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Some Remarks on First and Second Order Stochastic Processes Choice

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Abstract

A comparison between A(1) and A(2) processes, when used for describing the evolution in time of the global rate of return on investments made by an insurance company, is proposed. In particular, we compare the two processes analysing the parameter sensibility to the size of the sampling interval. An application shows the results. Finally the impact on the global riskiness of a whole life annuity portfolio is evaluated for both the two models.

Key words: A(1) model, A(2) model, covariance equivalence principle, investment risk, total riskiness of a life insurance portfolio, whole life annuity portfolio.

1. Introduction

The investment risk constitues a tool of great interest in financial and actuarial field; to approach the problem the use of appropriate and suitable stochastic models is obviously necessary.

Many contributions in the recent actuarial literature, dealing with the study of life insurance portfolios, propose valuation models based on actuarial techniques, in a stochastic mortality and interest environment. In the actuarial context, many authors consider the use of existing actuarial techniques more appropriate than the use of financial models (APT, CAPM, and the like). In fact, the assumptions holding in these models, such as frictionless trading, efficient markets and so on, are not satisfied in the insurance context (cfr. [5]). Moreover even if all the key assumptions were satisfied, they might be suitable only to put a market value on the liabilities and not for measuring the riskiness of a portfolio of policies (cfr. [16]).

In this framework, the study of the global riskiness connected to a portfolio of policies and its components, is one of the most discussed topics. As well known, the global riskiness of a portfolio of policies is due to the insurance risk and to the investment risk. The mortality risk tends to zero as the number of contracts in the portfolio increases but the investment risk does not, stemming from the random nature of the rate of return on investments made by the Company. That is the reason why many authors stress the importance of this component of risk.

We can recall [12], in which the interest rate is modelled by stationary autoregressive models of order one and two, [8], in which ARIMA (p,0,q) and ARIMA(p,1,q) processes are used to analyze actuarial functions, [4], in which the moments of the present value functions are computed when the force of interest is described by an ARIMA(p, d, q) process.

One of the most followed approaches for the investment risk component is to study the evolution in time of a unique global rate of return, that is the rate of return obtained by the investment of all the revenues according to a defined strategy made by the Insurance Company. The assets can be invested in common shares, bonds, mutual funds and the like obviously, because at any given time not all of the assets are subject to market fluctuations, the global rate of return will tend to be less volatile than the market in which the assets are invested [cfr. 16].

In this context, A(1) and A(2) models are largely used because of their characteristics of flexibility and manageability. Indeed those models depict very well the evolution in time of the global rate of return on investments and they are characterized by the mean reverting property which is significant in this kind of applications (cfr. 13,14).

The use of continuous stochastic models (working with stochastic differential equations) rather than discrete ones (working with difference equations) is more convenient because of the easier formalism and the better readability of the results.

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First order continuous processes are proposed in many papers: [1] and [2] model the integral of the force of interest by an Ornestein - Uhlenbeck process and a Wiener process respectively; in [4] an Ornestein - Uhlenbeck process is used to analyze the measures of the investment and insurance risk for a homogeneous life annuity portfolio; in [13] the first three moments of the present value of benefits for a portfolio of identical policies are proposed modelling the force of interest by an Ornestein- Uhlenbeck process.

Some contributions make use of the second order processes to model the global rate of return (cfr.[14] and [9]). Those processes, described by a second order linear stochastic differential equation, "combine the two effects of a tendency to continue a recent trend and of a mean reverting property" (cfr. 14), offering additional valuable information characterizing the system.

1.1. Aim and structure of the paper

Aim of this paper is to furnish some elements useful in the choice between A(1) and A(2) processes, when used for describing the evolution in time of the global rate of return on investments made by an insurance company.

We propose a comparison between the two models, studied when the size of the sampling interval varies. We prove that the same process, observed at different sampling intervals, shows a very different capacity to describe the analyzed phenomenon: in particular we note that if the size of the sampling interval decreases, a higher order of the process has to be preferred. From an economic point of view this can be explained considering that the value of the global rate of return at time *t* has a weaken relation with previous data at time *t-1*, *t-2*, etc.; vice-versa. When we consider closer data, the process shows a higher "memory" of the preceding observations which can be depicted by means of a higher order model.

The study is based on the analysis of the parameters of the model. As well known, each model is completely defined by its parameters; they are estimated by means of the historical data series. In our case, the past data are assumed to reflect the whole investment strategy of the company, including the allocation of assets and the degree of asset/liability matching (cfr. [16]). The problem we consider seems to be relevant when thinking that using a superior order continuous stochastic process involves many complicating elements both in analytical terms and in terms of the parameter calculation.

The research of the "minimum order" the stochastic process should have in order to consider it as a satisfactory representation of the analyzed phenomenon is, in this order of ideas, useful in the choice of the process to use: for this purpose we take into account the amplitude of the sampling interval available by the actuary in the time horizon he considers.

