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AUTHORS	Mary-Paz Arrieta Paredes
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THE TIMING OF CAPACITY EXPANSION INVESTMENTS IN OLIGOPOLY UNDER DEMAND UNCERTAINTY

Mary-Paz Arrieta Paredes*

Abstract

Since a flexibility value emerges in waiting to expand capacity, the impact of demand uncertainty in an oligopolistic industry leads to capacity expansion timing. The creation of growth opportunities is then the outcome of expanding capacity at optimal times. However, in our model different capacity size competitors interact not affecting each others, because assessing the impact of demand uncertainty on capacity expansion projects takes them to set up independently their optimal capacity expansion timing schedules. In equilibrium no firm expands capacity more often than any other. Under demand uncertainty simultaneity in capacity expansions is the only possible Markov Perfect Equilibrium.

Key words: Capacity expansion timing; Oligopolies.

JEL Classification: L11; L13; L25; L29.

I. Introduction

Capital assets investment poses a significant challenge under strategic competition. Theoretically speaking, assessment of risk in this context is more complex than in the single firm case. Furthermore, it is a stylised fact that, due to their usual irreversible nature, decisions such as Capacity Expansion (CE) investments generate large sunk costs, especially in capital-intensive industries.

We start from the basis that competitors want to expand existing plants capacity or build new plants by examining their products demand behaviour. Hence, incumbent firms make their expansions profitably enough to minimise the effects of demand downturns in the future by timing CEs optimally. The following paper is an effort to apply investment under uncertainty analysis to strategic CE investments.

Analysing CEs timing under demand uncertainty, we find that the underlying issue is the existence of growth options. Consequently, we define an equilibrium state as one in which all firms expand capacity with the same frequency, so that a timing equilibrium indicates that all firms in the industry are cultivating growth options.

In oligopoly, we obtain optimal expansion thresholds, determined by both financial and real variables. We then find the optimal CE schedule for each firm. Under demand uncertainty, the only possible CE timing equilibrium is that one where all firms in the industry expand capacity at the same optimal expansion threshold. Otherwise, no CE timing equilibrium is possible.

Assuming homogeneous product technologies, we assert that larger capacity firms can afford to expand capacity at lower product prices than smaller capacity firms, because scale economies allow them dispose of enough cash flow (and/or financial leverage). This makes sense since the more scale economies the higher turnover, and therefore the larger cash-flow (and/or financial leverage). The point is linking operations to revenues to capital budgeting.

First we order firms exogenously according to decreasing installed capacity size from I to N , such that an augmenting demand will trigger CE projects sooner for earlier order firms. We basically mean that firms can always be ordered such that the smallest one is defined as N . Hence firm N is the firm with least cash flow (and/or financial leverage), and therefore, the one requiring the highest demand realisation to make its CE project affordable.

Once firms calculate profits endogenously, each competitor will be able to obtain separately its optimal CE thresholds over the underlying asset demand, setting so its CE timing strat-

* University of Essex, England.

egy, as it maximises its CE value. In our model pre-emption does not emerge basically because the size of firms determines the order of expansion.

We achieve closed-form solutions that allow us to study the analytical properties of CE timing. Unlike mainstream strategic CE investment literature, in our model firms are able to decide the timing of CEs endogenously. This is the main advantage of endogenising demand uncertainty stochastically.

II. Background

Our most straightforward antecedent from the Industrial Organisation literature is due to Gilbert and Harris (1984). These authors study the timing of CE investments in the absence of demand uncertainty.

However, due to the lack of option values, Gilbert and Harris (1984) endogenise timing deterministically, whereas we do it stochastically. This changes the results in that firms will time CEs to take account of the ongoing demand uncertainty. Apart of this, in our model strategies are not history dependent, while Gilbert and Harris allow for some of their strategies to be history-dependent.

A much closer literature to our approach develops the Industrial Organisation implications of investment under uncertainty. The article from this strand of literature methodologically nearest to our article is Grenadier (1996). That paper models strategic expansion timing under demand uncertainty in Real Estate markets. Grenadier's paper focuses on the strategic exercise of American call options to explain strategic expansion processes. His model's development is analogous to ours in the use of option pricing techniques to assess the value of waiting to invest under demand uncertainty, but his focus is not on timing equilibrium.

We depart from the point that, in a market without demand uncertainty, there is no option value in waiting to expand capacity. In such a case, CE timing is faster than with the presence of demand uncertainty in both single and multiple-firm industries. As Grenadier finds, once we introduce demand uncertainty in duopoly, an option value of waiting to invest emerges for competitors. This makes the CE timing game slower.

However, Grenadier imposes a time-to-build constraint, which clearly distinguishes the outcomes of his model from ours. Like us Grenadier (1996) reaches a simultaneous equilibrium, but in contrast, he reaches a sequential equilibrium that we rule out by definition. Additionally, we are able to completely characterise the solutions of our model as Markov Perfect Equilibria.

