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Parametric analysis of the Bass model

Abstract

In this research, the authors explore the influence of the Bass model p, q parameters values on diffusion patterns and map p, q Euclidean space regions accordingly. The boundaries of four different sub-regions are classified and defined, in the region where both p, q are positive, according to the number of inflection point and peak of the non-cumulative sales curve. The researchers extend the p, q range beyond the common positive value restriction to regions where either p or q is negative. The case of negative p, which represents barriers to initial adoption, leads us to redefine the motivation for seeding, where seeding is essential to start the market rather than just for accelerating the diffusion. The case of negative q, caused by a declining motivation to adopt as the number of adopters increases, leads us to cases where the saturation of the market is at partial coverage rather than the usual full coverage at the long run. The authors develop a solution to the special case of $p + q = 0$, where the Bass solution cannot be used. Some differences are highlighted between the discrete time and continuous time flavors of the Bass model and the implication on the mapping. The distortion is presented, caused by the transition between continuous and discrete time forms, as a function of p, q values in the various regions.

Keywords: Bass model, mapping, diffusion patterns, discrete time, continuous time, seeding.
JEL Classification: M3.

Introduction

The theory of diffusion of innovation, proposed by Bass (1969), has been explored, implemented and extended by numerous researches. Some researchers extend the model to explicitly consider market or industry characteristics or behavior. Others examine implementation of the model on various cases and show that adding more factors improves the capability of the model to capture complex behavior in details. Another direction taken by some researchers was to explore the factors that influence parameters values estimation and forecasts accuracy. Some researches explore empirically how the Bass model parameters vary across products and markets, usually at the ranges when $p << q$ and $0.1 < q < 0.7$. Bass (1969) notes that there are two different categories of diffusion curve patterns. When $q > p$, the periodic sales grow until they reach a peak and, then, decline asymptotically to zero. When $q < p$, periodic sales decline asymptotically to zero starting from launch. Little attention has been directed since then to a further comprehensive exploration of the constraints and classification of the Bass model parameters and how they affect the adoption curve patterns.

Another issue that has attracted little attention in diffusion theory is the transition between the continuous time form and the discrete time form of the Bass model. While many researchers switch between these two flavors without further notice, attention needs to be paid to verifying, for each implementation, that this transition has little impact on parameters' values and on forecasts' accuracy.

In this paper, we perform a comprehensive exploration and classify and map the different diffusion patterns over the p, q space. The exploration is theoretic and is not limited to specific empiric cases. We refer to the different flavors (continuous vs. discrete) and highlight the differences in mapping, constraints and classification. We also define the conditions that need to be checked when switching between them.

1. Literature review

The diffusion of innovation model, known as the Bass model, has several flavors and numerous extensions. In this paper we focus on the basic model of Bass (1969), which has a simple analytic solution and not to extensions that are usually solved numerically. We brief the different basic models and highlight the differences between them. We use these models for mapping the patterns and constraints categories over the p, q space and show that each flavor has a slightly different map.

The Bass (1969) presents the diffusion of innovation dynamic equation:

$$f(t) = \frac{dF(t)}{dt} = \left(p + q \cdot F(t)\right) \cdot \left(1 - F(t)\right). \quad (1)$$

This equation represents the innovation and imitation influence on the remaining potential market. The periodic sales, or the rate of cumulative sales change, which is the derivative of the cumulative adoption, are proportional to the multiplication of the remaining market by the sum of innovation and imitation influence. The analytic continuous time solution that Bass presents to his equation is:

$$F(t) = \frac{1 - e^{-(p+q)\cdot t}}{1 + \frac{q}{p} e^{-(p+q)\cdot t}}. \quad (2)$$

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The cumulative sales function, as well as its derivative or periodic sales function, can be outlined as a graph where time is the X axis and sales, or sales rate, are the Y axis. The shapes of both curves, the cumulative and periodic sales, are determined by the values of the model parameters p and q. Srinivasan and Mason (1986) note that both p and q must be non-negative. Acemoglu and Ozdaglar (2009) note that the levels of p and q scale time, while the ratio q/p determines the overall shape of the curve.

Bass (1969) refers to the effect of the q/p ratio or q, p phase and differentiates between two categories. He notes that when $q/p > 1$, i.e., p, q phase = 45°, the product is successful and sales experience growth and then decline due to saturation. When $q/p < 1$, which represents an unsuccessful product, sales will start at a certain level and keep declining. Bass (1969) also presents the influence of $(p + q)$ values, between 0.3 and 0.9, on the growth rate. Sultan et al. (1990) performed a meta-analysis of 213 applications of diffusion models from 15 articles published between 1950 and 1980. They compare several parameters estimation methods (OLS, MLE, Bayesian and non-linear least square) and also how the number of sampling points influences accuracy. They found that the average p value is 0.03, while the average q value is 0.38. Van den Bulte (2002) explored how p and q vary across products and countries, based on a database containing 1586 sets of p and q parameters, from 113 papers published between January 1969 and May 2000. He explains that the parameters p and q provide information about the speed of diffusion. A high value for p indicates that the diffusion has a quick start, but also tapers off quickly. A high value of q indicates that the diffusion is slow at first, but accelerates after a while. He also notes that, when q is larger than p, the cumulative number of adopters follows the type of S-curve often observed for radically innovative product categories. When q is smaller than p, the cumulative number of adopters follows an inverse J-curve often observed for less risky innovations such as new grocery items, movies, and music CDs.

