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Stochastic Programming Methods in Asset-Liability Management

Michael Grebeck, Svetlozar Rachev

Abstract

This paper reviews some of the stochastic programming (SP) frameworks that are useful in applications to asset-liability management (ALM). Two such frameworks include recourse models and SP with decision rules. Recent advances also provide a representation for the Conditional Value-at-Risk risk measure that can be easily optimized in SP. Uncertainty in ALM stochastic programs is represented through discrete scenarios that are often generated through time-series methods. Sophisticated methods, such as those incorporating stable distributions, are needed to capture typical characteristics of financial data.

Key words: Stochastic programming, stable distributions, risk, uncertainty, time-series analysis.

1. Introduction

Asset-liability management attempts to find the optimal investment strategy under uncertainty in both the asset and liability streams. In the past, the two sides of the balance sheet have usually been separated, but simultaneous consideration of assets and liability can be very advantageous when they have common risk factors. Allocating assets such that they are highly correlated with the liabilities can increase returns and reduce risk.

Developed in the late 1970's, immunization is an earlier ALM method that is still very popular today. Bond immunization attempts to match the interest rate sensitivity of a bond portfolio with the interest rate sensitivity of a liability stream. The resulting allocation only hedges against a small shift in the term structure of interest rates. This technique fails to incorporate the stochastic nature of interest rates and is a single stage model with no transaction costs. Therefore, immunization is inadequate for the multistage and stochastic problems of ALM. This paper focuses on the more recent SP ALM models that attempt to capture the dynamic and uncertain characteristics of financial decision-making.

Stochastic programming is becoming more popular in finance as computing power increases. While multistage stochastic programs can adequately model dynamic and stochastic financial problems, realistic ALM models could rarely be solved until recently, and still some simplifications are usually needed to implement the problems. But now there have been enough advances that stochastic programming can obtain superior results to simple diversification or immunization.

The tradeoff between risk and reward is an important consideration in ALM. Two important measures of risk that depend on the tail of a loss distribution are the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR). Unlike VaR, recent results have shown that CVaR can be easily optimized in a convex program.

2. Scenario Generation

The stochastic programming formulations used in ALM rely on uncertainty approximated by a discrete set of scenarios organized in a tree structure. These scenarios are often generated by the discretization or simulation of a time-series model. To trust the solution of a stochastic program, these time-series models should include realistic characteristics of financial data such as heavy tails, high peaks, skewness, long-range dependence, and stochastic volatility. These characteristics can be captured by time-series methods incorporating stable distributions.

2.1. Stable Distributions

By the Central Limit Theorem, normalized sums of i.i.d. random variables with finite variance converge weakly to the normal random variable, and with infinite variance, and sums

converge weakly to a stable random variable[1]. This gives a theoretical basis for the use of stable distributions when heavy tails are present.

A random variable x has a stable distribution if for any $a > 0$ and $b > 0$ there exists $c > 0$ and $d \in \mathrm{R}$ such that:

$$ax_1 + bx_2 \overset{d}{=} cx + d\,, \tag{1}$$

where x_1 and x_2 are independent copies of x. Stable distributions are described by four parameters, α, β, σ, and μ, and are denoted by $S_\alpha(\sigma, \beta, \mu)$. When the index of stability, $\alpha \in (0,2]$, is small, the distribution has a high peak and heavy tails. Gaussian distributions are a subset of the class of all stable distributions and are obtained when $\alpha = 2$. The skewness parameter, β, determines if the distribution is skewed to the left $(\beta < 0)$ or the right $(\beta > 0)$. The scale parameter, σ, generalizes the notion of standard deviation, and the variation, σ^α, generalizes the notion of variance.

If $\alpha \in (0,2)$, the heavy tails can be described by:

$$\begin{aligned} &P(x > \lambda) \sim k_1 \lambda^{-\alpha}, \\ &P(x < -\lambda) \sim k_2 \lambda^{-\alpha}, \end{aligned} \quad \text{as } \lambda \to \infty\,, \tag{2}$$

for constants k_1 and k_2 that depend on α and β. In this case, the p-th absolute moment of x, $E|x|^p = \int_0^\infty P\left(|x|^p > \lambda\right) d\lambda$, is finite if and only if $p < \alpha$. If $\alpha = 2$, all absolute moments are finite. Models of financial data typically assume $\alpha \in (1,2]$, so it is possible to discuss expected returns.