On the other hand, our model is generated through Risk-Neutral Valuation Programming, then is solved as a stopping time game. The overall results portray Markov Perfect Equilibria (MPE) which, necessarily, are Subgame Perfect Nash Equilibria. The main benefit of employing this solution technique is the result that MPE characterise the solution to single agent stopping time problems.

In our model this translates into solving single-firm CE optimisation problems in an industry where firms, with asymmetric marginal profits, are waiting to expand capacity by assessing risk through option values. The timing of CE decisions will then be determined through optimal stopping time games, delimited by those marginal profits asymmetries.

Subsequently, we make use of some of Dutta and Rustichini's (1993) theorems to characterise the results of our model as an application of optimal stopping time games.

In the past, Capozza and Li (1994) have applied optimal-stopping rules to solve risk-neutral valuation programmes to time investment decisions, but without studying strategic effects, as we do here. Observe the main advantage of this kind of technique is its effectiveness as a strategic management tool, which is difficult to achieve with fixed-point models.

III. The Model: CE investments and risk neutral valuation

Preliminaries

Let us focus on homogeneous product industries¹. Uncertainty on demand, assumed to be industry-wide because of the non-differentiated character of product, is the result of the occurrence of events whose behaviour can be captured through an exogenous stochastic process that shifts market demand, and consequently, firms' CE timing through their profit functions.

For N -firms, denoted $i = 1, \dots, N$ where $N \geq 2$ is a finite number, suppose each firm i possesses K_i units of optimally chosen capacity, so that the N -firms can be ordered in decreasing installed capacity size from 1 to N . Each firm can make irreversible investments of exogenously determined sizes.

Without further loss of generality, let all CEs of firm i be the same size $\Delta K_i > 0$. Since the capacity of firm i is K_i , firm i will have $K_i(j) \equiv K_i + j\Delta K_i$ capacity after CE j , where $j \geq 1$ ($\Delta K_i = K_i(j) - K_i = j\Delta K_i$). The production function q of firm i is a function

$$q = q(K_i)$$

This will be the underlying production function of the model.

Assumption 1: *Production technologies are homogeneous across the industry.*
Then

$$K_1 \geq \dots \geq K_N \Rightarrow q(K_1) \geq \dots \geq q(K_N)$$

The industry's demand function is given by

$$P = \varphi P(Q), \quad (1)$$

where Q represents the total output flow of the industry such that $Q = \sum_{i=1}^N q(K_i) > 0$

and φ , the demand shift (stochastic process, $\varphi > 0$), follows the following Geometric Brownian Motion (Ito Diffusion Process):

$$\frac{d\varphi}{\varphi} = \alpha dt + \sigma dz. \quad (2)$$

Representing t time, α the expected growth rate of φ (growth rate parameter, $\alpha > 0$), σ the volatility of φ (standard deviation parameter) and z a Wiener process on φ .

Basically, equation (1) tells the deterministic demand function of the industry is subject to multiplicative shocks represented by the stochastic process in equation (2).

Assumption 2: *For simplicity, we assume there are no variable costs.*

Then, firm i 's profit flow, given the vector of rival capacities K_{-i} , is

$$\Pi_i(K_i, K_{-i}, \varphi) = Pq(K_i)$$

¹ Differentiated product industries will call for a separate treatment from the investment under uncertainty theoretical viewpoint.

$$\begin{aligned}
\Pi_i(K_i, K_{-i}, \varphi) &= \varphi P(Q) q(K_i) \\
&= \varphi P(q(K_1), \dots, q(K_N)) q(K_i) \\
\Pi_i(K_i, K_{-i}, \varphi) &= \varphi G(K_i, K_{-i}). \tag{3}
\end{aligned}$$

And firm i 's marginal profit flow is written as

$$\pi_i(K_i, K_{-i}, \varphi) = \varphi G'(K_i, K_{-i}),$$

where $G(K_i, K_{-i})$ denotes the capital revenue function and $G'(K_i, K_{-i})$ is the marginal revenue product of capital of firm i .

In fact,

$$G(K_i, K_{-i}) = P(Q) q(K_i) = P(q(K_1), \dots, q(K_N)) q(K_i).$$

By definition $P(Q) > 0$, below we show P is also positive in equation (1).

Assumption 3: Firm i 's marginal profit is always positive.

$$\pi_i(K_i, K_{-i}, \varphi) > 0 \Rightarrow \varphi G'(K_i, K_{-i}) > 0$$

Such that

$$\pi_1 \geq \dots \geq \pi_N.$$

Based on assumptions 1 to 3, we have to make clear that we will only pay attention to continuous operations. This is equivalent to say that firms are not likely to have losses, otherwise they may suspend operations or exit the industry, which are exogenous decisions (that is, these cannot be made from this model).

The firm's value in oligopoly

We focus in oligopolies. Observe in capital-intensive industries, it is usual to incur in large fixed costs. These may represent sunk costs, namely, once installed, machinery and capital equipment may be impossible to resell, and even if these were not, their second-hand value may be far less than the original one.