The paper is structured as follows: in section 2 the main features of A(1) and A(2) models are described, in section 3 the methodology adopted to estimate parameters is presented. Section 4 illustrates the procedure to choose the preferable model, in section 5 an application to compare the two models is proposed and finally, on the basis of the obtained results, in section 6 the total riskiness connected to a life annuity portfolio is computed.

2. A(1) and A(2) models for the global rate of return

2.1. The A(1) model: the Ornstein-Uhlenbeck process

Let us assume that the force of interest $\delta(t)$ is governed by an A(1) process of the Ornstein-Uhlenbeck type.

Let $\{\delta(t), 0 \le t < \infty\}$ have parameters β >0 and σ >0 and initial position δ_0, the process is given by the following differential equation:

$$d(\delta_t - \overline{\delta}) = -\beta(\delta_t - \overline{\delta})dt + \sigma dW(t), \tag{1}$$

where W_t is a Wiener process, β represents the force that brings the process towards the equilibrium position (mean reverting property), $\overline{\delta}$ is the long term mean of the process, σ is the diffusion coefficient.

The solution of equation (1) is (cfr. [7]):

$$(\delta_t - \overline{\delta}) = \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(t) \tag{2}$$

and, given the position $\delta_t = \delta_0$ (cfr. [17]), δ_{t+s} is a gaussian variable with mean:

$$E[\delta_{t+s} / \delta_t = \delta_0] = \delta_0 e^{-\beta s} \tag{3}$$

$$\operatorname{var}[[\delta_{t+s} / \delta_t = \delta_0]] = \sigma^2 \left(\frac{1 - e^{-2\beta s}}{2\beta} \right) \tag{4}$$

and:

$$\begin{aligned} cov[\delta_t, \delta_s] &= \sigma^2 E\left[e^{-\beta(s+t)} \left(\int_0^s e^{\beta u} dW(u) \right) \left(\int_0^t e^{\beta u} dW(u) \right) \right] = \\ &= \sigma^2 e^{-\beta(s+t)} E\left[\left(\int_0^{min(s,t)} e^{\beta u} dW(u) \right)^2 \right] = \sigma^2 e^{-\beta(s+t)} \frac{e^{2\beta min(s,t)} - 1}{2\beta}. \end{aligned} \tag{5}$$

We can write:

$$\operatorname{var}[\delta_t] = \frac{\sigma^2}{2\beta} \quad \text{when} \quad t \to \infty . \tag{6}$$

2.2. The A(2) model

Let us consider a second order stochastic process to model the force of interest, δ_t, governed by the stochastic differential equation (cfr. [14]):

$$\frac{d^2}{dt}(\delta_t - \overline{\delta}) + \alpha_1 \frac{d}{dt}(\delta_t - \overline{\delta}) + \alpha_0(\delta_t - \overline{\delta}) = \sigma dW_t) \tag{7}$$

with initial conditions δ_0 and $\delta_0^{'}$, where δ_t is known for *t<0,* δ is the long term mean of the process, W_t is a standard Wiener process and σ is the diffusion coefficient.

Equivalently we can write equation (7) as follows:

$$(D^2 - \alpha_1 D - \alpha_0)(\delta_t - \overline{\delta}) = Z(t)\,, \tag{8}$$

where $Z(t) = \sigma \dfrac{d}{dt} W_t$ is a stochastic process whose characteristics are:

- $E[Z(t)]=0$
- $E[Z(t)Z(t-u)] = \sigma^2 \delta(u)$.

The homogeneous equation related to (8):

$$(D^2 - \alpha_1 D - \alpha_0)(\delta_t - \delta) = 0, \tag{9}$$

has the following solutions:

$$\lambda_1 = \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_0}}{2}, \qquad \lambda_2 = \frac{\alpha_2 + \sqrt{\alpha_1^2 + 4\alpha_0}}{2}. \tag{10}$$

The solutions given by (10) can be real and distinct ($\alpha_1^2 > -4\alpha_0$), real and equal ($\alpha_1^2 = -4\alpha_0$) or complex conjugate ($\alpha_1^2 > -4\alpha_0$).

The autocovariance function is given by the following expression:

$$\gamma_k = m_1 e^{\lambda_1 k\Delta} + m_2 e^{\lambda_2 k\Delta}, \tag{11}$$

where:

$$m_1 = \frac{\lambda_2 \sigma^2}{2\lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2)}, \qquad m_2 = \frac{-\lambda_1 \sigma^2}{2\lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2)}. \tag{12}$$

The main feature of this model is the tendency to continue the recent trend of the time series used to estimate parameters. Moreover, the meaning of the parameters α_0 and α_1 are interesting: α_0 represents the restoring force bringing the process back to the equilibrium position and α_1 is a damping force which, for large values of *t*, brings the process back to its equilibrium position. Then we can argue that the process has a mean reverting property stronger than the one we find in first order models.

