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Enlarged Separation Portfolios and Financial Synthetics

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Abstract

This paper sets forth a wider-angle perspective to deal with arbitrage portfolios and financial synthetics by means of a very simple construct: the enlarged separation portfolio, which consists of a risk free asset and a market index. However, and this is a distinctive feature, at variance with the so called simple separation portfolio that lies on the Capital Market Line, the enlarged ones are foreign to the CML. On balance, these portfolios are cheaper and more feasible than other alternatives to cope with arbitrage and synthetics; besides, they only require of the standard setting that the Security Market Line world provides eventually. The main outcome of this paper grants that any mispriced asset can be synthesized with an enlarged separation portfolio, which in turn could be used to build up an arbitrage portfolio against a simple separation portfolio lying on the SML.

JEL: G11, G12.

Key words: separation portfolios, enlarged separation portfolios, arbitrage, synthetics, capital market line, security market line.

Introduction

It is usually understood that synthetic securities are patterns of cash flows built up from combining or decomposing sets of securities in order to replicate the cash flow streams of other distinctive securities[1]. Currently, financial engineers, by drawing heavily from derivatives markets widely resort to synthesizing securities, even portfolios, to design new financial assets and provide economic agents with risk-management shields.

However, another sort of synthetics are focused on this paper, namely those that can be attained when taking into account the risk-return profile of some asset we wish to synthesize, or against which we look for an arbitrage opportunity, within the framework provided by some equilibrium model performing as a benchmark, in this case the Security Market Line, SML[2].

The roadmap is the following: In section 1, we expand on enlarged separation portfolios and derive a lemma by which they contribute to set up useful arbitrage portfolios. Whereas section 2 does the groundwork with synthetics, it is for section 3 to show the technical constraints that simple separation portfolios meet when portfolio managers intend to use them as synthetizers. Section 4 shows how enlarged separation portfolios overcome those constraints and successfully perform as synthetics and, at the same time, it ties together the strands of analysis deployed by the former sections.

1. Enlarged Separation Portfolios

It is usually meant by a separation portfolio the one that consists only of risk-free asset (F) and the market portfolio (M). In vectorial format, a separation portfolio comes defined this way

$$\begin{cases} S_e = < x_F \,;\, x_M > \\ x_F + x_M = 1 \end{cases}.$$

These portfolios are the outstanding output of the Capital Market Line, CML, whose main assumptions further the Markowitz approach, by adding homogeneous expectations, a risk free

[1] A standard rendering stressing the financial swaps is in Marshall and Kapner (1993).

[2] Background at a general level in Elton-Gruber (1995), while Apreda (2001a, 2001b, 2003b) offers a detailed account of these topics.

asset, and equilibrium in the capital markets (the classic report can be found in Sharpe, 1970). In this setting, the portfolios lying on the CML are efficient and dominate the efficient frontier consisting of risky portfolios in the Markowitz´s sense.

We are going to design an outgrowth of those portfolios, which we will label enlarged separation portfolios.

Definition 1. Enlarged Separation Portfolios

By an **enlarged separation portfolio S_e** we mean a portfolio with the following features:

$$\begin{cases} S_e = < x_F ; x_M > \\ x_F + x_M \neq 1 \end{cases}.$$

In such context, F stands as the risk-free asset, and M as a suitable market index. It seems of the essence to keep in mind two remarks predicated upon this definition:

- Whereas in the CML´s world separation portfolios fulfill all the assumptions conveyed by the model, which makes them efficient in a mean-variance translation, enlarged separation portfolios do not require the model's assumptions to be met eventually.
- If we want to make operational the CML in the real world, we have to substitute the market portfolio for an index that proxies the former as a second best resource for the analysis. Enlarged portfolios do not need such adjustment since they have already been defined by means of a market index.

Now, we raise the following question: is there any close connection between enlarged separation portfolios and the plain separation portfolios that come out of the CML? To address this issue, let us suppose we have chosen a risk-free asset and a market index and draw up the portfolios $\mathbf{S_e}$ and **S**, namely

a) enlarged separation portfolio

$$\begin{cases} S_e = < x_F ; x_M > \\ x_F + x_M \neq 1 \end{cases};$$

b) plain separation portfolio

$$\begin{cases} S = < y_F ; y_M > \\ y_F + y_M = 1 \end{cases}.$$

The first relationship that comes up to our minds is that both portfolios have a similar structure, consisting of a proportion of risk-free asset on the one hand, and a proportion of market portfolio, proxied by an index, on the other.

