

“Fair split of profit generated by n parties”

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FAIR SPLIT OF PROFIT GENERATED BY n PARTIES

Abstract

The authors studied the process of merging insured groups, and the splitting of the profit that arises in the process due to the fact that the risk for the merged group is essentially reduced. There emerges a profit and there are various ways of splitting this profit between the combined groups. Techniques from game theory, in particular cooperative game theory turn out to be useful in splitting of the profit. The authors proceed in this paper to apply techniques of utility theory to study the possibility of a fair split of that profit. In this research, the authors consider a group of n parties $1, \dots, n$ such that each of them has a corresponding utility function $u_1(x), \dots, u_n(x)$. Given a positive amount of money C , a fair split of C is a vector $(c_1, \dots, c_n)$ in R^n , such that $c_1 + \dots + c_n = C$ and $u_1(c_1) = u_2(c_2) = \dots = u_n(c_n)$. The authors presume the utility functions to be normalized, that is $u_i(c) = 1$ for each party i , $i = 1, \dots, n$. The authors show that a fair split exists and is unique for any given set of utility functions $u_1(x), \dots, u_n(x)$, and for any given amount of money C . The existence theorem follows from observing simplexes. The uniqueness follows from the utility functions being strictly increasing. An example is given of normalizing some utility functions, and evaluating the fair split in special cases. In this article, the authors study the case of merging two groups (or more) of insured members, they provide an evaluation of the emerging benefit in the process, and the splitting of the benefit between the groups.

Keywords

utility function, risk, cooperative game, simplexes

JEL Classification C51, C57

INTRODUCTION

In the process of merging insured groups, there emerges a profit that arises in the process due to the fact that the risk for the merged groups is reduced. There arise various ways to split the emerging profit between the merging groups. Techniques from game theory, in particular cooperative games, will be applied to consider various ways to split the emerging profit. The authors apply techniques of utility theory to investigate various possible ways to split this profit.

The authors consider a group of n parties $1, \dots, n$ each of which has a corresponding utility function denoted by $u_1(x), \dots, u_n(x)$.

Given a positive amount of money C , the authors define a fair split of C to be a vector $(c_1, \dots, c_n)$ in R^n , such that $c_1 + \dots + c_n = C$ and $u_1(c_1) = u_2(c_2) = \dots = u_n(c_n)$.

The authors presume that all utility functions are normalized by $u_i(C) = 1$ for each party i , $i = 1, \dots, n$.

The authors show that a fair split exists and is unique for any given set of utility functions $u_1(x), \dots, u_n(x)$, and for any given amount of money C .

For the rest a utility function is a real function that is continuous and that is strictly increasing, and maps zero to zero. The positive real

numbers (including zero) are denoted by R_+ . Given a positive amount of money C , a *split* of C is a vector $(c_1, \dots, c_n) \in R_+^n$ so that $c_1 + \dots + c_n = C$.

Given n utility functions, $u_1(x), \dots, u_n(x)$, a *fair split* of C is a split for which the following equalities hold: $u_1(c_1) = u_2(c_2) = \dots = u_n(c_n)$.

The authors' first task is to show that a fair split exists and is unique, given the utility functions and the amount of money to split. The existence theorem (theorem 2) uses notations of simplexes and the following theorem from [KKM] with the notations and remarks:

For a finite set $S = \{x_1, \dots, x_m\} \subseteq R^n$, denote the *convex hull* of S as:

$$\text{Conv}(S) = \left\{ \sum_{i=1}^m \lambda_i x_i \mid \sum_{i=1}^m \lambda_i = 1 \text{ and for all } i \lambda_i \geq 0 \right\}.$$

Using the standard basis for R^n , $\{e_1, \dots, e_n\}$, the *standard simplex* $\Delta_{n:M}$ is the convex hull of $S = \{Me_1, \dots, Me_n\}$ and a *face* of $\Delta_{n:M}$ is the convex hull of a subset of S . The faces of $\Delta_{n:M}$ are denoted by $\Delta_{n:M}(T)$ where T is a subset of $\{1, 2, \dots, n\}$. In particular $\Delta_{n:M}(T)$ is the convex hull of $\{Me_i \mid i \in T\}$.

The proof of theorem 2.2 uses the famous theorem:

Theorem [KKM]: Let $A_1, \dots, A_n$ be closed subsets of $\Delta_{n:M}$. If for all $T \subseteq \{1, 2, \dots, n\}$ the following holds: $\Delta_{n:M}(T) \subseteq \bigcup_{i \in T} A_i$ then $\bigcap_{i=1}^n A_i \neq \emptyset$.

1. MAIN RESULTS

In the rest the authors presume n utility functions and a positive amount of money C .

The main theorem: There exists a unique fair split for the n utility functions and for the given amount of money C , that is there exists a unique vector $(c_1, \dots, c_n) \in R_+^n$ such that $u_i(c_i) = u_j(c_j)$ for all pairs i, j so that $1 \leq i < j \leq n$ so that the following equality holds $c_1 + \dots + c_n = C$.

