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A Coincident Financial Indicator for the Australian Stock Market

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Abstract

The stock market investment requires recognising the financial risk. Different economic indicators are normally used in this context. In this article we focus on developing a coincident financial indicator for the stock market, which can summarise the market participants' assessment of the state of the economy. This indicator is shown to have all the desirable characteristics and is suitable for investment decision making process. The methodology is illustrated using the Australian stock market as an example.

Introduction

The stock market has long been viewed as an indicator of the economic activity. Some researchers believe that large changes in stock prices are signals of subsequent economic swings. Hamilton and Lin (1996) empirically capture, in a non-linear framework, the basic precept that the anticipation of an economic downturn affects the stock market before the industrial output actually starts to decline. The theoretical reason why stock market might anticipate economic activity includes the underlying principle of stock price valuation. This suggests that the stock price reflects the expectations of the future of the economy. However, there are some researchers who do not support this notion. They point out the strong economic growth following the 1987 stock market crash.

In spite of this controversy it is important to understand the nature of interaction of the stock market and the business cycle. From the stock market investors' point of view the assessment of risk is very important and this risk changes as the business cycle passes through the different phases. The cornerstone of an investment process is the development of an investment strategy to take advantage of the changing risk of the financial market. An understanding of the major trends is, therefore, crucial for developing investment strategies.

In this context the investors rely on economic indicators. The indicators are identified as leading, coincident and lagging indicators of the business cycle. The aim of this article is to develop a coincident financial market indicator which can encapsulate the state of the economy in a single dynamic factor. This indicator would essentially proxy the state of the aggregate knowledge about the economy. The dynamic model we propose would also help extend the framework to generate one-step ahead forecast of the state of the economy. We demonstrate the statistical property of such a coincident indicator by applying it to the Australian stock market.

To build a coincident indicator of the kind mentioned above, we must extract information from several variables. Many economic and/or financial time series are found to be highly correlated, but not necessarily cointegrated. Economists have attempted to make use such observation to develop models that might be able to predict the likely direction of the particular market being investigated. It has also been found useful to be able to summarise a number of series into a smaller number and use that for prediction purposes. In this background the factor structure has been employed. Stock and Watson (1991) model of coincident economic indicator is a classic example. Since such a factor is essentially unobservable, it is well suited for modelling by an appropriate state space system.

In the next section we describe the process for building the coincident indicator along with the technical details (given in the appendix) for estimating such a model. This will be followed by a short description of the data set used for empirical study of the Australian stock market, analysis of the results and a summary of the article.

Model

One of the key decisions we have to make in modelling time series is the appropriate specification of the variance of the process. Several studies have applied the time varying conditional variance to capture the stylised facts. ARCH and GARCH or some variation of these models has been employed for this purpose. One important point to remember that in these set up the unconditional variance remains constant. Alternative specification of the time varying variance requires an unobserved Markov chain driving this and it can take on one of several possible values depending on the state that might occur. This specification also accounts for heteroscedasticity, but here the unconditional variance itself is changing. Thus the intuition behind the ARCH/GARCH processes and the state changing variance process are quite different.

We already indicated that the kind of dynamic factor structure we have in mind requires the use of state space framework to infer the unobserved component. Most traditional implementation of such a strategy assumes constant variance process. More recently, though, Chauvet (1998) and Chauvet and Potter (2000) have enhanced such a modelling strategy by incorporating heteroscedastic innovations. This not only allows them to incorporate different regimes the economy might pass through over a long period of time, say twenty years or more. In the following paragraphs we outline this modelling approach and apply that to infer the indicator variable for the financial market. Chauvet and Potter (2000) also exploit this inferred variable to alter portfolio composition in order to earn additional return. In this article, however, we simply focus on the statistical properties of the extracted coincident indicator.