But if we seek out a deeper linkage among them, it will be worth considering what would happen when we request the same level of systematic risk for both of them[1]:

$$\beta(S_e) = \beta(S). \tag{1}$$

As the plain separation portfolio also lies on the SML, it holds that

$$\beta(S) = x_F \cdot \beta_F + x_M \cdot \beta_M = x_M \tag{2}$$

since $\beta_F = 0$, and $\beta_M = 1$. Therefore,

[1] This will prove a crucial assumption for the construct this paper brings forth and the ensuing statements around arbitrage portfolios and synthetics.

$$\beta(S) = y_M .$$

Let us unfold the consequences of such a choice, since several points remain to be settled so far.

a) Since the plain portfolio belongs to the SML, it follows that

$$E[R(S)] = R_F + < E[R(M)] - R_F > . \beta(S) \quad (3)$$

and by (2)

$$E[R(S)] = R_F + < E[R(M)] - R_F > . y_M$$

that can be rewritten as

$$E[R(S)] = (1 - y_M) . R_F + E[R(M)] . y_M \quad (4)$$

b) For being S_e a portfolio of financial assets in the SML´s world, its risk-return profile fulfils the following constraints:

$$E[R(S)] = x_F . R_F + E[R(M)] . x_M$$

and also

$$\beta(S_e) = x_F . \beta_F + x_M . \beta_M = x_{M.}$$

On the other way, (1) and (2) lead[1] to

$$\beta(S_e) = \beta(S) = y_M = x_M$$

and so we get

$$E[R(S_e)] = x_F . R_F + E[R(M)] . y_{M.} \quad (5)$$

For being S_e an enlarged separation portfolio, we have

$$x_F + x_M \neq 1$$

but this is equivalent to the statement

$$x_F + x_M - \alpha = 1$$

for some $\alpha \neq 0$ that can be solved uniquely[2]. In this way,

$$x_F - \alpha = 1 - x_M = 1 - y_M$$

and, by means of (4), we reach to a new expression for y_F:

$$x_F - \alpha = 1 - x_M = 1 - y_M = y_F$$

and now we can brief this outcome as

$$x_F = y_F + \alpha .$$

By coming back to (5), we can write

$$E[R(S_e)] = (y_F + \alpha) . R_F + E[R(M)] . y_M$$

which is equivalent to

$$E[R(S_e)] = < y_F . R_F + E[R(M)] . y_M > + \alpha . R_F$$

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[1] See Apreda (2001a) on arbitrage portfolios for further information.

[2] If S $_e$ lay over the SML where S lives, then α is greater than zero. Otherwise, it would be less than zero.

but the expression between brackets is (3). Hence

$$E[\, R(\, S_{e}\,)] \;=\; E[\, R(\, S\,)] \;+\; \alpha\,.\, R_{F} \qquad (6)$$

c) Let us give some heed to (6), because it improves our understanding of the whole subject.

In the first place, when $\alpha > 0$, it means that S_e lies over the SML, showing a higher return than ***S*** and becoming therefore cheaper than the latter.

In the second place, when $\alpha < 0$, it means that S_e lies under the SML, showing a lower return than S and becoming so more expensive than the latter.

All in all, these remarks adds up the following statement: in both cases we can set up arbitrage portfolios[1].

When S_e becomes cheaper, we would have

$$\Delta P \;=\; <\, x_{long}\;;\; x_{short}\,> \;=\; <\, x\,(\,S_{e}\,)\,;\; x\,(\,S\,)\,>\,,$$

when S_e becomes more expensive, we would have

$$\Delta P \;=\; <\, x_{long}\;;\; x_{short}\,> \;=\; <\, x\,(\,S\,)\,;\; x\,(\,S_{e}\,)\,>\,.$$

Throughout this line of analysis, from a) to c), we have already proved the following lemma.

Lemma 1 ***Given an enlarged separation portfolio S_e we can always proceed to build up arbitrage portfolios whenever we choose a plain separation portfolio S for which it holds***

$$\beta(S_{e}) \;=\; \beta(S)\,.$$

Besides, the corollary that comes next takes advantage of (6) and explains why it is that a separation portfolio cannot live in the SML.