This condition is often known as capital specificity, which gives place to different degrees of irreversibility depending on the type of uncertainty faced: The wider the uncertainty is, the more the will be irreversibility. For instance, in the petroleum refining industry capital specificity is typical, and demand uncertainty is likely to be wide due to the non-differentiated character of product.

Assumption 4: In this CE's timing model investments are completely irreversible.

Suppose now that the rest of firms in the oligopoly have already expanded capacity at least once, such that these are only incumbent firms. For the time being, in this model entry is blockaded. Using vector notation, the total value of firm i in terms of physical capacity $Z_i(K_i(j), K_{-i}, \varphi)$ is:

$$Z_i(K_i(j), K_{-i}, \varphi) = V_i(K_i(j), K_{-i}, \varphi) + W_i(K_i(j), K_{-i}, \varphi), \tag{4}$$

where $V_i(K_i(j), K_{-i}, \varphi)$ denotes profits flows on capacity in place and $W_i(K_i(j), K_{-i}, \varphi)$ denotes the net present value of additional capacity not yet purchased.

Note equation (4) is a wider definition than the one in equation (3). Such a partition in two terms aims to show how, given other firms' capacity, under demand uncertainty firm i 's cur-

rent CE investments will affect posterior ones by taking into account potential CEs net present value¹. The concept of growth option is in this way captured in equation (4): The unit(s) of capacity added today would pave the way to add further units in the future².

Assume momentarily that we can number units of capacity infinitely in the order they are installed, with each unit installed one after another. Suppose firm i has installed units 1 through n so far. Then, omitting φ momentarily

$$Z_i(K_i(j), K_{-i}) = V_i(K_i(j), K_{-i}) + W_i(K_i(j), K_{-i}). \quad (5)$$

We can rewrite equation (5) by adding the change in value of firm i 's installed units to the change in value of its projects to install further units. Namely:

$$\begin{aligned} Z_i(K_i(j), K_{-i}) &= \Delta V_i(K_i(1), K_{-i}) + \dots + \Delta V_i(K_i(n), K_{-i}) \\ &+ \Delta W_i(K_i(n+1), K_{-i}) + \Delta W_i(K_i(n+2), K_{-i}) + \dots \end{aligned} \quad (6)$$

Equation (6) poses the main issue of growth options: If firm i bought and installed unit $n+1$, of value $\Delta V_i(K_i(n+1), K_{-i})$, at an optimal time endogenously determined, at what other optimal time endogenously determined should this exercise its option, worth $\Delta W_i(K_i(n+2), K_{-i})$, to buy and install unit $n+2$, which will eventually have a value of $\Delta V_i(K_i(n+2), K_{-i})$, and so on?³

The key notion is, if firms investment projects are irreversible, agents must make investment timing decisions that trade-off the extra returns from early commitment against the benefits of tailoring their investment decisions to the ongoing uncertainty. Then uncertainty can depress current investment by making waiting for information more attractive. This gives place to what is called the flexibility value of waiting to invest.

Once a CE decision is made, at least the cost of investment is sunk and can not be recovered, should there be regret. Therefore firm i will wait as long as possible for information on demand to arrive before making any CE decision, because this firm will be assessing CE options before committing resources.

This insight can be directly applied to any kind of capital investment decision involving irreversibility, especially in those environments where sunk costs are large⁴. We hold that in an infinite large horizon CE options are exercised to install capacity as growth options, renouncing to the flexibility value of waiting to expand capacity. The total value of firm i in oligopoly Z consequently is given by

$$Z_i(K_i(j), K_{-i}) = \sum_{j=0}^n \Delta V_i(K_i(j), K_{-i}) + \sum_{j=n+1}^{\infty} \Delta W_i(K_i(j), K_{-i}). \quad (7)$$

Choosing an arbitrary interval of time dt equal to Δt and studying when its limit tends to 0, in continuous time (7) results in

¹ Pindyck (1988) departs from an equation similar to (4). For the single firm, he examines the implications of irreversible investments with respect to capacity choice, capacity utilisation, firm value and long run marginal cost.

² Carliss Baldwin (1982) and later Pindyck (1991) explain the existence of growth options through investment timing decisions at single firm level.

³ Pindyck (1988) solves this as an investment timing optimisation problem for the single firm. In fact, we are doing a similar work by applying the same methodology, to the multiple-firm case though.

⁴ He and Pindyck (1992) prove that when capacity is flexible (this means different processing units can be used indistinctly for producing different outputs), the single product and the multi-product firm are solving the same irreversible investment problem.

$$Z_i(x, \varphi) = \int_0^n \Delta V_i(x, \varphi) dx + \int_n^\infty \Delta W_i(x, \varphi) dx. \quad (8)$$

Such that the profit flow up to n is $\Pi_i(K_i, K_{-i}, \varphi)$. Equation (8) can be interpreted as the analytical dissection of equation (4), where x represents total industry capacity.