Based upon the related literature we rely on four financial market variables that have been reported to be commoving and these are, equity market excess return, proxy for the market volatility, short-term interest rate and the price-earning ratio. These four variables are believed to be collectively summarising the state of the economy. As we find the short-term interest rate and the price-earning ratios are non-stationary, we use the first difference of the log of these variables in the model. The preliminary analysis of the principal component of these four variables indicate one dominant factor, hence we set up the model to account for this unobserved factor with a given dynamic characteristic. Additional details of the data and their sources are given the data section.

The model is, thus, based around the unobserved dynamic factor with the following structure:

$$\lambda_t = \alpha_0 + \alpha_1 S_t + \phi\lambda_{t-1} + \eta_{S_t}, \eta_{S_t} \sim N\left(0, \sigma_{St}^2\right), \tag{1}$$

where, $S_t = \{0,1\}$ is 1 is the Markov switching variable indicating the state of the financial market at any given month. This evolves with its own transition probability property. This is discussed further in the appendix. The innovation variance also depends on the state of the financial market as does the mean of the unobserved factor. This factor is essentially a non-linear proxy of the stock market risk. It captures the swings in the stock market at the monthly frequency based upon the participants' perception about the changes in the state of the market.

Our model assumes that the observations of the four financial market variables are related to this dynamic factor as well as its own idiosyncratic innovations. Under this proposal the measurement process is given by,

$$\begin{bmatrix} y_{ret} \\ y_{vol} \\ y_{ir} \\ y_{pe} \end{bmatrix} = \begin{bmatrix} \beta_{ret} \\ \beta_{vol} \\ \beta_{ir} \\ \beta_{pe} \end{bmatrix} \lambda_t + \begin{bmatrix} \nu_{ret} \\ \nu_{vol} \\ \nu_{ir} \\ \nu_{pe} \end{bmatrix}. \tag{2}$$

The idiosyncratic innovations are assumed to have an AR(1) structure of their own as described by,

$$\nu_{i,t} = \theta_i \nu_{i,t-1} + \varepsilon_i, \varepsilon_i \sim N\left(0, \sigma_i^2\right). \tag{3}$$

Here either return, volatility, interest rate or the price-earning variables are represented. We also assume that these innovations are uncorrelated between themselves as well as with the *i* innovation of the factor. With a little thought we could put this set of equations in the state space form which is then directly comparable to the model estimation procedure discussed in the appendix. The measurement equation of the state space form is, therefore,

$$\begin{bmatrix} y_{ret} \\ y_{vol} \\ y_{ir} \\ y_{pe} \end{bmatrix} = \begin{bmatrix} \beta_{ret} & 1 & 0 & 0 & 0 \\ \beta_{vol} & 0 & 1 & 0 & 0 \\ \beta_{ir} & 0 & 0 & 1 & 0 \\ \beta_{pe} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_t \\ \nu_{ret} \\ \nu_{vol} \\ \nu_{ir} \\ \nu_{pe} \end{bmatrix}, \tag{4}$$

and the state transition equation becomes,

$$\begin{bmatrix} \lambda_t \\ \nu_{ret,t} \\ \nu_{vol,t} \\ \nu_{ir,t} \\ \nu_{pe,t} \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 S_{S_t} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi & 0 & 0 & 0 & 0 \\ 0 & \theta_{ret} & 0 & 0 & 0 \\ 0 & 0 & \theta_{vol} & 0 & 0 \\ 0 & 0 & 0 & \theta_{ir} & 0 \\ 0 & 0 & 0 & 0 & \theta_{pe} \end{bmatrix} \begin{bmatrix} \lambda_{t-1} \\ \nu_{ret,t-1} \\ \nu_{vol,t-1} \\ \nu_{ir,t-1} \\ \nu_{pe,t-1} \end{bmatrix} + \begin{bmatrix} \eta_{S_t} \\ \varepsilon_{ret} \\ \varepsilon_{vol} \\ \varepsilon_{ir} \\ \varepsilon_{pe} \end{bmatrix}. \tag{5}$$

The estimation of the unknown parameters of this model is achieved via the numerical maximisation of the prediction error form of the likelihood function as described in the appendix.