In general, there is no closed form density or distribution function for stable random variables. However, the characteristic function is known and can be used to calculate densities through fast Fourier transform methods.

2.2. Financial Time-Series

The traditional autoregressive moving average (ARMA) model is often adequate for short-term prediction of stationary time-series. However, the conditional homoscedasticity property of ARMA states that the conditional variance is constant and independent of past observations. Therefore, ARMA cannot model volatility clusters.

ARMA is also inadequate for capturing long-range dependence (LRD). In the case of innovations with finite variance, LRD of data can be described through the decay of the autocorrelation functions $\rho_n = \mathrm{Corr}(x_0, x_n),\ n = 0,1,\ldots$. A common definition of LRD is:

$$\rho_n \sim cn^{-h} \quad \text{as} \quad n \to \infty\,, \tag{3}$$

for some $c > 0$ and $0 < h < 1$. It is known that ARMA processes have an exponential decay of the correlation and hence cannot model LRD. A fractional integrated ARMA can model this dependence.

The generalized autoregressive conditional heteroscedastic (GARCH) model is used to capture the conditional heteroscedasticity, or clustered volatilities, in time-series with constant unconditional variance. The following is the GARCH model for ε_t:

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[1] In the special case of random variables with infinite variance and α=2, the sum converges weakly to a random variable that is not only stable, but also normal.

$$\varepsilon_t = c_t u_t, \quad u_t \overset{\text{iid}}{\sim} N(0,1), \tag{4}$$

where the process for the conditional variance c_t^2 is:

$$c_t^2 = \alpha_0 + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j c_{t-j}^2, \quad c_t > 0, \tag{5}$$

with all α_i and β_j nonnegative constants. The effects of the squared innovations decay exponentially in a GARCH process. Similar to ARMA processes, a fractional integrated GARCH (FIGARCH) can model LRD. In FIGARCH, the past shocks to the conditional variance c_t^2 decay at a slower hyperbolic rate.

In ARMA-GARCH, the innovations of ARMA follow a GARCH process. In addition to volatility clusters, ARMA-GARCH provides unconditional heavy-tailed distributions with high peaks. However, the GARCH filtered residuals still possess heavy tails, so the next logical step is to include non-normal innovations. Three of the suggested non-normal distributions for u_t are the generalized error distribution, Student's t-distribution, and stable distribution.

Stable GARCH has advantages over other fat-tailed distributions. The first is the theoretical properties mentioned earlier. The second is that the stable distribution can also model conditional skewness. The asymmetric stable GARCH process for ε_t is:

$$\varepsilon_t = \mu_t + c_t u_t, \; u_t \overset{\text{iid}}{\sim} S_\alpha(1, \beta, 0), \tag{6}$$

where the process for c_t follows:

$$c_t^\delta = \alpha_0 + \sum_{i=1}^{r} \alpha_i \left| \varepsilon_{t-i} - \mu_{t-i} \right|^\delta + \sum_{j=1}^{s} \beta_j c_{t-j}^\delta, \; c_t > 0, \tag{7}$$

with all α_i and β_j nonnegative constants. The usual assumption is that $1 < \alpha < 2$, but this is not very restrictive since most financial time-series have a finite mean. The stable GARCH equations are often stated with $\delta = 1$. Since $\alpha > 1$, this assumption guarantees the first moments of c_t are finite. This would not be the case if $\delta \geq \alpha$ as the first moments of c_t would be infinite for any $\alpha < 2$. A full treatment on GARCH with stable distributions can be found in Rachev and Mittnik (2000).

3. Recourse Models

Stochastic programming with recourse is a general formulation in which many ALM applications fit.

3.1. Two-Stage Recourse

The two-stage recourse problem allows a recourse decision made after uncertainty is realized. The first stage is a vector of initial decisions $x_1 \in \mathrm{R}_+^{n_1}$ made when there is a known distribution of future uncertainty. The second stage decisions $x_2 \in \mathrm{R}_+^{n_2}$ adapt after the uncertainty is realized. For instance, consider an asset allocation problem: The first stage is the initial portfolio, the uncertainty is in the stock prices, and the recourse stage is portfolio adjustments. This two-stage recourse problem finds the best first stage allocations for the given distribution of future stock movements.