Corollary ***An enlarged separation portfolio S_e cannot lie on the Security Market Line.***

Proof: Let us suppose that

$$x_{F} + x_{M} \;=\; 1 \;+\; \alpha \;\;,\quad \alpha > 0$$

it follows that

$$x_{F} \;=\; 1 \;- x_{M} \;+\; \alpha$$

On the other hand, the expected return of S_e comes assessed by

$$E[R(S_{e})] \;= x_{F}\,.\, R_{F} \;+\; x_{M}\,.\, E[R_{M}]$$

that means

$$E[R(S_{e})] \;= (1 \;- x_{M} \;+\; \alpha\,)\,.\, R(F) \;+\; x_{M}\,.\, E[\, R_{M}\,]$$

which leads to

$$E[R(S_{e})\,] \;= \alpha\;.\, R(F) \;+ <(1 \;- x_{M})\,.\, R_{F} \;+\; x_{M}\,.\, E[\, R(M)\,] >$$

that can be rewritten as

$$E[R(S_{e})] \;= \alpha\;.\, R(F) \;+ <\; R_{F} \;+\; (\, E[\, R(M)] - R_{F}\,)\,.\, x_{M} >$$

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[1] We say that ΔP is an arbitrage portfolio when these features hold: a) $x_{long} + x_{short} = 0$; $x_{long} = +1$; $x_{short} = -1$; b) $\beta(\Delta P) = 0$; c) $E[\, R(\Delta P)\,] > 0$. In other words, it is self-financing, riskless and profitable. More background on this is in Apreda (2001a).

now, what we find within brackets amounts to the expected return of an asset that has as its beta the value x_M .

Furthermore, any asset with this beta and living in the SML would have such expected return, and not other. But $\alpha > 0$, which prevents S_e from lying on the SML.

2. Synthetic Securities

For any financial asset or portfolio A, its risk-return profile RR in the SML world comes defined by the vector

$$RR(A) \quad = \quad < \beta(A) \; ; \; E[R(A)] \quad > .$$

We are moving forward so as to make explicit what a synthetic will mean in the context of this section.

Definition 2. Synthetic Portfolios

By a ***synthetic portfolio P of the asset A,*** *it is meant a portfolio*

$$P \;=\; < \; x_1 \, ; x_2 \, ; x_3 \, ; \;\; \ldots\ldots \; ; x_N \; >$$

so that its vectorial risk-return profile fulfills the following boundary condition:

$$RR(P) = < \beta(P); E[R(P)] >= \begin{cases} \beta(P) = \beta(A) \\ E[R(P)] = E[R(A)] \end{cases}.$$

Although mathematically we can find synthetics for A under almost any circumstances (we have to solve a system of two equations with N unknowns), it goes without saying that one thing is to get a theoretical solution and quite another to being able to come across with a down-to-earth synthetic, because of transaction costs and the always pervasive market microstructure (Apreda, 2001a, 2001b, 2003a, 2003b).

3. Separation Portfolios as Synthetics

In this section, we expand on the kind of synthetics a simple separation portfolio can furnish, leaving for the next section the case for enlarged separation portfolios. The development will make apparent that getting access to simple or enlarged separation portfolios depends on whether security ***A*** belongs or not to the Security Market Line.

Lemma 2 ***If A belongs to the SML, then there is always a separation portfolio S such that***

$$\beta(S) \;=\; \beta(A)$$

which qualifies as a synthetic of A.

Proof: When A lies on the SML its risk-return profile comes out of

$$E[R(A)] \;=\; R(F) \;+\; < \; E[R(M)] \; - \; R(F) \; > \; \times \; \beta(A) \, .$$

On the other hand, separation portfolios

$$S \;=\; < \; x_F \; ; \; x_M >$$

lie on the Capital Market Line, it holds by (2) that

$$\beta(S) \;=\; x_M$$

then, by choosing

$$\beta(S) \;=\; \beta(A)$$

it follows that

$$S \;=\; < x_F \; ; \; x_M > \;=\; < 1 - \beta(S) \; ; \; \beta(S) \;> \;=\; < 1 - \beta(A) \; ; \; \beta(A) \;>$$

which, in fact, qualifies as a good synthetic for A since

$$E[\, R(S) \,] \;=\; x_F \times R(F) \;+\; x_M \times E[R(M)] \,,$$

$$E[\, R(S) \,] \;=\; (\, 1 - \beta(A) \,) \times R(F) \;+\; \beta(A) \;\times E[R(M)] \,,$$

$$E[\, R(S) \,] \;=\; R(F) \;+\; < E[\, R(M) \,] \;-\; R(F) \;> \;\times\; \beta(A) \;=\; E[R(A)] \,.$$