3. The parameter estimation

In this section we estimate the parameters of the continuous A(1) model (β, σ^2) and of the A(2) one (α_0, α_1, σ^2).

On the basis of the data observed at uniform sampled intervals, we need first to find the discrete representation of the chosen continuous processes, then we must establish the appropriate parametric relations between the discrete and the continuous models. For this purpose, we remember that the autocovariance function of the sampled discrete models must coincide with that of the continuous process at all sampling points. It can be shown (cfr. [11]) that the discrete models corresponding to the chosen A(1) model and to the A(2) one are an AR(1) process and an ARMA(2,1) model respectively.

The estimation of AR(1) and ARMA(2,1) parameters allows us to calculate the parameters of the continuous processes by means of the established parametric relations.

3.1. The discrete representation of the A(1) model: the AR(1) model

The AR1 model is described by the following non homogeneous difference equation:

$$(\delta_t - \overline{\delta}) = \phi(\delta_{t-1} - \overline{\delta}) + \sigma_a^2 a_t, \tag{13}$$

which expresses the auto-regressive dependence of order one; a_t's are normal mutually independent variables with zero mean and variance σ_a^2.

Looking at (13) we can understand the meaning of the parameter ϕ : if ϕ takes low (high) values, the process will be characterized by a weak (strong) auto-regressive dependence.

3.2. The discrete representation of the A(2) model: the ARMA(2,1) model

The ARMA (2, 1) model is described by the following difference equation:

$$(\delta_t - \delta) = \phi_1(\delta_{t-1} - \overline{\delta}) + \phi_2(\delta_{t-2} - \overline{\delta}) + a_t - \theta_1 a_{t-1}, \tag{14}$$

where a_t expresses the "shock" affecting the system at time *t* and causes a difference between the previewed values and the effectively observed ones.

The set of the N random variables a_t (t =0 1, 2,….N) is characterised by a multivariate normal distribution, and each variable has zero mean and variance equal to σ_a^2.

Looking at (14), we observe that δ_t is characterised by an "auto-regressive" dependence of order two expressed by the addends $\phi_1\delta_{t-1}$ e $\phi_2\delta_{t-2}$. We observe, moreover, a dependence of order one of the process δ_t on a_t, expressed by the addend $\theta_1 a_{t-1}$.

The solution of (14) can be written as follows (cfr.[11]):

$$(\delta_t - \overline{\delta}) = \sum_{j=0}^{\infty}\left[\left(\frac{\omega_1 - \theta_1}{\omega_1 - \omega_2}\right)\omega_1^j + \left(\frac{\omega_2 - \theta_1}{\omega_2 - \omega_1}\right)\omega_2^j\right]a_{t-j}\ , \tag{15}$$

where:

$$\omega_1, \omega_2 = \frac{1}{2}\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2} \tag{16}$$

with:

$$\begin{aligned} \phi_1 &= \omega_1 + \omega_2, \\ \phi_2 &= -\omega_1\omega_2. \end{aligned} \tag{17}$$

3.3. Parametric relations between discrete and continuous processes

a) AR(1) and A(1) models

On the basis of the well known covariance equivalence principle and being Δ the size of the sampling interval, (for more details see [11] and [17]), it happens that:

$$\phi = e^{-\beta\Delta} \tag{18}$$

and

$$\sigma_a^2 = \frac{\sigma^2}{2\beta}(1-\phi^2)\cdot \tag{19}$$

From (18) and (19) we easily obtain the parameters of the A(1) process :

$$\beta = -\frac{\ln \phi}{\Delta}, \tag{20}$$

$$\sigma^2 = \frac{2\beta\sigma_a^2}{\left(1-\phi^2\right)}.$$

b) ARMA(2,1) and A(2) models

On the basis of the covariance equivalence principle, we can assert that the ARMA(2,1) and the A(2) models are equivalent if , being Δ the size of the sampling interval (cfr. [11] and [15]), it happens that:

$$\phi_1 = \omega_1 + \omega_2 = e^{\lambda_1\Delta} + e^{\lambda_2\Delta} \tag{21}$$

$$\phi_2 = -\omega_1\omega_2 = -e^{(\lambda_1+\lambda_2)\Delta} \tag{22}$$

$$\theta_1 = -P \pm \sqrt{P^2 - 1} \tag{23}$$

with:

$$\begin{aligned} P &= \frac{bsenh(2a\Delta) - senh(2b\Delta)}{2asenh(b\Delta)\cosh(a\Delta) - 2bsenh(a\Delta)\cosh(b\Delta)}, \quad \alpha_1^2 \geq 4\alpha_0 \\ P &= \frac{bsenh(2a\Delta) - asen(2b)}{2asen(b\Delta)\cosh(a\Delta) - 2bsenh(a\Delta)\cos(b\Delta)}, \quad \alpha_1^2 < 4\alpha_0 \end{aligned} \tag{24}$$

where, on the basis of (10)

$$a = \frac{\alpha_1}{2} \quad \text{and} \quad b = \frac{\sqrt{\alpha_1^2 + 4\alpha_0}}{2}$$

To assure that the process is invertible, the parameter θ_1 must take value between –1 and 1 (cfr. [11]).