Our analysis concentrates on Markov Perfect Equilibrium (MPE) in pure strategies. Note a Markovian pure strategy is a strategy that depends only on the current level of the state variable, and not the past levels. The analysis will show that an MPE takes the following form for each firm: Expand capacity as soon as the state variable is above some critical threshold, otherwise do not expand capacity. The task now is to determine the critical thresholds.

Next, we explain the Risk-Neutral valuation programme we use to optimise equation (8) and the optimal stopping time solution we find to determine the critical thresholds for firm i .

The optimal capacity expansion threshold

Let us assume at this stage there are increasing or at least constant returns to CEs at plant level, such that product is growing at least as capacity, to incentive lumpy additions. This means that $q_i(aK_i) \geq aq_i(K_i)$ for all $a \geq 0$, that is, homogeneity of at least degree zero exists. Suppose firm i increases its capital stock from K_i to $K_i(j)$ as φ varies over dt from φ to $\varphi + d\varphi$.

According to our theory, given other firms' capacities, expanding capacity from K_i to $K_i(j)$, would generate an immediate profit flow, eliminating the option value of waiting to expand capacity. We assert firm i can reach $K_i(j)$ capacity at $jk\Delta K_i$ sunk costs of the depleted option value, where k represents unitary fixed costs that are positive and exogenously determined. Hence, we set the following firm i 's optimisation problem w for any interval $(t, T]$

$$w^i(K_i(j), \varphi + d\varphi) = \max_{\{X_t | X_0 = K_0^i, K_0^{-i}\}} \mathbb{E} \left(\int_0^t e^{-r\tau} \Pi^i(X_\tau, \varphi_\tau) d\tau + e^{-r(T-t)} \Omega^i(X_T, \varphi_T) | \varphi_0 = \varphi \right)$$

subject to $\int_0^\infty e^{-rt} jk\Delta K_i dt,$ (9)

where X_τ denotes the capacity of all firms in the industry at time t and r the risk-free rate; note that in order for firms to exercise their options at some finite expected time t we need $r > \alpha$. Ω denotes a forced terminal profit for firm i between t and a virtually final time T and crystallises the calculation of firm i 's growth options.

The left-hand side of equation (9) indicates that firm i expands K to $K_i(j)$ and then the problem re-starts at a new uncertainty level $\varphi + d\varphi$. Given other firms' capacity at time t , the first term of the right-hand side in equation (9) represents the present value of ΔK_i for firm i . The second term represents the present value of any further ΔK_i added by firm i , given other firms' capacity at time t .

Since φ is a stochastic process, the whole right-hand side of equation (9) embodies an option value. Basically, this means that the profit flows of firm i 's CE projects can be reproduced in the financial markets.

Assumption 5: In this oligopoly firms are risk-neutral, or equivalently, risk in φ has zero correlation with the overall futures risk of the underlying product market.

Subsequently, to time the CE investment, firm i will calculate recursively the CE expected value, denoted by operator ε in equation (9), over dt as

$$w^i(K_i, \varphi) = \varphi G(K_i, K_{-i})dt + e^{-rdt} \left\{ \varepsilon[w^i(K_i(j), \varphi + d\varphi)] - k(K_i(j) - K_i) \right\}. \quad (10)$$

The solution to equation (10) (expected value function or Bellman equation) in our context prescribes the formulation of an optimal stopping time problem.

Therefore, firm i will choose $K_i(j)$ to maximise the right-hand side of equation (10). Then the resulting maximum will be the value of the Bellman equation each time firm i expands capacity. Since irreversibility implies $K_i(j) \geq K_i$, substituting $K_i(j) = K_i$ in equation (10)

$$w^i(K_i, \varphi) = \varphi G(K_i, K_{-i})dt + e^{-rdt} \left\{ \varepsilon[w^i(K_i, \varphi + d\varphi)] \right\} \quad (11)$$

we obtain the “first” initial value of $w^i(K_i, \varphi)$.

To determine the path the option value to expand capacity follows, we derive the right hand side of equation (11) by Ito’s lemma. Dividing through by dt and making zero dt terms of second and higher order, we get:

$$\frac{1}{2} \sigma^2 \varphi^2 w_{\varphi\varphi}^i(K_i, \varphi) + \alpha \varphi w_{\varphi}^i(K_i, \varphi) - r w^i(K_i, \varphi) + \varphi G(K_i, K_{-i})_i = 0. \quad (12)$$

Solving differential equation (12), we obtain:

$$w^i(K_i, \varphi) = A_1(K_i) \varphi^{\beta_1} + A_2(K_i) \varphi^{\beta_2},$$

where β_1 and β_2 are the following (finite) exponents

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} > 1 \quad (13)$$

and

$$\beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} < 0.$$

$A_1(K_i)$ and $A_2(K_i)$ represent solution coefficients (below we will describe their economic interpretation).

The complete solution to equation (12) is given by

$$w^i(K_i, \varphi) = A_1(K_i) \varphi^{\beta_1} + A_2(K_i) \varphi^{\beta_2} + \frac{\varphi G(K_i, K_{-i})}{\delta} - k K_i, \quad (14)$$

where δ ($\delta = r - \alpha > 0$) stands for a dividend rate (“convenience” yield) on φ , a positive constant term.