By the same token,

$$\beta(S) \;=\; x_F \times \beta(F) \;+\; x_M \times \beta(\, M\,)$$

But the risk-free asset has a beta equal to zero while the market portfolio has a beta equal to one. Hence,

$$\beta(S) \;=\; x_M \;=\; \beta(A)$$

The question arises as to whether we could find a feasible plain separation portfolio acting as a synthetic for any mispriced asset. The following statement makes clear that such environment might not be attainable.

Lemma 3 ***If A does not belong to the SML, and S is a separation portfolio with the same expected return as A, it follows that***

$$\beta(S) \;\neq\; \beta(A)$$

Proof: If A does not belong to the SML, then its expected return differs from the one ruled by the SML:

$$E[\, R(A) \,] \;\neq\; E[\, R(A) \,]_{SML} \,.$$

Let us suppose that

$$E[\, R(A) \,] \;>\; E[\, R(A) \,]_{SML} \,.$$

There are two separation portfolios relevant here: first, the one lying on the SML

$$S \;=\; < x_F ; \; x_M >$$

whose expected return is $E[\, R(A) \,]_{SML}$. Furthermore, it has the same beta as A.

And secondly, we have to take into account the separation portfolio S' whose expected return is

$$E[\, R(A) \,].$$

Although S' also belongs to the SML, it does so at the cost of having a higher beta. To prove this last statement, we shift to the capital market line (CML), where it holds

$$\begin{cases} E[R(S')] = R(F) + \{< E[R(M)] - R(F) > / \sigma(M)\} \times \sigma(S') \\ E[R(S)] = R(F) + \{< E[R(M)] - R(F) > / \sigma(M)\} \times \sigma(S) \end{cases} .$$

Subtracting the second equation from the first one, and rearranging, we get:

$$\{\, E[\, R(S') \,] - E[\, R(S) \,] \,\} \;/\; \{\, \sigma(S') - \sigma(S) \,\} \;=$$

$$=\; \{\, E[\, R(M) \,] - R(F) \,\} \;/\; \sigma(M) \;\}.$$

And solving for $\sigma(S')$

$$\{\, E[\, R(S') \,] - E[\, R(S) \,] \,\} \times \sigma(M) \;/\; \{\, E[\, R(M) \,] - R(F) \,\} \;+\; \sigma(S) \;=$$

$$= \sigma(S')$$

Moreover, for every separation portfolio lying on the CML[1]

$$\sigma(S) = \beta(S) \times \sigma(M)$$

that can be replaced in the former relationship to get, after leaving out $\sigma(M)$ from both sides:

$$\beta(S') = \beta(S) + \{ E[R(S')] - E[R(S)] \} / \{ E[R(M)] - R(F) \}$$

There being the differential rate of return between S' and S greater than zero, it follows

$$E[R(A)] > E[R(A)]_{SML} \Rightarrow \beta(S') > \beta(S)$$

In the same way,

$$E[R(A)] < E[R(A)]_{SML} \Rightarrow \beta(S') < \beta(S)$$

4. Enlarged Separation Portfolios As Synthetics

In the context of lemma 3, it is worth highlighting two outcomes:

a) Provided security A does not belong to the SML, there will not be any separation portfolio acting as a synthetic of A.
b) If we had required that A and the separation portfolio S both share the same beta, then the expected return of S would not have fit that of A, because S is to lie on the SML.

If one fails to understand these features, then the shortcomings of plain information portfolios would not be fully grasped. It is for the following lemma to provide a positive answer to this problem, through the introduction of enlarged separation portfolios.

Lemma 4 ***If A does not belong to the SML, we can find an enlarged separation portfolio to perform as a synthetic of A.***

Proof: Let us suppose that A is a mispriced asset with respect to the SML, with a higher expected return than the predicted one

$$E[R(A)] > E[R(A)]_{SML} \Rightarrow E[R(A)] = E[R(A)]_{SML} + \alpha \ , \quad \alpha > 0 \tag{7}$$

and we are going to build up a portfolio S_e

$$S_e = < x_F \ ; \ x_M > \tag{8}$$

consisting of a free-risk asset and the market portfolio that will prove to become not only an enlarged separation portfolio but A's synthetic as well.