From (22) and (10) it follows:

$$\alpha_1 = -\ln(-\phi_2), \tag{25}$$

being, from (22), $\phi_2 < 0$

Finally, for the diffusion coefficient we have:

$$\begin{aligned} \sigma^2 &= \frac{bsenh(2a\Delta) - asenh(2b\Delta)}{2ab(a^2 - b^2)(1+\theta_1^2)e^{2a}}\sigma^2, \quad \alpha_1^2 \geq 4\alpha_0, \\ \sigma^2 &= \frac{bsenh(2a\Delta) - asen(2b\Delta)}{2ab(a^2 + b^2)(1+\theta_1^2)}\sigma^2, \quad \alpha_1^2 < 4\alpha_0. \end{aligned} \tag{26}$$

3.3. The methodology to estimate parameters

a) AR(1) model

In order to estimate the parameters of the AR(1) model we can apply the conditional least squares method (cfr. [11]) obtaining for the two parameters, respectively:

$$\phi = \frac{\sum_{t=2}^{N}\left(\delta_t - \bar{\delta}\right)\left(\delta_{t-1} - \bar{\delta}\right)}{\sum_{t=2}^{N}\left(\delta_t - \bar{\delta}\right)^2}, \tag{27}$$

$$\overline{\sigma}_a^2 = \frac{\sum_{t=2}^{N}\left[\left(\delta_t - \overline{\delta}\right) - \phi\left(\delta_{t-1} - \overline{\delta}\right)\right]^2}{N-1} = \frac{RSS}{N-1}, \tag{28}$$

where N represents the number of observations, δ is the long term mean of the process and RSS is the residual sum of squares of $a_t^{'}$. $\hat{\varphi}$ and $\hat{\sigma}_a^2$ completely describe the model. The estimation of the discrete parameters allows us to calculate, using (20), the parameters of the A(1) model.

b) ARMA(2,1) model

Considering an ARMA(2,1) model, the residual sum of squares of a_t (t = 1,2,3….N) is given by the following expression:

$$RSS = \sum_{t=3}^{N} a_2^t = \sum_{t=3}^{N}\left[(\delta_t - \overline{\delta}) - \phi_1(\delta_{t-1} - \overline{\delta}) - \phi_2(\delta_{t-2} - \overline{\delta}) + \theta_1 a_{t-1}\right]^2. \tag{29}$$

The procedure to estimate the parameters of the ARMA(2,1) model is more elaborate than the AR(1) one because, in order to calculate the minimal value of the *RSS*, it is necessary to operate a non-linear regression method. The reason is that the second order auto-regressive dependence, which characterises the ARMA(2,1) model, involves that the difference equation (14) is not linear in the parameters ϕ_1, ϕ_2 e θ_1.

The non-linear regression implies the application of an iterating method: the procedure, in fact, begins with the choice of two values of α_0 and α_1. Those values will have to be negative to make the system stable (cfr.[15]).

On the basis of (21), (22) and (23) we can calculate the parameters ϕ_1, ϕ_2 e θ_1 and, approximating the value of $\overline{\delta}$ by the simple average of the observed data, it is possible to compute the RSS value using (29).

The procedure stops when the values of α_0 and α_1, minimising the residual sum of the squares of a_t (t = 1,2,3….N), are found.

Being the estimated value of σ_a^2 given by the following expression:

$$\sigma_a^2 = \frac{\sum_{t=3}^{N}\left[(\delta_t - \overline{\delta}) - \phi_1(\delta_{t-1} - \overline{\delta}) - \phi_2(\delta_{t-2} - \overline{\delta}) + \theta_1 a_{t-1}\right]^2}{\text{N - 2}}. \tag{30}$$

On the basis of (26) we can calculate the value of the diffusion coefficient σ.

To implement the estimation algorithm we developed a software using GUPTA Sql 32, a well known language for rapid prototyping.

4. The choice of the preferable model: methodological issues

As said before, obtaining a superior order in continuous stochastic processes involves many complicating elements, both for the analytical complexity and for the calculus of the parameters of the model. The increasing analytical difficulty can be seen in the previous section comparing the first two orders: the recursive research of the parameters becomes more and more complex as far as the computational profile is concerned too.