Data

The main sources of the data are tabulated below. The data spans the period from January 1980 to September 2004.

	Data Set Codes from DataStream			
	Price Index	Dividend Yield	P/E Ratio	Interest Rate
	(PI)	(DY)	(PE)	(IR)
Australia	TOTXTAU(PI)	TOTXTAU(DY)	TOTXTAU(PE)	ADBR090(IR)

Excess return = $xs_t \equiv \left(\ln\left[PI_t + PI_{t-1} * DY_t\right] - \ln\left[PI_{t-1}\right] - \ln\left[1 + IR_t / 12\right]\right) * 12$

Volatility = $\sqrt{\left(xs_t - xs_{avg}\right)^2}$

Empirical Results

The model outlined above has nineteen unknown parameters to be estimated. We find it convenient for the likelihood maximisation if we standardise the response parameter for the excess return i.e. in that case other response parameters are essentially relative values with respect to that of the excess return series. It is also intuitively meaningful to standardise against the excess return series since it is one of the most important quantities for investment managers. Thus, the number of parameters to be estimated is reduced to eighteen. $\beta_{ret} = 1$.

In Table 1 we show all the eighteen parameter estimates along with their standard errors. The AR(1) coefficient (φ) of the coincident index is not statically significant. It tends to indicate that the mean of this indicator series switches between 0.08 in state 0 to -0.53 in state 1. The state 0 is easily identified as the low variance regime and the state 1 represents the high variance regime. The difference in the variances in these two states is quite large. The transition probabilities suggest that the average duration of state 0 is 13.5 months and that of the state 1 is 3.1 months. This observation compares well with empirical regularity that the episodes of instability do not last long. From the properties of the ergodic Markov chain we estimate the steady state probabilities of the state 0 as 0.81 and that of state 1 as 0.19.

Since we set, it implies that the scale of the coincident indicator is same as that of the excess return. The signs of the response coefficients for change in interest rate and volatility suggest that positive changes in these variables would impact negatively the coincident indicator. Similarly positive change in price-earnings ratio would impact positively the indicator. These are all intuitive as well as empirically consistent behaviour of the market. The estimated variances of the four financial market variables are close to their sample values. This establishes the correctness of the model estimation process. $\beta_{ret} = 1$.

Table 2 lists several diagnostic statistics and the interpretation of these entries are in the notes below the table. It shows that the model captures the heteroscedasticity in the series and the recursive *t* statistics support the model specification. As expected the extracted coincident indicator is highly correlated with all the components. It is particularly high with the excess return and the market volatility. The signs of the correlations are also intuitively correct. This result confirms the efficacy of the non-linear model from which we inferred the unobserved coincident indicator.

We now focus on the graphs in Figure 1. The probability plot of the state variable being 0 reveals some interesting facts. The global macro events that affected the stock markets have been clearly identified. For example, during the October 1987 stock market crash the probability of the stock market being in low volatility state is very low and it shows up in the plot. Also, for the Australian stock market the September 1997 Asian financial crisis is a major event and the plot reveals that the probability at that time to be in a low volatility state is quite low. Similarly, following the event of September 2001, this probability is also very low. Along with this probability plot the coincident indicator is also behaving as intuitively expected. During the major events mentioned above the indicator dives quite low. Since the indicator is in the same scale as that of the excess return, it implies that the model is picking up the current economic trends quite successfully.

In order to understand how this coincident indicator may help in investment management we need to look at the expression from which this coincident indicator plot has been constructed. The equation (A.5) of the appendix shows the way it has been recursively computed. It has the characteristic of being one-step ahead projection and thus might be useful in formulating portfolio rebalancing strategy in order to gain additional excess return from the stock market. In this article we do not pursue this aspect of the investigation. However the analysis and the methodological contribution of this article can be adopted and extended in that regard.