For a given initial stage decision vector x_1, the best recourse decision is found through the second stage problem:

$$\begin{aligned} \min_{x_2} \quad & q_2(x_2,\xi) \\ \text{s.t.} \quad & A_2(\xi)x_2 = b_2(\xi) - B_2(\xi)x_1, \\ & x_2 \in \mathrm{R}_+^{n_2}, \end{aligned} \tag{8}$$

where $q_2(x_2,\xi)$ is the cost of the decision x_2 for the given realization of the second stage uncertainty ξ. With the optimal objective value of the second stage problem denoted by $Q(x_1,\xi)$, the full two-stage recourse problem is:

$$\begin{aligned} \min_{x_1} \quad & q_1(x_1) + E[Q(x_1,\xi)] \\ \text{s.t.} \quad & A_1x_1 = b_1, \\ & x_1 \in R_+^{n_1}. \end{aligned} \tag{9}$$

Cash flow balancing between stages and an initial wealth restriction are typical examples of constraints found in (8) and (9).

Many applications of recourse models in ALM are linear programs. In this case, functions are linear in the decision variables:

$$q_1(x_1) = c^T x_1, \; q_2(x_2,\xi) = q(\xi)^T x_2, \tag{10}$$

where $c \in \mathrm{R}^{n_1}$, and for a given ξ, $q(\xi) \in \mathrm{R}^{n_2}$.

After scenarios have been generated, the recourse problem is converted into a deterministic equivalent form. Assume there are S paths in the scenario tree and each scenario s has probability p^s for $s = 1,\ldots,S$. The two-stage recourse model is then:

$$\begin{aligned} \min_{x_1,x_2} \quad & q_1(x_1) + \sum_{s=1}^{S} p^s q_2^s(x_2^s) \\ \text{s.t.} \quad & A_1x_1 = b_1, \\ & B_2^s x_1 + A_2^s x_2^s = b_2^s, \qquad s = 1,\ldots,S, \\ & x_1 \in \mathrm{R}_+^{n_1}, x_2 \in \mathrm{R}_+^{n_2}. \end{aligned} \tag{11}$$

The scenario s corresponds to a realization of uncertainty ξ that determines $q_2(\cdot,\xi)$, $B_2(\xi)$, $A_2(\xi)$, and $b_2(\xi)$. These are denoted by $q_2^s(\cdot)$, B_2^s, A_2^s, and b_2^s, respectively.

3.2. Multistage Recourse

The stage in a recourse problem does not necessarily correspond to a time period. In a two-stage problem, the second stage may unrealistically contain many time periods. For instance, consider the asset allocation problem where the initial portfolio is constructed at time zero, and the second stage corresponds to time periods 1 through τ. If the portfolio is adjusted at time τ', $0 < \tau' < \tau$, the re-allocation is made with knowledge of future uncertainty that should not yet be realized. A multistage recourse program can provide a more realistic model, but it is more complex and can be difficult to solve. As in the two-stage problem, the initial decision vector x_1 is determined before the first realization of uncertainty ξ_1, and the second stage decisions x_2 are then made based on x_1 and ξ_1. In the T-stage problem, this process continues for the uncertainties ξ_t, $t = 1,\ldots,T-1$, and the decision vectors x_t, $t = 1,\ldots,T$.

The T-stage recourse program seeks to find the optimal decisions $\{x_t\}_{t=1}^{T}$ where $x_t \in \mathrm{R}_+^{n_t}$. Let the uncertainty up to stage t, for $t = 1,\ldots,T-1$, be denoted by $\xi^t = \{\xi_j\}_{j=1}^{t}$, where each ξ_t is the realized uncertainty between stage j and $j+1$. The following is a common form of the multistage model:

$$\begin{array}{lll} & \min\limits_{x_1} q_1(x_1) + E_{\xi_1}\left[\min\limits_{x_2} q_2(x_2,\xi^1) + \cdots + E_{\xi_{T-1}|\xi^{T-2}}\left[\min\limits_{x_T} q_T(x_T,\xi^{T-1})\right]\cdots\right] & \\ \text{s.t.} & A_1 x_1 & = b_1, \\ & B_2(\xi^1)x_1 + A_2(\xi^1)x_2 & = b_2(\xi^1), \\ & \ddots & \vdots \\ & B_T(\xi^{T-1})x_{T-1} + A_T(\xi^{T-1})x_T & = b_T(\xi^{T-1}). \end{array} \tag{12}$$