For this reason in the present contribution our aim is the research of the “minimum order” the stochastic process should have to be considered a satisfactory representation of the analysed phenomena. The following applications refer only to the first two orders, with the possibility of extending this type of analysis to the first generic *n* orders.

To establish a preferability order between A(1) and A(2) models, we use the same historical data series considering different sampling intervals and comparing the values of the characteristic parameters for each sampling interval taken into consideration.

As the first step we must verify that the parameter values observe the basic assumptions of each model. To this aim we must study the sensibility of the parameters to the sampling interval (cfr. [11]).

Let us consider the limiting cases of the sampled discrete AR(1) model for extreme values of Δ.

Looking at equation (18) we observe that $\phi \to 0$ as $\Delta \to \infty$ and in this case the equation (13) becomes a sequence of uncorrelated variables that is an AR(0) model:

$$\delta_t - \overline{\delta} = a_t, \tag{31}$$

with variance $\sigma_a^2 \to \dfrac{\sigma^2}{2\beta}$ (see equation 20).

On the other hand, as $\Delta \to 0$, $\phi \to 1$ and equation (13) becomes:

$$\nabla(\delta_t - \overline{\delta}) = a_t \tag{32}$$

that is a random walk with variance $\sigma_a^2 \to 0$.

Another important matter is to analyse the behaviour of the parameter β.

In fact, as the parameter ϕ measures the auto-regressive dependence of order one in the AR(1) model, the parameter β expresses the capability of the continuous process A(1) to return to its equilibrium position, the speed depending on the magnitude of β. Looking at (18) we observe that $\phi \to 0$ as $\beta \to \infty$ and equation (13) becomes a sequence of uncorrelated variables with variance $\sigma_a^2 \to 0$. It follows that, when β increases, the resistance of the system to change increases too and a single disturbance affects the system for a short time; in other words, observations at even short intervals will be uncorrelated. On the other hand, as $\beta \to 0$, $\phi \to 1$ and the process become a random walk with variance $\sigma_a^2 \to \sigma^2\Delta$. In fact, when β decreases, the resistance of the system to change decreases too, that is a single disturbance affects the response for a long time (the process has a "long memory") and observations will be highly correlated ($\phi \to 1$).

Moreover, comparing (1) and (13) we can observe that AR(1) and A(1) processes are equivalent as $\varphi \to (1-\beta)$ and the development in series of (18) confirms that β must take low values different from zero, in order to have the same process.

Let us now consider the limiting cases of the sampled discrete ARMA(2,1) model for extreme values of Δ. Looking at (21) and (22) we observe that if $\Delta \to 0$,$\phi_1 \to 2$, $\phi_2 \to -1$ and it is demonstrated that (cfr. [11]) $\theta_1 \to -2+\sqrt{3}$ and $\sigma_a^2 \to \dfrac{\sigma_z^2\Delta^3}{6(2-\sqrt{3})}$. In this case, the values of ϕ_1 and ϕ_2 parameters describe a boundary stability condition of the system, static and dynamic stability occurring when $\phi_2 > -1$ and $\phi_1 \pm \phi_2 < 1$ (cfr.[11])

On the other hand, if $\Delta \to \infty$, $\phi_1 \to 0$, $\phi_2 \to 0$, equation (14) takes the form

$$\delta_t - \delta = a_t \tag{33}$$

that is a white noise with variance $\sigma_a^2 \to \dfrac{-\sigma^2}{2\lambda_1\lambda_2(\lambda_1 \pm \lambda_2)}$ and $\theta_1 \to 0$.

It is interesting the behaviour of the parameter α_1. In fact, as underlined in section 2.2, it represents a damping force which, for large values of t, brings the process to its equilibrium position giving the process a mean reverting property stronger than the one we find in first order models because it acts along with a restoring force (α_0) bringing the process back to its equilibrium too.

Looking at (25) we observe that if $\alpha_1 \to -\infty$, $\phi_2 \to 0$, $\phi_1 \to 1$, equation (14) is a random walk. It follows that, as α_1 increases in absolute value, the damping effect increases too and a stochastic shock will be quickly damped by the system. The consequence is that the process will have a short memory and the auto regressive dependence of order 2 tends to be null.

On the other hand, if $\alpha_1 \to 0$, $\phi_2 \to -1$ and $\phi_1 \to 2e^{\sqrt{\alpha_0\Delta}}$, the damping effect decreases and parameter ϕ_2 takes a high value.