Summary

Investment strategy in stock markets requires a clear understanding of the state of the economy and the likely move in the following period. In this respect the coincident indicator plays an important role by not only encapsulating the aggregate knowledge of the economy, but it also helps reducing the number of variables required for the decision making process. In this article we illustrate the process of building a coincident financial indicator for the stock market in a non-linear environment. The statistical properties of the indicator support the expected characteristics and its possible role in investment decision making process. We also outline the necessary algorithmic details with emphasis on intuition.

Table 1

Parameter Estimates for the Markov Switching Coincidence Index Model

	Australia
α_C	0.08526 (0.03008)
α_1	-0.52703 (0.15946)
ϕ	-0.03824 (0.03644)
$\sigma^2_{\eta C}$	2.8E-06 (2.2E-07)
$\sigma^2_{\eta 1}$	1.14823 (0.25991)
σ^2_{ret}	0.21389 (0.02039)
σ^2_{vol}	0.16100 (0.01589)
σ^2_{ir}	0.00170 (0.00022)
σ^2_{pe}	0.00235 (0.00021)
β_{vol}	-0.76517 (0.05383)
β_{ir}	-0.03439 (0.00800)
β_{pe}	0.08096 (0.00651)
θ_{ret}	-0.09521 (0.06083)
θ_{vol}	0.68059 (0.04396)
θ_{ir}	0.37921 (0.03245)
θ_{pe}	-0.01978 (0.02411)
p_{00}	0.92571 (0.02335)
p_{11}	0.67860 (0.08618)

Maximum likelihood estimates of the parameters are reported here. The numbers in parentheses are the standard errors computed from the diagonal elements of the final covariance matrix.

Table 2

Diagnostic Statistics for the Residuals of the Measurement Equations

	Portmanteau	ARCH	KS Test	Recursive T
Australia				
Return	0.211	0.999	0.082	0.560
Volatility	0.005	0.990	0.118	0.620
T Bill	0.001	0.015	0.150	0.620
P/E	0.767	0.991	0.062	0.460

Entries are p-values for the respective statistics except for the KS statistic. These diagnostics are computed from the recursive residual of the corresponding measurement equation. The null hypothesis in portmanteau test is that the residuals are serially uncorrelated. The ARCH test checks for no serial correlations in the squared residual up to lag 26. Both these tests are applicable to recursive residuals as explained in Wells (1996, page 27). If the model is correctly specified then Recursive T has a Student's t-distribution (see Harvey (1990, page 157)). KS statistic represents the Kolmogorov-Smirnov test statistic for normality. 95% significance level in this test is 0.079. When KS statistic is less than 0.079 the null hypothesis of normality cannot be rejected at the indicated level of significance.

Table 3

Descriptive Statistics

	CI	Excess Ret.	Volatility	Δ(P/E)	Δ(IR)
Sample Mean	-0.0471	0.0191	0.4573	0.0010	-0.0021
Std. Deviation	0.4976	0.6774	2.2090	0.0642	0.0706
Corr. with CI		0.813	-0.822	0.727	-0.252