Again, linear programs are often considered in ALM where the costs q_t are linear in the decisions x_t. Another very common addition to the above program is to set various upper and lower bounds on the decision vectors x_t.

The split-variable formulation is a deterministic equivalent form that lends itself to decomposition and parallel implementation. Allowing all decisions to be scenario dependent creates S independent subproblems. Non-anticipatory constraints must be explicitly considered in this formulation: For any two scenarios s and s' with a common path up to and including stage t, the constraints $x_j^s = x_j^{s'}$, for $j = 1,\ldots,t$, must be enforced.

Many multistage applications in ALM can be posed as stochastic generalized networks. This means that each scenario subproblem of the split-variable formulation has a generalized network structure. Parallel implementations of highly efficient network algorithms can provide substantial computational advantages; however, some aspects of a desired application may destroy the network structure.

Another deterministic formulation of the multistage program is the arborescent form that implicitly includes the non-anticipatory constraints, but this form will not preserve any network structure present.

Additional recourses including solution techniques for two-stage and multistage linear stochastic programs with recourse are found in Dupačová *et al.* (2002), Birge and Louveaux (1997), and Censor and Zenios (1997).

3.3. Successful Applications

The model in Kusy and Ziemba (1986) is an earlier application of a two-stage stochastic linear program with simple recourse for the Vancouver City Savings Credit Union. The main features include: changing yield spreads over time, synchronization of cash flows by matching maturities of assets with expected cash outflows, simultaneous consideration of assets and liabilities to satisfy accounting principles and match liquidities, various transaction costs, uncertain cash flows arising from uncertainty in withdrawal claims and deposits, uncertain interest rates, and legal and policy constraints. While the model contains many multi-period aspects, it is not completely dynamic. Also, scenarios are independent over time and are limited to high, average, and low returns. However, even with many simplifications, the model generated superior policies.

Another successful application is the Russell-Yasuda Kasai model for a Japanese insurance company. A six-stage recourse model incorporates complex liabilities and regulations. A general description of this model is found in Carino *et al.* (1999).

4. Stochastic Programming with Decision Rules

Decision rules determine asset allocations and other money management decisions in each time period and do not change over the time horizon. One popular decision rule is referred to

as fixed-mixed: The wealth invested in each asset class is constant. A typical fixed-mixed rule is 60% invested in stocks and 40% invested in bonds. In the beginning of each time period, the stocks and bonds are bought or sold to keep the desired ratio.

Using decision rules significantly reduces the number of decision variable as compared to the previous multistage SP formulations. Optimizing over decision rules is less computationally intense than the large-scale linear programs found in these other SP methods; however, most decision rules result in non-convex optimization problems. If a nonlinear programming algorithm is used, only a local solution is found, so this algorithm would need to be re-started at many points. Alternatively, a global search algorithm may be used. In either case, there is no guarantee that these methods can find the global solution.

In addition to being easier to implement, the quality of the optimal decision rule is generally good. Any decision rule can be tested with out-of-sample scenarios, and confidence limits on risk measures can be constructed.

The presence of heavy tails in multistage asset allocation with fixed-mixed decision rules is examined in Tokat *et al*. (2003). The decision rule determines the wealth invested in a Treasury bill and the S&P500. The returns of the S&P500 are generated under a normal assumption and under a stable assumption. When the level of risk aversion is high or low, the computational results illustrate that the allocations are close under both assumptions. With moderate risk aversion, the stable assumption results in less invested in the risky S&P500 and a higher certainty equivalent final wealth.