On the basis of the previous considerations with the following table we report the intervals for the parameters observing the basic assumptions:

A(1)	AR(1)	A(2)	ARMA(2,1)
$0 < \beta < 1$	$0 < \phi < 1$	$-\infty < \alpha_1 < 0$	$-1 < \phi_2 < 0$

The second step of our analysis is to consider the value of the parameters ϕ and ϕ_2 parameters of AR(1) and ARMA(2,1) model respectively. If the analysis shows a strong memory effect, that is the second order parameter ϕ_2 takes significant values, the A(2) model will be preferable. Otherwise the A(1) model will be considered sufficient because in this case the use of the A(2) model would require more elements of complications in term of calculus, and this without considerable benefits. Referring to a generic *n* orders models, a limit for the parameter ϕ_n often used in the practice is 0.1 (cfr.[10] and [11]): if ϕ_n takes positive values, an AR(n-1) model will be chosen when $\phi_n < 0.1$, the AR(n-1) model will be chosen if $\phi_n > -0.1$, otherwise. In our case, if $\phi_2 > -0.1$ the A(1) model will be considered sufficient.

5. Applications

5.1. A(1) and A(2) models: a comparison

In this section we propose some applications to test the parameter sensibility to the size of the sampling interval Δ and, therefore, to compare the two models. For this purpose we collected the data relating to annual bond funds rates for the period of 1988-98 and we worked with seven historical data series referring to bond fund rates; they are related respectively to half-yearly, four-monthly, quarterly, bimonthly, monthly, weekly, daily plotted bond fund rates.

Using the estimation procedure described in section 3, we obtained the results resumed in the following tables: Table 1 illustrates the parameter values of AR(1) and A(1) models respectively while Table 2 shows the parameter values of ARMA(2,1) and A(2) models.

Table1

Parameters value of A1 and AR1 models estimated using annual bond fund rates for the period of 1988-98

Surveys:	ϕ	β	σ_a^2	σ^2
Daily	0.996	0.0043	0.1436	0.1442
Weekly	0.965	0.0356	0.0196	0.0203
Monthly	0.909	0.0949	0.0063	0.0069
Bimonthly	0.795	0.2289	0.0046	0.0057
Four-monthly	0.642	0.4439	0.0036	0.0055
Quarterly	0.590	0.5273	0.0034	0.0055
Half-yearly	0.316	1.1519	0.0017	0.0045

Table2

Parameters value of A2 and ARMA(2,1) models estimated using annual bond fund rates for the period of 1988-98

Surveys:	α_0	α_1	ϕ_1	ϕ_2	θ_1	σ^2_a	σ^2
Daily	-0.1388	-0.1388	1.658294	-0.65924	-0.26794	0.646739	0.67383
Weekly	-0.9615	-0.9615	1.035798	-0.5411	-0.26790	0.49128	0.0020995
Monthly	-4.1667	-1.25	0.724958	-0.09697	-0.26793	0.00926	0.0010431
Bimonthly	-8.333	-2.5	0.027096	-0.01550	-0.26790	0.00529	0.00005534
Four-monthly	-12.5	-3.5	0.099568	-0.00247	-0.26786	0.0039	$2.7292*10^{-6}$
Quarterly	-16.667	-4.667	0.036627	-0.00033	-0.26780	0.0036	$1.1616*10^{-7}$
Half-yearly	-25.0	-7.0	0.008164	-0.00001	-0.26773	0.001798	$7.4485*10^{-10}$

The obtained results relieve and confirm the theoretical observations of section 4.

Referring to Table 1 we can observe that:

a) the parameter $\hat{\varphi}$ decreases as the number of surveys considered in the decade diminishes. Therefore, on the basis of (18) the parameter β increases;

b) the parameters $\hat{\sigma}_a^2$ and σ^2 decrease when the size of the sampling interval increases.

The consideration (**a**) underlines and confirms the influence of the size of the sampling interval on the structure of the model: as the sampling interval decreases the process tends to a random-walk, on the other hand as the sampling interval increases the process tends to an AR(0) model.

Also the behaviour of parameter β confirms the observations in section 4: as β increases, parameter ϕ decreases and the process is characterized by a "short memory". On the other hand, as the parameter β decreases the processes has a "long memory" because it will be influenced for a long time by a stochastic shock and parameter $\phi \to 1$.

The consideration (**b**) implies that as the number of observations decreases we have (see equation (**31**)):

$$\sigma^2 \to 2\beta\sigma_a^2 \cdot$$

Moreover, increasing the number of observations, we verified that $\sigma_a^2 \to \sigma^2\Delta$ being Δ the size of the sampling interval.

Referring to Table 2, we can notice that:

a) both values of the parameters ϕ_1 and ϕ_2 tend to zero as the number of the surveys considered in the decade diminishes and parameter α_1 increases in absolute value;

b) the values of the parameters σ_a^2 and σ^2 decrease when the size of the sampling interval gets larger.

Also in this case the consideration (**a**) points out the influence of the sampling interval size on the structure of the model. The values in Table 1 point out that the random component prevails the auto-regressive one, as the sampling interval gets larger and the process tends to be a white noise (see equation (33)).

The behaviour of parameter α confirms that as it increases in absolute value, the auto-regressive dependence of order two gets weaker ($\phi_2 \to 0$).

