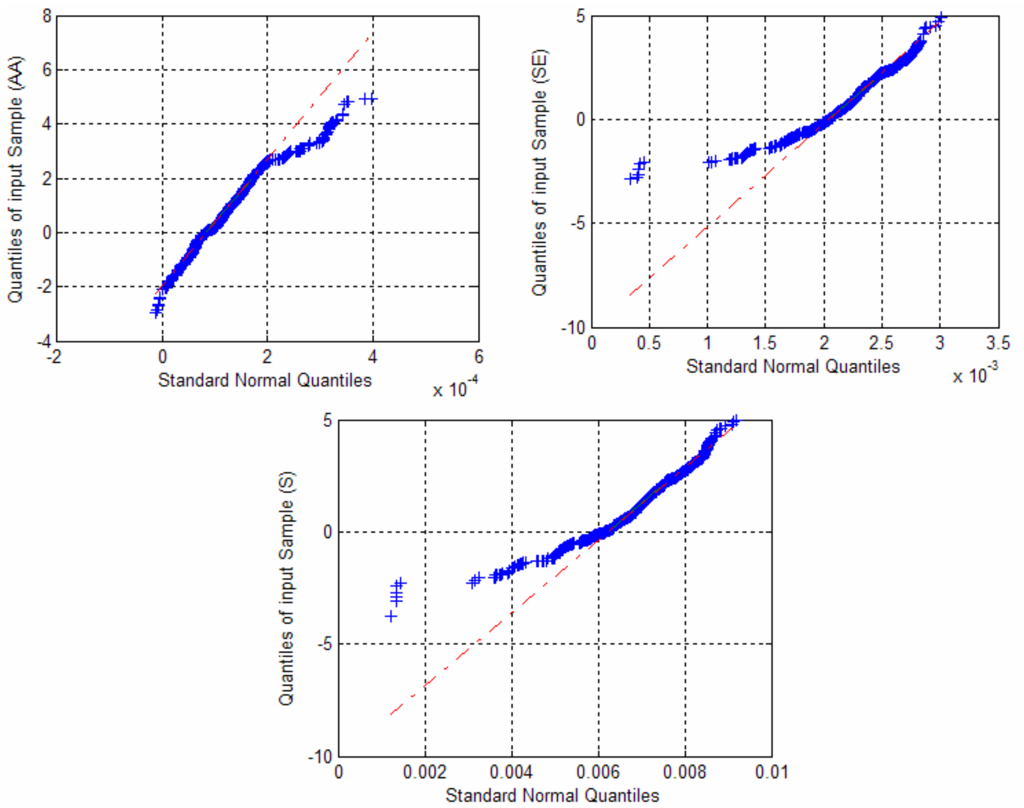

Fig. 3. Quantile-quantile (QQ) plots of the AA, SE and S quantiles and corresponding the normal (Gaussian) quantiles in the BTEP

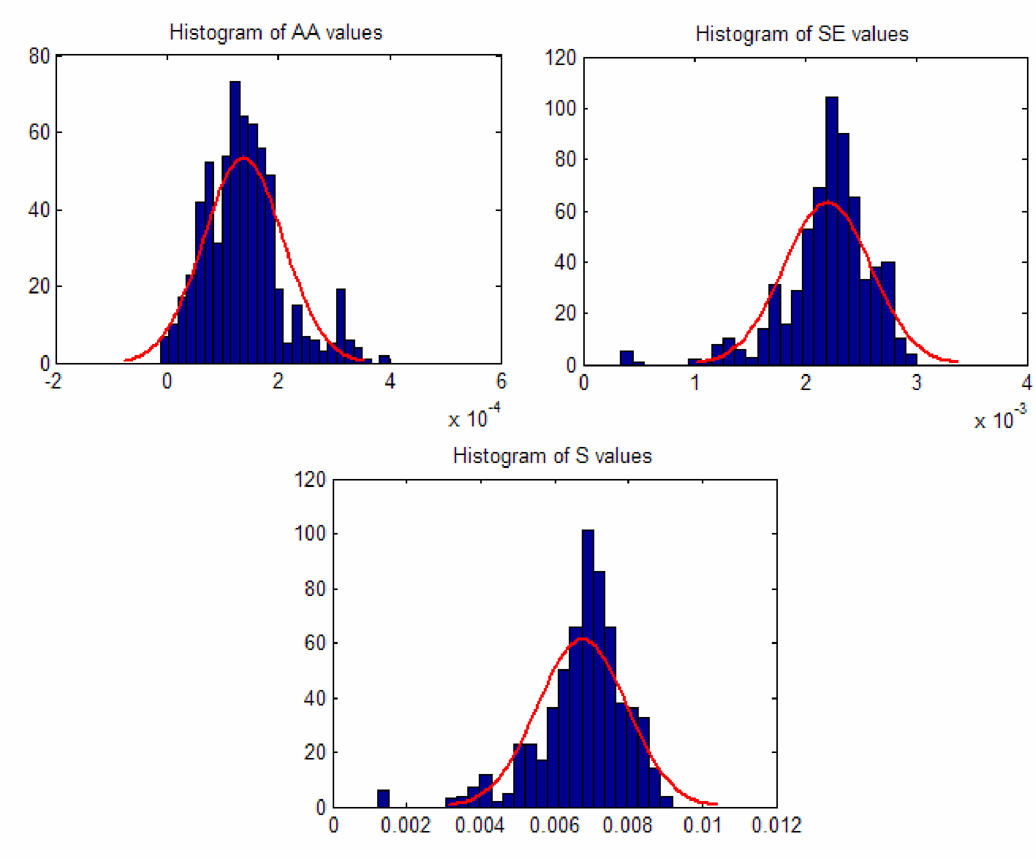

Fig. 4. Histogram of AA, SE and S values and the normal density fit

Graphs show that the analyzed observations exhibit heavier tails than that the normal. The fit of stable non-Gaussian distribution is now applied to the observations and the parameters of stable distribution are obtained. Table 4 presents obtained parameters and K-S statistics for the normal and stable non-Gaussian cases. The mean values of annualized AA, SE and S are also presented in Table 4.

Table 4

Estimated parameters alpha, beta of AA, SE, and S sequences and K-S distances under the normal and the stable distribution in the BTEP

	Alphas	Betas	K-S distances (normal case)	K-S distances (stable case)	Mean annualized values (%)
AA	1.4599	0.5932	0.7594	0.0664	1.36%
SE	1.3624	-0.4191	0.7563	0.0537	21.99%
S	1.3670	-0.4307	0.7705	0.0569	67.35%

Table 4 shows that the K-S distances in the stable case are 10 times smaller than the K-S distances in the Gaussian case for the analyzed sequences. So showing clearly that the stable fit outperforms the Gaussian one.

5. Conclusions

In this study, we further develop performance attribution methods introducing new optimization models based on ETL-risk measure. We determine the most profitable model for portfolio optimization, which best outperformed the benchmark portfolio. In addition, we analyze the distributional properties of Asset Allocation (AA), Selection Effect (SE) and Total Expected Value, Added by portfolio managers (S), and strongly reject for those sequences the normality assumption in favor of the stable Paretian Hypothesis.

In the future, we expect to confirm the obtained results on a large dataset and further develop suggested models.

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