5. Conditional Value-at-Risk

5.1. Definitions

Value-at-Risk (VaR) is a frequently used measure of risk for financial institutions and regulators. For a given confidence level α, VaR is the maximum loss that is exceeded no more than $(1-\alpha)$% of the time. Its ease of understanding helps to make VaR a popular risk measure.

The following notations and definitions of VaR and CVaR resemble mostly those of Rockafellar and Uryasev (2002). Let $x \in \mathrm{R}^n$ be a vector of decisions and $L(x)$ be a random variable representing a loss for each *x*. For example, $L(x)$ may be linear in *x*:

$$L(x) = x_1 Y_1 + \ldots + x_n Y_n . \tag{13}$$

Here, Y_i could be a random variable representing a loss (or negative return) or an individual asset. If the distribution function of $L(x)$ is:

$$\Psi_L(x,\zeta) = P[L(x) \le \zeta], \tag{14}$$

then for a given decision *x*, the VaR at confidence level α is given by:

$$\mathrm{VaR}_\alpha(x) = \inf\{\zeta \mid \Psi_L(x,\zeta) \ge \alpha\}. \tag{15}$$

Another important tail measure of risk is the Conditional Value-at-Risk (CVaR). While it is not widely used in finance, it has properties that make it a very logical alternative to VaR. Define a random variable $T_\alpha(x)$ on the α-tail of the loss $L(x)$ through the distribution function:

$$\Psi_{T_\alpha}(x,\zeta) = \begin{cases} 0 & \zeta < \mathrm{VaR}_\alpha(x) \\ \dfrac{\Psi_L(x,\zeta) - \alpha}{1-\alpha} & \zeta \ge \mathrm{VaR}_\alpha(x) \end{cases} \tag{16}$$

For a given decision *x*, the CVaR at confidence level α is the mean of the tail random variable $T_\alpha(x)$ with distribution function (16):

$$\mathrm{CVaR}_\alpha(x) = E\left[T_\alpha(x)\right]. \tag{17}$$

If there is no discontinuity in the distribution function of $L(x)$ at $\mathrm{VaR}_\alpha(x)$, CVaR is just the conditional expectation beyond VaR.

To help define a sensible risk measure, Artzner *et al.* (1999) introduce properties that are required of a *coherent* risk measure; however, VaR does not satisfy these properties in general. As is well known, VaR is not sub-additive: Examples have been constructed where the VaR of the sum of two portfolios is greater than the sum of individual VaRs. Lack of subadditivity is very undesirable because diversification is not promoted.

5.2. Equivalent CVaR Representation

A lack of convexity of VaR contributes to numerical difficulties in optimization. VaR is easy to work with when normality of distributions is assumed, but financial data is typically heavy-tailed. In addition, VaR is non-convex and non-smooth in the case of discrete distributions of scenario trees. On the other hand, CVaR has a representation that is practical in minimization problems regardless of distributional assumptions. However, minimization of CVaR may produce very different solutions than minimization of VaR: VaR minimization may stretch the tail of the distribution beyond VaR resulting in a poor CVaR value.

To begin, define the function:

$$\Gamma_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha} E\left[\left[L(x)-\zeta\right]^+\right], \tag{18}$$

then CVaR is expressed as a minimization through the following result: $\Gamma_\alpha(x,\cdot)$ is finite and continuous with:

$$\mathrm{CVaR}_\alpha(x) = \min_{\zeta\in\mathrm{R}} \Gamma_\alpha(x,\zeta)\cdot \tag{19}$$

As a corollary, it can be shown that if $L(x)$ is convex in *x*, then $\mathrm{CVaR}_\alpha(x)$ is convex in *x* and $\Gamma_\alpha(x,\zeta)$ is jointly convex in (x,ζ). In addition, if a constraint set *X* is convex, the next result produces a convex minimization problem in (x,ζ): Minimizing $\mathrm{CVaR}_\alpha(x)$ with respect to $x \in X$ is equivalent to minimizing $\Gamma_\alpha(x,\zeta)$ with respect to $(x,\zeta) \in X \times \mathrm{R}$, *i.e.*

$$\min_{x\in X} \mathrm{CVaR}_\alpha(x) = \min_{(x,\zeta)\in X\times\mathrm{R}} \Gamma_\alpha(x,\zeta)\cdot \tag{20}$$

The proofs of these results are found in Rockafellar and Uryasev (2002). There are very similar results for CVaR constraints in optimization problems.