The consideration (**b**) allows us to observe that, as the number of the observations increases, the value of the parameter σ^2 increases too. We can observe that, in this case, $\sigma_a^2 \to \frac{\sigma^2 \Delta^3}{6(2-\sqrt{3})}$ (see section 4).

Remarks

On the basis of the previous results, now we want to compare the two models for setting an order of preferability. The first consideration we can do is that if the second order parameters take high values, showing a strong memory effect, the A (2) model is preferable. Otherwise the A (1) model can be considered sufficient to describe the phenomenon.

Looking at Table 2 we observe that closer surveys show a higher "memory effect" and a tendency of the process to continue the recent trend (it depends not only by the last value but also by the preceding ones). In fact, we observe that $\phi_2 \to -1$ as the sampling interval gets smaller, being –1 the value expressing the higher second order auto-regressive dependence. In particular, if we look at daily and weekly observations, parameter ϕ_2 takes values lower than -0.1 showing a high memory effect (see section 4). Even if we consider monthly observations, the parameter ϕ_2 takes a high value ($\cong -0.1$). On the other hand, looking at bimonthly, four-monthly, quarterly and half yearly observations, we point out a quite null value of parameter ϕ_2 (> -0.1).

We can conclude that for daily, weekly and monthly observations, the A(2) model is preferable. On the other hand, as the sampling interval increases (bimonthly, four-monthly, quarterly and half yearly observations) the use of an A(1) model can be considered sufficient being $\phi > 0.1$.

Moreover, we observe that, when the sampling interval is too large (one year), the mean reverting property is too weak and the A(1) model does not work well too.

As said above, the proposed applications refer only to the first two orders, but it is possible to extend this type of analysis to the first generic *n* orders.

Several methods are used with the intent of carrying out an ex-ante analysis of preference among estimated orders without the calculus of continuous parameters (cfr. [3], [10] and [11]). However the parameter values of the continuous processes are extremely important because they give additional information about the memory of the process.

5.2. A whole life annuity portfolio: the total riskiness evaluation

In this section we evaluate the total riskiness connected to a whole life annuity portfolio. It is well known that the global riskiness connected to a portfolio is the sum of two components: insurance risk, depending on the mortality of policy-owners, and investment risk, related to the random nature of rate of return on investments.

We describe the evolution in time of the rate of return on investments by means of A(1) and A(2) models, in order to put in evidence the different impact the two models present on the portfolio global riskiness valuation.

Let us consider a portfolio of *c* identical whole life annuity policies issued to *c* independent lives aged *x* paying a unique premium at issue. The benefit payable by the\ \at the end of each year, if the life insured is alive, is supposed to be equal to 1.

Let Z_i be the random variable denoting the present value of the i-th (i=1, 2,….c) policy (cfr.[4]), we can write:

$$Z_i = \begin{cases} 0 & if\ T_i = 0 \\ \sum_{h=1}^{T_i} e^{-y(h)} & if\ T_i = 1,2...., \end{cases} \tag{34}$$

where:

- T_i is the random variable representing the curtate future life time of the i-*th* life insured .

- $y(t) = \int_0^t \delta_s ds$, δ_s being the stochastic instantaneous rate of return used to discount payments.

We suppose that the following assumptions hold (cfr. [13]):

the random variables T_i are independent and identically distributed;

given the knowledge of *y(h)* for h=1,2,3... ,the $Z_i^{'} s$ are independent and identically distributed;

the random variables $T_i^{'} s$ are independent of *y(h).*

We indicate by *Z(c)* the total present value for the entire portfolio of *c* annuities:

$$Z(c) = \sum_{i=1}^{c} Z_i \cdot \tag{35}$$

It is well known that the global riskiness connected to the portfolio can be measured by the variance of *Z(c)* (cfr. [13], [16]). Therefore we need the first two moments of *Z(c),* given by the following expressions (cfr. [4]):

$$\begin{aligned} &E[Z(c)] = cE[Z_i] \\ &E[Z(c)^2] = cE[Z_i^2] + c(c-1)E[Z_i Z_j], \end{aligned} \tag{36}$$

where:

$$E[Z_i] = \sum_{h=1}^{\infty} {}_h p_x E[e^{-y(h)}],$$

$$E[Z_i^2] = \sum_{h=1}^{\infty} {}_h p_x E[e^{-2y(h)}] + 2\sum_{h=2}^{\infty_i} {}_h p_x \sum_{r=1}^{h-1} E[e^{-y(h)} e^{-y(r)}],$$

$$E[Z_i \cdot Z_i] = \sum_{h=1}^{n_i}\sum_{k=1}^{n_i} {}_h p_{x\,k} p_x E[e^{-y(h)} e^{-y(k)}] \cdot$$

Whether we use an A(2) model or an A(1) one, the random variable y(k) is normally distributed and the discounting factor $e^{-y(k)}$ is log- normally distributed with parameters E[y(k)] and Var[y(k)] .