CI : Coincident Indicator

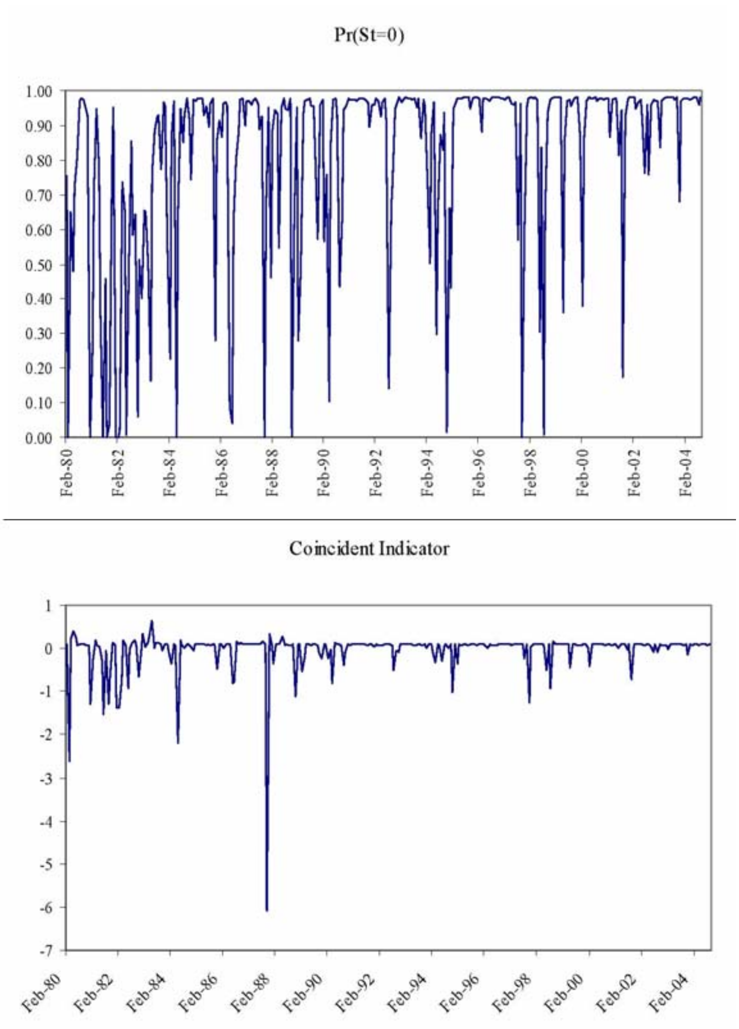

Fig. 1. Estimated State Probabilities and the Coincident Indicator

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Appendix A

State Space Model Estimation Algorithm
State Dynamic Subject to First Order Markov Chain Evolution

We discuss the problem of estimation of a state space model when only the state equation is subjected to multiple regimes and the switch between regimes takes place according to a first order Markov chain. Both the mean and the variance part of the state equation may be subjected to this influence. Following the steps of this algorithm is important in understanding the accompanying program logic. Without the presence of this Markov chain, the state space system can be estimated by using the standard Kalman filter recursion and updating algorithms that would be used to derive the prediction error form of the likelihood function.

The main issues that have to be addressed due to the Markov chain driving the state transition are: 1) The probabilities of being in a particular state assuming that the system was in a given state in the previous time step, 2) How to cope with the exploding number of states to be accounted for as each observation is processed. For example, given that there are only two states possible, then at each time step there is a two-fold increase of number of state to account for. This implies that for a 100-time step observation of a given system there will be at the end states to be dealt with. This is clearly impractical. Hence, there is a need for approximating the system with a sensible approach. The algorithm discussed below deals with the first issue following the approach suggested in Hamilton (1989) and the second issue is addressed by the algorithm suggested in Kim (1994). Kim's procedure collapses the number of states to the previous number by a probability weighting scheme. Thus for a two-state Markov chain, we will always deal with two states after each observation input is processed. 2^{100}

The general structures to reference with respect to these algorithms we focus on the system given in equation (A.1) as the measurement equation and the equation (A.2) as the state equation. Obviously, all the matrices and vectors are of compatible dimensions. The 2-state Markov chain $S_t = \{0,1\} \text{is } 1$ is used as a suffix to explicitly recognise those variables that may depend on the state we are in at any time. The equation (A.3) also states that the innovations of the measurement and the state equations are uncorrelated.

$$y_t = H\beta_t + Az_t + e_t \,, \tag{A.1}$$

$$\beta_t = M_{S_t} + F_{S_t}\beta_{t-1} + G_{S_t}\nu_t \text{, and} \tag{A.2}$$

$$\begin{pmatrix} e_t \\ \nu_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} R & 0 \\ 0 & Q_{S_t} \end{pmatrix}\right). \tag{A.3}$$

The covariance matrix of the innovations in measurement equation is given by R and that of the state equation is given by. This representation is somewhat general and not all the elements of the system could be present for a given problem. For example, the component is the measurement equation suggests possible presence of endogenous variables entering the measurement process through the coefficient matrix. $Q_{S_t} Az_t z_t A$.