Lilien et al. (2000) formulate a discrete difference equation to model diffusion of innovation, used by many previous and later of the Bass model extensions.

$$\begin{aligned} f_d(n) &= \Delta F_d(n) = F_d(n+1) - F_d(n) = \\ &= X(n)\cdot\left(p_d + q_d \cdot F_d(n)\right)\cdot\left(1 - F_d(n)\right), \end{aligned} \tag{3}$$

where $F_d(n)$ is the cumulative adoption and $f_d(n)$ is the periodic sales at period n.

We use the notation p_d and q_d for the discrete time model parameters to distinguish them from the p, q parameters of the continuous time form. The $X(n)$ function can capture many factors the original Bass model ignores such as advertisement of price changes. When $X(n) = 1$, it converges back to the original Bass model. For example, Bass et al. (1994) propose including the influence of advertisement of price changes by using:

$$x(n) = 1 + \alpha\frac{\Pr(n) - \Pr(n-1)}{\Pr(n-1)} + \beta\cdot Max\left\{0, \frac{A(n) - A(n-1)}{A(n-1)}\right\}. \tag{4}$$

When α coefficient captures the percentage increase in diffusion speed resulting from a 1% decrease in price, $Pr(n)$ is the price in period n, β coefficient captures the percentage increase in diffusion speed resulting from a 1% increase in advertising and $A(n)$ is advertising in period n. The main advantage of the discrete model is that it can be solved numerically, in cases where there is no analytic solution. While Bass (1969) presents an analytic general solution for the continuous time differential equation (1), neither he nor others propose an analytic solution for the discrete time difference equation (3). Bass (1969) does refer implicitly to the discrete model by providing an insight that the likelihood of a purchase at time t, $P(n)$ is calculated by:

$$P(n) = \frac{f_d(n)}{1 - F_d(n)} = p_d + q_d \cdot F_d(n). \tag{5}$$

While many researchers switch between the discrete model, based on a difference equation (3), and the continuous time model, based on the differential equation (1), the transition is not trivial. Van den Bulte and Lilien (1997) show that OLS or NLS estimations of the continuous Bass model parameters using discrete time data are biased and that they change systematically as one extends the number of observations. An analytic discrete time solution for the Bass continuous time differential equation (1), developed by Satoh (2001), is:

$$F(n) = \frac{1 - \left(\dfrac{1 - (p_s + q_s)}{1 + (p_s + q_s)}\right)^{\frac{n}{2}}}{1 + \dfrac{q_s}{p_s}\left(\dfrac{1 - (p_s + q_s)}{1 + (p_s + q_s)}\right)^{\frac{n}{2}}}. \tag{6}$$

However, this solution does not solve the difference equation (3). Satoh (2001) notes that the relation between his discrete time Bass model parameters p_s and q_s and the corresponding continuous time Bass model parameters p and q is:

$$k = \frac{1}{p+q}\cdot\frac{1 - e^{-2(p+q)}}{1 + e^{-2(p+q)}}\ ;\ \begin{aligned} p_s &= k\cdot p \\ q_s &= k\cdot q \end{aligned}. \tag{7}$$

Table 1 compares the Bass solution (2) for typical values $p = 0.01$, $q = 0.3$, Satoh solution (6) for the corresponding parameters values $p_S = 0.0097$, $q_S =$

0.2907 and a numeric solution for the Lilien equation (3) that has minimal RMSE with them with p_d = 0.0138, q_d = 0.2865. While the parameters values and forecasts of all three flavors are very close, we still notice that, while Satoh solutions perfectly match Bass, Lilien solution is very close but not identical.