When $L(x)$ has a discrete distribution arising from, for example, a scenario tree or sampling, equation (18) becomes:

$$\tilde{\Gamma}_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha}\sum_{s=1}^{S} p^s\left[L^s(x)-\zeta\right]^+, \tag{21}$$

where the random variable $L(x)$ takes the value $L^s(x)$ with probability p^s for $s = 1,\ldots,S$. Furthermore, if $L(x)$ is linear in *x*, then $\tilde{\Gamma}_\alpha$ is convex and piecewise linear.

5.3. CVaR Risk-Reward Optimization

To apply the above results to the single period asset allocation problem, define the constraint set:

$$X = \left\{ x \in \mathrm{R}^n \middle| \sum_{j=1}^{n} x_j = 1, x_j \geq 0, j = 1,\ldots,n \right\}, \tag{22}$$

where $x \in X$ represents positions in n assets. The random return on these assets at the end of the time period is represented by $r = (r_1,\ldots,r_n)^T$, and the total negative return of the portfolio is then given by:

$$L(x) = -x^T r\,. \tag{23}$$

If the mean of r is given by the vector μ, the risk-reward problem is:

$$\min_{x \in X} \mathrm{CVaR}_\alpha(x) \quad \text{s.t.} \quad x^T \mu \geq \mu_0\,, \tag{24}$$

where μ_0 is the required portfolio return. If the uncertainty in the return is given through the set of scenarios $\{r^1,\ldots,r^S\}$, the equivalent formulation is:

$$\begin{aligned} \min_{x,\zeta} \quad & \zeta + \frac{1}{1-\alpha}\sum_{s=1}^{S} p^s \left[-x^T r^s - \zeta\right]^+ \\ \text{s.t.} \quad & x^T \mu \geq \mu_0, \\ & x \in X, \zeta \in \mathrm{R}, \end{aligned} \tag{25}$$

and by introducing auxiliary variables y^s, $s = 1,\ldots,S$, a linear program results:

$$\begin{aligned} \min_{x,\zeta,y^s} \quad & \zeta + \frac{1}{1-\alpha}\sum_{s=1}^{S} p^s y^s \\ \text{s.t.} \quad & x^T \mu \geq \mu_0, \\ & x^T r^s + \zeta + y^s \geq 0, \quad s = 1,\ldots S, \\ & y^s \geq 0, \qquad\qquad s = 1,\ldots S, \\ & x \in X, \zeta \in \mathrm{R}. \end{aligned} \tag{26}$$

This optimization program is used to compare hedging strategies for international asset allocation in Topaloglou *et al.* (2002).

6. Conclusions and Current Research

Stochastic programming captures the stochastic and dynamic nature of ALM problems more realistically than the current standard methods of immunization. However, due to the computational complexity of SP models, it is only until recently that SP has gained applicability in industry. Still, some simplifications such as decision rules are often needed in implementation.

Stochastic programming ALM relies on uncertainty modeled through a discrete set of scenarios. Although there has been some work studying the effects of stable distributions in SP, ALM applications and case studies have been fairly limited in incorporating the various characteristics of financial time-series. This is one area of current research being pursued by the authors.

It is known that GARCH and other time-series methods create scenarios that severely over forecast the unconditional volatility at distant time periods. This creates a problem because ALM models should include scenarios far into the future. At longer horizons, it appears more appropriate to generate scenarios that match the future volatilities implied by the market data. This is a second area of ongoing research: Construct scenario trees that use historical time-series, such as stable GARCH, for early time periods, but generate scenarios for later time periods that agree with the market data.

CVaR has the desirable property of coherence that is lacking in other common risk measures such as VaR and variance. In addition, recent advances have shown that CVaR has an equiva-

lent representation that lends itself to convex optimization in stochastic programming. A third area of current research is to analyze the effects of the distributional assumptions on the optimal allocations of ALM models with CVaR constraints. CVaR is closely related to the expected loss in the tail of the distribution, so the stable assumption will provide a more accurate risk measurement than the normal assumption. Therefore, the stable assumption should improve the optimal allocations.

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