We next focus on the prediction and the updating equation of the basic Kalman filter assuming that in the previous time slot $S_{t-1} =$ and the next time step it is. At this stage we define the probability transition matrix for the Markov chain variable. This is given by, $S_{t-1} =$.

$$p = \begin{bmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{bmatrix}. \tag{A.4}$$

Following Harvey (1991) and Kim and Nelson (1999) and assuming that we are moving from state realisation of to state realisation, the relevant equations are given below. ij

Prediction:

$$\beta^{i,j}_{t|t-1} = M_j + F_j \beta^{i}_{t-1|t-1}, \tag{A.5}$$

$$P^{i,j}_{t|t-1} = F_j P^{i}_{t-1|t-1} F_j' + G_j Q_j G_j', \tag{A.6}$$

$$\eta^{i,j}_{t|t-1} = y_t - H\beta^{i,j}_{t|t-1} - Az_t, \tag{A.7}$$

$$f^{i,j}_{t|t-1} = HP^{i,j}_{t|t-1} H' + R. \tag{A.8}$$

Updating:

$$\beta^{i,j}_{t|t} = \beta^{i,j}_{t|t-1} + P^{i,j}_{t|t-1} H \left[f^{i,j}_{t|t-1}\right]^{-1} \eta^{i,j}_{t|t-1}, \tag{A.9}$$

$$P^{i,j}_{t|t} = \left(I - P^{i,j}_{t|t-1} H' \left[f^{i,j}_{t|t-1}\right]^{-1} H\right) P^{i,j}_{t|t-1}. \tag{A.10}$$

In the above equations, is the state vector estimated based upon information at time (*t*-1), and the equation (A.5) states how it would evolve if at time *t* the state realisation happens to be *j*. Similar interpretation applies to the estimate of the state covariance matrix, with respect to the equation (A.6). The equation (A.7) describes the forecast error at time *t* when the state realisation is *j* assuming the previous state was *i* at time (*t*-1). The equation (A.8) gives the covariance of the forecast error just discussed above. Thus, the equations (A.7) and (A.8) would provide the input required to build the state dependent conditional density of observations. The updating equations propel the equations (A.5) and (A.6) based upon the observations just made and makes it ready for use at the next time step. Thus, the basic nature of the Kalman filter is preserved; only these are now state contingent. Obviously, for this recursive procedure to work, we need to supply the prior starting values for. We use the method discussed in Kim and Nelson (1999, p. 27). $\beta^{i}_{t-1|t-1}$ $P^{i}_{t-1|t-1}$ $\beta_{0|0}, P_{0|0}$.

For a two state Markov process, this recursion in the filter produces (posteriors for and when moving from (*t*-1) to *t*. Kim (1994) develops the following approximation where by taking appropriate weighted average over the states at (t-1) from which the particular state at *t* could be reached, this can be reduced to. (We define the probability weighting as, 2×2) $\beta^{i,j}_{t|t}$ $P^{i,j}_{t|t}$ 2).

$$\Gamma^{i,j} = \frac{\Pr\left[S_{t-1} = i, S_t = j \middle| \psi_t\right]}{\Pr\left[S_t = j \middle| \psi_t\right]}, \tag{A.11}$$

where, is the information available at time *t*. Therefore, the approximation for the state vector is, ψ_t

$$\beta^{j}_{t|t} = \sum_{i=0}^{1} \beta^{i,j}_{t|t} \times \Gamma^{i,j} \tag{A.12}$$

and the approximation for is, $P^{j}_{t|t}$

$$P^{j}_{t|t} = \sum_{i=0}^{1} \left[P^{i,j}_{t|t} + \left(\beta^{j}_{t|t} - \beta^{i,j}_{t|t}\right)\left(\beta^{j}_{t|t} - \beta^{i,j}_{t|t}\right)\right] \times \Gamma^{i,j}. \tag{A.13}$$