Since β_2 may make w^i fluctuate infinitely, we have to impose the boundary condition $w(0) = 0$, which implies making $A_2(K_i) = 0$. Then equation (14) becomes

$$w^i(K_i, \varphi) = A_1(K_i) \varphi^{\beta_1} + \frac{\varphi G(K_i, K_{-i})}{\delta} - kK_i. \quad (15)$$

At CE j^{th} , writing $\varphi = \varphi(K_i)$, the necessary condition for optimality over (15), called the value matching condition, imposes that the additional value of expanding capacity be equal to its additional cost, namely

$$A_1(K_i(j)) \varphi(K_i(j))^{\beta_1} + \frac{\varphi(K_i(j)) G(K_i(j), K_{-i})}{\delta} = jk\Delta K_i. \quad (16)$$

The sufficient condition for optimality over (15), called the smooth pasting condition, requires that

$$\beta_1 A_1(K_i(j)) \varphi(K_i(j))^{\beta_1-1} + \frac{G(K_i(j), K_{-i})}{\delta} = 0. \quad (17)$$

Solving for equations (16) and (17) we obtain the closed-form solutions of $\varphi_i^*(K_i(j))$ and $A_i(K_i(j))$

$$\varphi_i^*(K_i(j)) = \frac{\beta_1 j \delta k \Delta K_i}{(\beta_1 - 1) G_i(K_i(j), K_{-i})}$$

and

$$A_1(K_i(j)) = - \left(\frac{\beta_1 - 1}{jk\Delta K_i} \right)^{\beta_1-1} \left(\frac{G(K_i(j), K_{-i})}{\beta_1 \delta} \right)^{\beta_1}. \quad (18)$$

The first equation in (18) describes firm i 's optimal expansion threshold for adding $j\Delta K_i$ lump sum of capacity. The ratio $\frac{\beta_1}{\beta_1 - 1}$ indicates the gap needed to trigger this CE under demand uncertainty. The second equation in (18), plugged back into equation (15) at round j^{th} becomes the value of waiting to expand capacity that firm i is giving up when decides to add $j\Delta K_i$ lump-sum of capacity, namely $A_1(K_i(j)) \varphi(K_i(j))^{\beta_1}$. Therefore, equation (15) represents the flexibility value function of firm i .

Observe equations (18) only make sense if the marginal profit of additional capacity is positive, which by assumption 3 holds. Therefore $\varphi_i^*(K_i(j)) > 0$ and $A_1(K_i(j)) < 0$.

IV. Characterisation of CE strategies: The endogenisation of CEs timing

In order to understand the exposition, we consider first an industry with two firms, i.e. $i = 1, 2$. Later on we will generalise to the N -firm industry case. Both firms hold the option to add lump-sum capacity when facing demand uncertainty. Because of the Markov nature of strategies, all subsequent rounds of investments are independent of the first round; then it is sufficient to analyse the first round of CEs, so we set the number of rounds to $j = 1$.

Firms will compete on the addition of lumpy capacity as follows: The first firm to expand capacity is firm 1, which will produce relatively more than firm 2, which will see the value of its plant(s) affected. Firm 2's capacity is rendered (relatively) less profitable because of the presence of a larger rival. If and when firm 2 exercises its CE option, this will gain value from increasing its current capacity.

From Dutta and Rustichini (1993), we expect that firm 2's optimal strategy will be to set up a CE timing threshold for P consistent with the behaviour of φ , namely φ_2^* , which was obtained in equation (18) and we restate here as:

$$\varphi_2^* = \varphi_2^*(K_2(1)) = \frac{\beta_1 \delta k \Delta K_2}{(\beta_1 - 1)G(K_1(1), K_2(1))}. \quad (19)$$

The threshold shown in equation (19) represents the solution to the optimal stopping time problem of firm 2 given firm 1 has already expanded capacity, in duopoly. We have to point out that model discrepancies basically due to the absence of time-to-build constraints in our model, take us to obtain a different expression of firm 2's optimal CE threshold from Grenadier (1996).

Given that firm 1 have already expanded capacity, the CEs timing is endogenised as follows:

If the observed $\varphi > \varphi_2^*$, firm 2 will expand capacity at once since this wants to be first, obtaining in that way the net present value of its CE

$$\frac{\varphi(K_2(1))G(K_1(1), K_2(1))}{\delta} - k\Delta K_2. \quad (20)$$

Equation (20) shows firm 2's discounted profit flows net of expansion costs.

If the observed $\varphi \leq \varphi_2^*$, firm 2 will wait until φ_2^* is first hit because its threshold is above firm 1's. Hence, in the moment φ_2^* is hit, firm 2 will get

$$\frac{\varphi_2^* G(K_1(1), K_2(1))}{\delta} - k\Delta K_2.$$

However, since firm 2 has to wait until φ_2^* is hit, the expected present value of its CE project will be:

$$\varepsilon \left[e^{-r\Gamma} \right] \left[\frac{\varphi_2^* G(K_1(1), K_2(1))}{\delta} - k\Delta K_2 \right], \quad (21)$$

where Γ represents the (random) first time when φ reaches φ_2^* ¹. Note in equation (21) the net present value is calculated at the optimal Γ , that is, when firm 2 reaches its optimal threshold. This means that the discount is taken at the moment when φ hits φ_2^* from below, and present values expectation are calculated as the waiting time gets closer to 0.