Consequently we get:

$$E[e^{-my(k)}] = \exp(-mE[(y(k)] + 0.5m^2 V[y(k)] \tag{37}$$

and:

$$\begin{aligned} E[e^{-y(h)} e^{-y(k)}] = \exp(-&E[y(k)] - E[y(h)] + \\ &0.5(V[y(k)] + V[y(h)] + \\ &2\operatorname{cov}[y(h), y(k)]) \end{aligned} \tag{38}$$

varying the expressions for the parameters E[y(k)] and V[y(k)] in dependence of the model for the rate of return used in the application.

In the first case, considering the A1 model (cfr.[13]) we get:

$$E[y(k)] = \delta t + (\delta_0 - \delta)(\frac{1-e^{-\beta}}{\beta}) \tag{39}$$

$$\text{cov (y(h),y(k)=} \frac{\sigma^2}{\beta^2}\min(h,k) + \frac{\sigma^2}{2\beta^3}(-2 + 2e^{-\beta h} + 2e^{-\beta kt} - e^{-\beta|k-h|} - e^{\beta(h+k)}), \tag{40}$$

where δ is the long term mean of the process which can be approximated by the simple average of the observed data and δ_0 can be approximated by the last known value of the historical data series used for valuation (cfr. [15]).

On the other hand, considering an A2 model we have (cfr.[14]):

$$E[y(k)] = \delta k + \frac{\lambda_2(\delta_0 - \delta) - \delta_0^{'}}{\lambda_2 - \lambda_1}(e^{\lambda_1 k} - 1) + \frac{\lambda(\delta_0 - \delta) - \delta_0^{'}}{\lambda_2(\lambda_1 - \lambda_2)}(e^{\lambda_2 k} - 1) \tag{41}$$

$$\begin{aligned} \operatorname{cov} y(h), y(k) = {} & \frac{\sigma^2}{\lambda_1^3\lambda_2(\lambda_2 - \lambda_1)}(1 - e^{\lambda_1 h} - e^{\lambda_2 k}) + \frac{\sigma^2}{\lambda_1\lambda_2^3(\lambda_1 - \lambda_2)}((1 - e^{\lambda_1 h} - e^{\lambda_2 k}) + \\ & \frac{\sigma^2}{2(\ \lambda_1^2 - \lambda_2^2)}\left[\frac{1}{\lambda_2^3}(e^{\lambda_2(k-h)} + 2\lambda_2 h) - \frac{1}{\lambda_1^3}(e^{\lambda_1(k-h)} + 2\lambda_1 h)\right] + \\ & \frac{\sigma^2}{2(\lambda_1 - \lambda_2)^2}\left[\frac{1}{\lambda_2^3}(e^{\lambda_2(h+k)}) + \frac{1}{\lambda_1^3}(e^{\lambda_1(h+k)})\right] - \\ & \frac{\sigma^2}{\lambda_1\lambda_2(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}\left[e^{\lambda_1 h + \lambda_2 k} + e^{\lambda_2 h + \lambda_1 k}\right], \qquad h \le k \end{aligned} \tag{42}$$

where λ_1 and λ_2 are the solutions of the equation (8), δ is the long term mean of the process approximated by the simple average of the observed data, δ_0 is the last known value of the historical data series and $\delta_0^{'}$ can be approximated by the difference of the two most recent values of the historical data series (cfr. [15]).

Using formula (40) we can easily calculate the parameter var(y(k)).

In our applications, we recalled only formulas (41) and (42) referring to the case of real and distinct roots (for the case of equal and real roots and complex roots see [14]).

Combining the results about the discounting factors and formulas (36) we can calculate the total variance of Z(c) in the case of both a stochastic rate of return described by an A(1) process and by an A(2) one.

Numerical illustrations

Let us consider an illustrative whole life annuity portfolio of 1000 policies with age at issue equal to 45. The mortality table used are "Italian male 1970-72".

The following table shows the values of the total variance connected to illustrative portfolio, computed by means of the previous formulas.

Considering one year time horizon, we calculate seven values of the variance corresponding to the seven historical data series used to estimate the parameters of the two models in section 5.1.

Table3

The values of the total variance connected to illustrative portfolio

Surveys	Var[Z(c)]	Model
Daily	656 974	A(2)
Weekly	897 790	A(2)
Monthly	882 072	A(2)
Bimonthly	3 900 410	A(1)
Four-monthly	1 268 060	A(1)
Quarterly	1 223 380	A(1)
Half yearly	995 229	A(1)

In particular we calculate each value of the variance using the preferable model on the basis of the results of section 5.1. Indeed we use the A(2) model for daily, weekly and monthly observations and the A(1) model for bimonthly, four-monthly quarterly and half yearly observations.

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