The equations (A.12) and (A.13) describe the nature of approximation applied to collapse the posteriors to posteriors with the help of the probability weighting factor. The detailed derivation of this could be found in Kim and Nelson (1999, p. 101). The probability terms necessary to achieve this can be obtained as follows: (2×2)(2)

$$\Gamma^{i,j} = \frac{\Pr\left[S_{t-1} = i, S_t = j \middle| \psi_t\right]}{\Pr\left[S_t = j \middle| \psi_t\right]} = \frac{\Pr\left[y_t, S_{t-1} = i, S_t = j \middle| \psi_{t-1}\right]}{\Pr\left[y_t \middle| \psi_{t-1}\right]} = \left(\frac{\Pr\left[y_t \middle| S_{t-1} = i, S_t = j \middle| \psi_{t-1}\right]}{\Pr\left[y_t \middle| \psi_{t-1}\right]}\right) \times \Pr\left[S_{t-1} = i, S_t = j \middle| \psi_{t-1}\right], \tag{A.14}$$

where as before i. With the help of the forecast error in the prediction relations we can now construct the numerator (in the parentheses) of the last term of equation (A.14) as, = 0, 1 and j = 0, 1. Wit

$$\Pr\left[y_t \middle| S_{t-1} = i, S_t = j \middle| \psi_{t-1}\right] = \frac{1}{\sqrt{2\pi\left(f_{t|t-1}^{i,j}\right)}} \exp\left\{-\frac{1}{2}\eta_{t|t-1}^{i,j}{}'\left(f_{t|t-1}^{i,j}\right)^{-1}\eta_{t|t-1}^{i,j}\right\}, \tag{A.15}$$

and $\Pr\left[y_t \middle| \psi_{t-1}\right]m$ may be expressed as,

$$\Pr\left[y_t \middle| \psi_t\right] = \sum_{i=0}^{1}\sum_{j=0}^{1} \Pr\left[y_t \middle| S_{t-1} = i, S_t = j \middle| \psi_{t-1}\right] \times \Pr\left[S_{t-1} = i, S_t = j \middle| \psi_{t-1}\right]. \tag{A.16}$$

It may be recognised that the last product term in equation (A.14) is the transition probability. Furthermore, the separation of the joint probability in equation (A.14) is possible due to the Markov assumption. The equation (A.16) shows how to propagate the probability information as new observation is processed. This also gives the log likelihood function that has to be maximised with respect to all the unknown parameters in the model by using some suitable numerical optimisation routine.

The logic of propagation of the probability information requires starting values at time 0. This is based on steady state probabilities of the assumed ergodic Markov chain. In this context we adopt the steps outlined in Kim and Nelson (1999, p. 71).

This Markov switching state space model generate, during the estimation process, the conditional variance of the forecast error given by equation (A.8) based upon a given state realisation. Using the probability of the state occurring as discussed above, we could easily construct the conditional variance of the state process. The conditional variance is thus given by,

$$\sum_{i=0}^{1}\sum_{j=0}^{1} \Pr\left[S_{t-1} = i, S_t = j \middle| \psi_t\right] \times f_{t|t-1}^{i,j}. \tag{A.17}$$

In a similar manner the estimate of probability weighted forecast error could be generated by using (A.7). This generated error series may then be analysed for model diagnostics tests. Furthermore, we make inference of the expected state vector based on the relation given by equation (A.17) but the last term is replaced by $\beta_{t|t}^{i,j}$ from the equation (A.9).