Firstly, we choose a positive x'_F so that

$$\alpha = x'_F \times R(F)$$

Secondly, let S be certain separation portfolio with the same beta as A, lying on the SML. That is to say:

$$\begin{cases} S =< x''_F \, ; x''_M >=< 1 - x''_M \, ; x''_M > \\ E[R(S)] = E[R(A)]_{SML} = x''_F \times R(F) + x''_M \times E[R(M)] \\ \beta(S) = \beta(A) = x''_M \end{cases} \tag{9}$$

Now we can make explicit **the S_e structure** in (8), by taking up:

$$\boldsymbol{S_e = < x_F \ ; \ x_M > \ = \ < x''_F + x'_F, x''_M >} . \tag{10}$$

[1] For a full development of this statement, see Apreda (2001b).

We need to prove, firstly, that S_e is a synthetic of A and, secondly, a truly separation portfolio.

a) S_e is a synthetic of A.

Applying definition 1,

$$E[\,R(S_e)\,] = x_F \times R(F) + x_M \times E[\,R(M)]$$

by (10) it holds that

$$E[\,R(S_e)\,] = (\,x''_F + x'_F) \times R(F) + x''_M \times E[\,R(M)\,].$$

Taking advantage of (9), we get

$$E[\,R(S_e)\,] = (\,1 - x''_M + x'_F) \times R(F) + x''_M \times E[\,R(M)\,]$$

by rearranging:

$$E[\,R(S_e)\,] = x'_F \times R(F) + R(F) + x''_M \times < E[\,R(M)\,] - R(F) >$$

and by (7) and (9),

$$E[\,R(S_e)\,] = E[\,R(A)\,]_{SML} + \alpha = E[\,R(A)\,]. \quad (11)$$

What is more,

$$\beta(S_e) = x_F \times \beta(F) + x_M \times \beta(M),$$

$$\beta(S_e) = x_F \times 0 + \beta(A) \times 1 = \beta(A).$$

So then, S_e is a synthetic for A

b) S_e is an enlarged separation portfolio since (10) holds,

$$S_e = < x_F\,;\, x_M > = < x''_F + x'_F, x''_M > .$$

Now, $x'_F > 0$, so we get

$$x''_F + x'_F + x''_M = 1 + x'_F > 1\,.$$

By the same token, if asset A were overpriced, we would arrive at the same conclusion, but in this case $\alpha < 0$.

The former lemma has brought about a separation portfolio that is designed to fit as a synthetic, albeit it does not belong to the SML. In fact, there is even a stronger result, shown in the corollary to Lemma 1: no separation portfolio could lie on the SML.

It remains to be settled a final issue: we should figure out the connection between mispriced assets, synthetics and arbitrage portfolios. The following statement will put all these pieces together.

Lemma 5 ***If A is a mispriced asset (or portfolio), we can find an enlarged separation portfolio to perform as synthetic to A. Afterwards, we can set an arbitrage portfolio consisting of the enlarged separation portfolio and a plain separation portfolio.***

Proof: It is a straightforward task. Firstly, from lemma 4 we get the enlarged separation portfolio we need. Secondly, from lemma 1 we set up the arbitrage portfolio we are seeking for.

Conclusions

Plain separation portfolios are suitable vehicles to set up arbitrage portfolios, any time a mispriced asset (or portfolio) is presumed to foster an arbitrage opportunity.

These portfolios, however, fail in most cases whenever we try to synthesize a mispriced asset (or portfolio). This paper brought forth the notion of enlarged separation portfolios and proved three outcomes that seem useful for theory and practice:

a) If we have an enlarged separation portfolio, then an arbitrage portfolio against a plain separation portfolio in the SML is always feasible.
b) If we have a mispriced asset (or portfolio), then we can find a separation portfolio that becomes its synthetic.
c) If we have a mispriced asset (or portfolio), then we can synthesize it with an enlarged separation portfolio, from which an arbitrage portfolio can be designed against a plain portfolio in the SML.

It's worth remarking that, either when arbitraging or synthesizing, these portfolios are cheaper, simpler and more feasible than other financial alternatives.

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