Proof: There are two claims, the first is that of the existence of a fair split and the second is that of the uniqueness of the fair split for the n given utility functions and the given amount of money C . For the existence the authors need to provide a vector $(c_1, \dots, c_n) \in R_+^n$ such that $u_i(c_i) = u_j(c_j)$ for all pairs i, j so that $1 \leq i < j \leq n$ and so that the equality $c_1 + \dots + c_n = C$ holds.

After proving the existence of a fair split the authors will show its uniqueness.

The authors suggest a more general result than the sole existence one.

Theorem 2.2: There exists a fair split for n real functions $u_1(c), \dots, u_n(c)$ that are continuous and $u_i(x) > 0$ for all $x > 0$ for $i = 1, \dots, n$, that map zero to zero. That is, for a given amount of money C , there exists a vector $(c_1, \dots, c_n) \in R_+^n$ such that $u_i(c_i) = u_j(c_j)$ for all pairs i, j so that $1 \leq i < j \leq n$ and so that the following equality holds $c_1 + \dots + c_n = C$.

Notice that we omitted the assumption that the functions are strictly increasing, hence the result that a fair split exist will follow from the following proof of theorem 2.

The proof of theorem 2.2 which implies the proof of the existence of a fair split: Let $u_1, \dots, u_n$ be n real functions that are continuous, map zero to zero and such that $u_i(x) > 0$ for all $x > 0$ for all $i = 1, \dots, n$. For every positive C there is a vector $(c_1, \dots, c_n) \in R_+^n$ such that $u_i(c_i) = u_j(c_j)$ for all i, j and $c_1 + \dots + c_n = C$.

The authors emphasize that from theorem 2.2 the existence of a fair split follows since the utility functions are increasing, map zero to zero and therefore are positive for positive values. Nevertheless, the strict monotonicity of the utility functions is not used in the proof. In fact, there might be general cases, which are out of our context, when strict monotonicity or even monotonicity cannot be assumed. Given a positive amount of money C , the set of *splits* of C is the standard simplex $\Delta_{n:C}$, therefore a fair split is a vector in $\Delta_{n:C}$. The proof the authors give here characterizes the set of fair splits as an intersection of specific subsets of $\Delta_{n:C}$, and this intersection turns to be non-empty by using theorem 1. Proof of theorem 2. For $1 \leq i \leq n$ denote the following closed non-empty subsets of $\Delta_{n:C}$:

$$A_i = \{(x_1, \dots, x_n) \in \Delta_{n:C} \mid u_i(x_i) \geq u_j(x_j) \text{ for all } j\}.$$

A fair split $(c_1, \dots, c_n)$ of C is a vector in $\Delta_{n:C}$ such that $u_i(x_i) = u_j(x_j)$ for all i, j .

Therefore fair splits are exactly the vectors which belong to the intersection of the sets $A_1, \dots, A_n$. Hence, we prove that this intersection is not empty and we do so by showing that the hypothesis of theorem 1 holds. Let T be a subset of $\{1, 2, \dots, n\}$ and let $v = (x_1, x_2, \dots, x_n)$ be a vector in the face $F = \Delta_{n:C}(T)$. The authors show that v is contained in $\bigcup_{i \in T} A_i$ and hence show that the hypothesis holds. Denote the index k for which $u_k(x_k)$ is maximal among $u_1(x_1), \dots, u_n(x_n)$. Then $v \in A_k$. For indexes $j \notin T$ we have $x_j = 0$ thus $u_j(x_j) = u_j(0) = 0$. For indexes $i \in T$ we have $x_i \geq 0$ and there is an index $r \in T$ such that $x_r > 0$. For this index $u_r(x_r) > 0$. Therefore, the index k (where the maximum is attained) is in T and the following holds $v \in A_k \subseteq \bigcup_{i \in T} A_i$ and therefore $F \subseteq \bigcup_{i \in T} A_i$.

Next the authors turn to the uniqueness that follows from the utility functions being strictly increasing, which is a monotonicity property. This property turns out to be crucial in deriving the proof of the uniqueness of a fair split.

Proof of the uniqueness of a fair split: Using the fact that a utility function is strictly increasing, we will show that this implies that a fair split is unique.

Let $v = (c_1, c_2, \dots, c_n) \in \Delta_{n:C}$ be a fair split and let $w = (x_1, x_2, \dots, x_n)$ be any other vector in $\Delta_{n:C}$, then there follows that w is not a fair split: since $w \neq v$ there is an index i for which $x_i \neq c_i$, and the authors assume without loss of generality that $x_i < c_i$. Since the utility functions are strictly increasing this implies that $u_i(x_i) < u_i(c_i)$. The sum of coordinates of both vectors is C and therefore the following inequality holds:

$$\sum_{j \neq i} x_j > \sum_{j \neq i} c_j.$$

It follows that there exists an index k , $k \neq i$ for which the inequality $x_k > c_k$ holds. The utility functions being strictly increasing imply that $u_k(c_k) < u_k(x_k)$ and hence:

$$u_i(x_i) < u_i(c_i) = u_k(c_k) < u_k(x_k).$$

In particular, there results the inequality $u_i(x_i) \neq u_k(x_k)$ so w is not a fair split.