Therefore, firm 2's CE value will be:

¹ Γ is the definition of the virtual final time T in equation (9).

$$W_2(K_1(1), K_2(1), \varphi) = \begin{cases} \frac{\varphi(K_2(1))G(K_1(1), K_2(1))}{\delta} - k\Delta K_2 & \text{if } \varphi > \varphi_2^* \\ \left(\frac{\varphi}{\varphi_2^*}\right)^{\beta_1} \left(\frac{\varphi_2^* G(K_1(1), K_2(1))}{\delta} - k\Delta K_2 \right) & \text{if } \varphi \leq \varphi_2^* \end{cases} \quad (22)$$

Working backwards, firm 1 will put in practice the following CE timing strategy:

If the observed $\varphi \geq \varphi_2^*$, firm 1 will not want to be left behind and will expand capacity right before firm 2 does. Then its CE value will be:

$$\frac{\varphi(K_1(1))G(K_1(1), K_2(1))}{\delta} - k\Delta K_1 > 0.$$

If the observed $\varphi < \varphi_1^*$, firm 1 will wait as earns the value of waiting to expand capacity given by

$$B_1(K_1(1))\varphi(K_1(1))^{\beta_1} > 0.$$

But if $\varphi_1^* \leq \varphi < \varphi_2^*$, firm 1 will be assessing the following flexibility value function

$$C_1(K_1(1))\varphi(K_1(1))^{\beta_1} + \frac{\varphi(K_1(1))G(K_1(1), K_2)}{\delta} - k\Delta K_1.$$

Therefore, firm 1's CE value will be:

$$W_1(K_1(1), K_2, \varphi) = \begin{cases} B_1(K_1(1))\varphi(K_1(1))^{\beta_1} & \text{if } \varphi < \varphi_1^* \\ C_1(K_1(1))\varphi(K_1(1))^{\beta_1} + \frac{\varphi(K_1(1))G(K_1(1), K_2)}{\delta} - k\Delta K_1 & \text{if } \varphi_1^* \leq \varphi < \varphi_2^* \\ \frac{\varphi(K_1(1))G(K_1(1), K_2(1))}{\delta} - k\Delta K_1 & \text{if } \varphi \geq \varphi_2^* \end{cases} \quad (23)$$

From which firm 1 obtains its optimal CE threshold:

$$\varphi_1^* = \frac{\beta_1 \delta k \Delta K_1}{(\beta_1 - 1)G(K_1(1), K_2)}. \quad (24)$$

Besides,

$$B_1(K_1(1)) = C_1(K_1(1)) = - \left(\frac{\beta_1 - 1}{jk\Delta K_1} \right)^{\beta_1 - 1} \left(\frac{G(K_1(1), K_2)}{\beta_1 \delta} \right)^{\beta_1}. \quad (25)$$

Once firm 1 expands capacity, its total value equals the sum of the profits on its existing capacity and the value of its additional capacity. That is, firm 1's payoff is:

$$Z_1(K_1(1), K_2, \varphi) = V_1(K_1(1), K_2, \varphi) + W_1(K_1(1), K_2, \varphi). \quad (26)$$

Analogously, once firm 2 expands capacity, this will receive a payoff $Z_2(K_1(1), K_2(1), \varphi)$ of

$$Z_2(K_1(1), K_2(1), \varphi) = V_2(K_1(1), K_2(1), \varphi) + W_2(K_1(1), K_2(1), \varphi). \quad (27)$$

Observe, equations (26) and (27), the values of firm 1 and 2 in terms of physical capacity under demand uncertainty, are effectively equation (4) written in duopoly.

V. The existence of equilibrium: Capacity Expansion Timing Equilibrium

DEFINITION: A CE Timing Equilibrium (CETE) is that one in which all firms in the industry expand capacity with the same frequency, that is, no firm expands capacity more often than any other.

Since all the firms held the same growth options, the existence of a CETE guarantees that all firms in the industry always grew at the same pace. Proposition 1 summarises firm i 's optimal CE strategy.

PROPOSITION 1: Under demand uncertainty, the optimal expansion threshold of any firm i at CE j is given by

$$\varphi_i^*(K_i(j)) = \frac{\beta_1 j \delta k \Delta K_i}{(\beta_1 - 1)G(K_i(j), K_{-i})}. \quad (28)$$

Proof: φ_i^* is the unique threshold solution of the optimal stopping time problem posed in equation (9) ■

Consequently,

PROPOSITION 2: The optimal strategy under demand uncertainty is to expand capacity the first moment that φ equals or exceeds the trigger parameter φ_i^* . That is, the optimal Markov CE time j of firm i , Γ_i^* , can be written as:

$$\Gamma_i^* = \inf \left\{ t > 0; \varphi(t) \geq \varphi_i^*(K_i(j)); \varphi_i^*(K_i(j)) = \frac{\beta_1 j \delta k \Delta K_i}{(\beta_1 - 1)G(K_i(j), K_{-i})} \right\}. \quad (29)$$

Proof: Γ_i^* is optimal because of proposition 1. Besides, to prove that Γ_i^* is Markov, we invoke Dutta and Rustichini's (1993) characterisation, according to which the Optimal Stopping Equilibria are a subset of the MPE of the game ■

We continue analysing the 2-firm case. The N -firm case will just restate the results studied. Then, by definition of CETE¹,

THEOREM 1: The following describes a timing Markov Perfect Equilibrium (MPE) in pure strategies: If $q_1'(K_1) = q_2'(K_2)$ both firms will exercise simultaneously their growth options at Γ_i^* with probability 1 and non-simultaneously with probability 0.

Proof: First, to show this is a Nash equilibrium, when φ hits φ_i^* then $\varphi = \varphi_i^*$, so that according to proposition 2, optimal payoffs are achieved in equations (26) and (27); below or above these thresholds, payoffs are not optimal. Moreover, we have shown, calculating the optimal

¹ We have to mention the proof on the existence of this equilibrium is summarised in theorem 0 of Dutta and Rustichini (1993).

expansion threshold in equation (18), that assumption 3 assures that $\varphi_i^* > 0$. Consequently, P will be positive by the definition of industry demand in equation (1). Proposition 2 proves that output price is positive in CETE.

Besides, since firm 1 and firm 2 will be compelled to expand capacity whenever their respective φ_i^* are reached, the only way they could time CEs in equilibrium, that is, the only way one of them could not expand capacity more often than the other, is if both always expand capacity at the same time. This means $\varphi_1^* = \varphi_2^*$. Then, at $j = 1$

$$\frac{\Delta K_1}{G(K_1(1), K_2)} = \frac{\Delta K_2}{G(K_1(1), K_2(1))} \Rightarrow \frac{1}{\frac{P(Q)(q(K_1(1)) - q(K_1))}{\Delta K_1}} = \frac{1}{\frac{P(Q)(q(K_2(1)) - q(K_2))}{\Delta K_2}}.$$

$$\text{Thus } \frac{1}{\frac{\Delta q(K_1)}{\Delta K_1}} = \frac{1}{\frac{\Delta q(K_2)}{\Delta K_2}}. \text{ Therefore, in the limit}$$

$$\frac{1}{q'(K_1)} = \frac{1}{q'(K_2)} \Leftrightarrow q'(K_1) = q'(K_2),$$

where:

$q'(K_i)$: MPC_i : Marginal Product of Capital of firm i

Which, by assumption 1, follows.

Hence, because both firms are maximising their payoffs, they will expand with probability 1 simultaneously at Γ_i^* , and with probability 0 at any other time ■

Note how the strategies work: firm i expands capacity as soon as $\varphi \geq \varphi_i^*$. Hence if the initial value of $\varphi = \varphi(0)$ is greater than φ_2^* , both firms expand capacity immediately. If $\varphi(0) < \varphi_1^*$, both firms wait before expanding capacity. If $\varphi_1^* \leq \varphi(0) < \varphi_2^*$, then firm 1 expands capacity immediately while firm 2 waits to expand capacity.

Therefore in our model actions can be taken simultaneously or sequentially, depending on the initial value of φ . This is distinct from whether the equilibrium is simultaneous or sequential.

The equilibrium is sequential if the thresholds are distinct, i.e., if $\varphi_1^* < \varphi_2^*$, but we rule this case out by the definition of CETE, because in this instance firms 1 could expand capacity twice or more before firm 2. The equilibrium is simultaneous if the thresholds are equal. The latter occurs if

$$\frac{\Delta K_1}{G(K_1(1), K_2)} = \frac{\Delta K_2}{G(K_1(1), K_2(1))}.$$

Observe this will hold in this model only if there are constant returns to scale in the production technology of the industry, that is if

$$\varphi_i^* = \frac{\beta_1 \delta k \Delta K_i}{(\beta_1 - 1)P(Q)(q(K_i(1)) - q(K_1))} = \frac{\beta_1 \delta k \Delta K_i}{(\beta_1 - 1)P(Q)\Delta q(K_i)}$$

$$\varphi_i^* = \frac{\beta_1 \delta k}{(\beta_1 - 1)P(Q)}.$$

Then, both firms (and all firms, in the N -firm case) expand capacity at the same moment, irrespective of their capacity and CE sizes, given their MPCs equal.

Note that in such a case firms 1 and 2's thresholds would be the same, even though these are asymmetric firms formulating asymmetric problems. The explanation is that firm 2's behaviour is unaffected by firm 1's. After firm 1 has moved, firm 2 faces a single-agent optimization problem that is independent of exactly when firm 1 has moved, since it relies only on the fact that firm 1 has already moved.