The authors are greatfull to our colleague Professor Baruch Granovski who suggested a different approach to prove that under the assumptions of theorem 2, and considering the normalized $u_i(c_i)$, that is $u_i(C) = 1$ for all i , $i = 1, \dots, n$.

Professor Baruch Granovski derives a different proof of theorems 2 as follows:

Alternative proof suggested by Professor Baruch Granovski: Let u_i , $i = 1, \dots, n$ be strictly increasing and continuous functions mapping $[0, C]$ to $[0, 1]$. Then there is a vector $(x_1, \dots, x_n) \in \Delta_{n:C}$ such that $u_i(x_i) = u_j(x_j)$ for $1 \leq i, j \leq n$.

Since u_i are bijective the inverse functions u_i^{-1} are also continuous and strictly increasing. Also, $u_i^{-1}(0) = 0$ and $u_i^{-1}(1) = C$.

Define the function

$$g(x) = u_1^{-1}(x) + \dots + u_n^{-1}(x).$$

Then, g is continuous, $g(0) = 0$ and $g(1) = nC$. Hence, by the mean value theorem there is a $0 \leq z \leq 1$ such that

$$g(z) = u_1^{-1}(z) + \dots + u_n^{-1}(z) = C.$$

Consequently, Professor Baruch Granovski defines $x_i = u_i^{-1}(z)$ then $(x_1, \dots, x_n)$ to be in Δ_{nC} and $u_i(x_i) = z$ for all $i = 1, \dots, n$ which proves the theorem as stated.

2. EXAMPLE

Assume that three persons join in a common interest to split between them in a fair way the sum of C ,

$$C = 148,847.86\$$$

Each of the three persons holds a personal utility function.

The first one have the utility function $u_1(x) = \ln(1+x)$, the second one have the utility function $u_2(x) = \sqrt{x}$, and the third one have the utility function $u_3(x) = x$.

One easily verifies that $u_1(x)$, $u_2(x)$, $u_3(x)$ are all utility functions.

In order to reach a fair split the authors normalize the three utility functions so that $a_1 u_1(x)$, $a_2 u_2(x)$, $a_3 u_3(x)$ will be the normalized utility function respectively so that the value for each person has the same value at their maximal value, say the value 1. That is the following will hold for each of the normalized utility functions $u_i(c)$,

where $u_i(c) = a_i u_i(C) = 1$ for $i = 1, 2, 3$.

After the normalization is achieved for the three persons in our case the authors have the three utility functions in their normalized forms:

$$\text{For the first person } u_1(x) = \frac{\ln(1+x)}{\ln(1+C)},$$

$$\text{for the second person } u_2(x) = \frac{\sqrt{x}}{\sqrt{C}},$$

$$\text{and for the third person } u_3(x) = \frac{x}{C}.$$

From theorem 2.2 a fair split exists, that is there exists a vector (c_1, c_2, c_3) with positive entries c_i , $c_i > 0$ for $i = 1, 2, 3$ that satisfy the following equations:

$$u_1(c_1) = u_2(c_2) = u_3(c_3),$$

$$c_1 + c_2 + c_3 = C = 148,847.86,$$

$$\frac{\ln(1+c_1)}{1+C} = \frac{\sqrt{c_2}}{\sqrt{C}} = \frac{c_3}{C}.$$

We use MATLAB to get the solution:

$$c_1 = 1490.82,$$

$$c_2 = 56,032.06,$$

$$c_3 = 91,324.98.$$

CONCLUSION

The ideas and the method as described in this article provide a model for an insurance company to bid for an insurance portfolio of a group of insured members, in which case one expects the emergence of benefit that results from adding the group's and the company's portfolio, and one may derive options to improve the winning prospect of the company's offer for the bid.

This article also suggests how to study the case of merging of insurance portfolios, as in the case of one insurance company bidding to buy another insurance company, or two insurance companies that consider merging to one, or two insurance companies that consider to manage jointly their portfolios. The article discusses options of evaluating the emerging benefit in the process of joining the two insurance portfolios to one. The results may provide a range of offers to bid that may improve the winning prospects, or of how to split the emerging benefit between two merging companies.

In a similar way this article provides ideas for a model for two groups of insured members, each with an insurance portfolio, that consider merging their portfolios. The merger is reasonable if there re-

sults a benefit in the process. Here the authors consider evaluations of emerging benefit and how to split it between the two groups.

Notice that the ideas and the results of this article may turn useful to split its overall expenses (e.g. administrative load, water and electricity expenses, various taxes etc.) in a company fairly between its several departments by charging overhead for each of its various departments to fairly determine the part of each of the various departments of the company in the overall expenses of the company. One starts by considering the overall expenses for each department separately, then one derives the emerging saving of the overall expenses for all departments when operating united under the company within a single administration. Finally, one uses the results of this article and the ideas as presented in it to split the emerging saving to the various departments.

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