Finally, let us suppose CE sizes (ΔK_i) were not the same. One way of allowing firms choosing their CE sizes would be the following (see Dixit (1995) for an extended explanation of the methodology here used). Observe adding only one unit of capacity under demand uncertainty, the optimal CE threshold can be written as:

$$\varphi_i^*(K_i) = \frac{\beta_1 \delta k}{(\beta_1 - 1)G'(K_i)}. \quad (30)$$

Let K_i^n denote firm i 's installed capacity in plant n and K_i^{n+j} denote firm i 's lump-sum projected CE of exogenously determined size $K_i^{n+m} - K_i^n$. By construction, the marginal revenue product of capital is:

$$G'(K_1^{n+m}, \dots, K_N^{n+m}) = \frac{G(K_1^{n+m}, \dots, K_N^{n+m}) - G(K_1^n, \dots, K_N^n)}{K_i^{n+m} - K_i^n}. \quad (31)$$

Substituting (31) into (30), we get

$$\varphi_i^*(\Delta K_i) = \frac{\beta_1 \delta}{(\beta_1 - 1)} \left[\frac{(K_i^{n+m} - K_i^n)k}{(G(K_1^{n+m}, \dots, K_N^{n+m}) - G(K_1^n, \dots, K_N^n))} \right]$$

or equivalently

$$\varphi_i^*(\Delta K_i) = \frac{\beta_1 \delta}{(\beta_1 - 1)} \frac{k \Delta K_i}{(\zeta - \eta Q) \Delta q_i(K_i)}, \quad (32)$$

where

$$K_i^{n+m} - K_i^n = \Delta K_i;$$

$$G(K_1^{n+m}, \dots, K_N^{n+m}) - G(K_1^n, \dots, K_N^n) = (\zeta - \eta Q) \Delta q(K_i) = \Delta G;$$

and, $\varphi_i^*(\Delta K_i)$: Threshold for the lump-sum CE (φ_i^* , in short).

We have to underline that the similarity of thresholds observed in the model with fixed CE sizes will sustain letting CE sizes varying, since in either case capacity asymmetric firms will still be setting their optimal CEs thresholds independently of each others, as they are endogenising timing optimally.

VI. The N-firm case

In the N -firm case we basically extend the number of firms in the industry from 2 to a finite N . Nonetheless, all the insights developed in the above sections are sustained. We endogenise CE timing straightforwardly. Remember, because of the Markov nature of strategies, all subsequent rounds of investments are independent of the first round; then it is sufficient to analyse the first round of CEs, so we set the number of rounds to $j = 1$.

Therefore, at CE $j = 1$

$$\begin{aligned}\varphi_N^*(K_N(1)) &= \frac{\beta_1 j \delta k \Delta K_i}{(\beta_1 - 1) G_N(K_1(1), \dots, K_N(1))}, \\ \varphi_1^*(K_1(1)) &= \frac{\beta_1 j \delta k \Delta K_i}{(\beta_1 - 1) G_1(K_1(1), \dots, K_N(1))}.\end{aligned}\quad (33)$$

Propositions 1 and 2 are generalisations that extend to the N -firm case. Consequently, under demand uncertainty, a CETE will be reached if and only if firms equal their marginal products of capital. That is,

THEOREM 2: The following describes a timing MPE in pure strategies: If

$$q_1'(K_1) = \dots = q_N'(K_N)$$

all firms will exercise their growth options simultaneously at Γ_i^* with probability 1 and non-simultaneously with probability 0.

Proof: It follows from theorem 1's proof. ■

VII. Conclusions

In the model this paper has developed, the main consequence of demand uncertainty has been the generation of growth options, which have been shown to impact firms' CE timing decisions. As the investment under uncertainty literature would anticipate, investment rules here derived involve the concurrent calculation of both CE values and timing.

According to our theory, out of CETE, larger capacity size firms can grow faster than smaller capacity size firms, which can even crowd these out of the industry. The absence of CETE affects industry growth, because larger capacity size firms can expand more often and therefore can get even larger than smaller capacity size firms.

In our model, if the industry follows a CETE, small capacity size firms will be able to grow as much as large capacity size firms, because no firm will expand capacity more often than any other. As a matter of fact, in this industry firms may equalise sizes at some point by following simultaneous CETE.

Besides, the model in this paper exposed could explain how firms tend to grow at different paces in industries experiencing increasing returns to scale, but grow more or less at the same pace in industries experiencing constant returns to scale. This is because industries already settle will tend to grow slower than industries at early stages.

We have to remark its many empirical applications, in comparison to fixed-point type models, especially managerial applications. Clearly, it would be easier to find analysable data in commodity industries.

As extensions, it would be interesting to study this model dropping some of the assumptions established. Particularly, for industries like the petroleum refining one it would be appealing allowing for the presence of variable costs, since this would bring up the role that crude oil prices play in CE timing decisions, especially if their stochastic nature is contemplated[‡].

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