Table 1. A comparison between Bass, Satoh and Lilien solution

t	Bass *F*(t)	Satoh *F*(*n*)	Lilien $F_d(n)$
0	0	0	0
1	0.011588	0.011588	0.013808
2	0.026960	0.026960	0.031327
3	0.047166	0.047166	0.053396
4	0.073399	0.073399	0.080948
5	0.106923	0.106923	0.114953
6	0.148907	0.148907	0.156323
7	0.200171	0.200171	0.205758
8	0.260871	0.260871	0.263546
9	0.330179	0.330179	0.329323
10	0.406107	0.406107	0.401864
11	0.485607	0.485607	0.478990
12	0.565000	0.565000	0.557684
13	0.640625	0.640625	0.634465
14	0.709486	0.709486	0.705958
15	0.769662	0.769662	0.769491
16	0.820388	0.820388	0.823493
17	0.861864	0.861864	0.867574
18	0.894941	0.894941	0.902319
19	0.920798	0.920798	0.928920
20	0.940698	0.940698	0.948819

The cumulative sales of Satoh (2001) from (6) perfectly match the continuous time Bass sales solution from (2) for $t = 0, 1, 2\ldots n$. Satoh (2001) compares several methods for estimating the discrete time Bass model parameters for nine data sets, and concludes that, for most cases, when the time interval is small enough, the exact solution of the discrete Bass model provides a very good approximation of the solution of the conventional Bass model. He also proves that, when using the transformation (7) to p and q, a solution of the discrete Bass model (6) provides identical values to the solution of the continuous model (2). For two cases (out of nine) where the estimated value of the p parameter is negative, he notes that the wrong sign indicates that the data are not appropriate for the Bass model.

2. Mapping the Bass model parameters

We perform the mapping separately for each flavor of the Bass model. First, we develop a formula of the periodic sales inflection points' times, for the continuous time form, and classify the diffusion patterns according to the number of inflection point and whether there is peak. Second, we map each category classified to a sub-region of the positive p and q values of the Euclidean space. Then, we extend the map to include regions where either p or q is negative, which add some insight about seeding and about market saturation. For the discrete time model, equation (3), which is more restricted, we remap the p, q space. The map is similar but has additional constraints about the absolute values of p, q and their sum. We redo the mapping for the Satoh model which is a discrete time solution (6, 7) for the continuous time mode of equation (1, 3).

2.1. Mapping the Bass model parameters ratio space. Bass (1969) and Van den Bulte (2002) distinguish between cases where $q > p$, and periodic sales have a peak after launch, to cases where $q < p$ and periodic sales keep declining since launch. The periodic sales peak time, as calculated by Bass (1969), is:

$$t^* = \left[\frac{1}{p+q}\right]\ln\left(\frac{q}{p}\right). \tag{8}$$

From (8), we see that when $q > p$, the peak time is positive. When $q < p$, peak time is negative, thus, sales keep declining since launch time (t = 0). There are no sales before product launch.

The inflection points' times (see Appendix A) are calculated as:

$$t^{**} = \frac{\ln\left(\frac{q}{p}\right) - \ln(2 \pm \sqrt{3})}{p+q} = t^* \pm \frac{\ln(2+\sqrt{3})}{p+q} \tag{9}$$

Result 1: the time between the inflection points and between them to peak depends only on (p + q) and not on (p/q).

Since sales begin at $t = 0$, when the inflection points or peak times are negative, the periodic, or non-cumulative, sales curve does not include them. Figure 1 presents the p, q space four regions (A to D) and periodic sales curves. We use the approximation $2+\sqrt{3}=3.73$ for convenience.

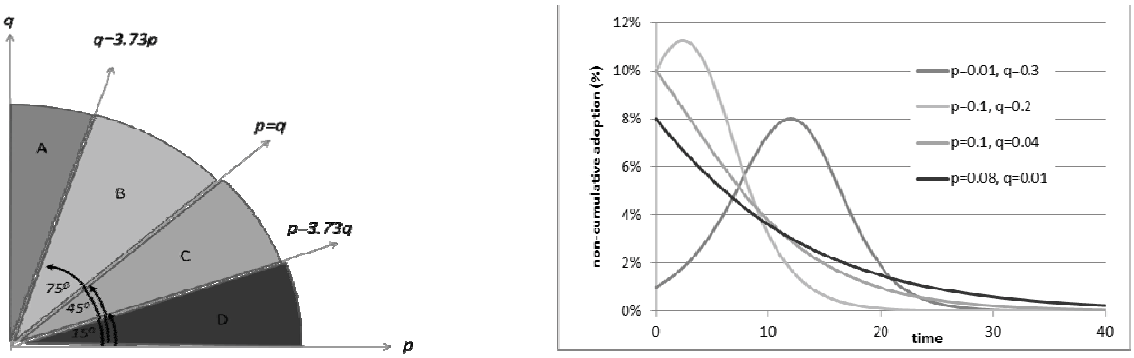


Fig. 1. p, q space four regions and periodic sales curves

The parameters p and q are usually considered to be constrained to positive values. In some researches (i.e., Srinivasan and Mason, 1986; Rafi and Akthar, 2011), p or q (but not both) may be 0. Bass (1969) states explicitly that, when estimating the value of q through regression of other methods, it must be positive in order for the model to make sense. Jiang et al. (2006) mention that non-positive values of p are not plausible.

Chandrasekaran and Telli's (2008) empirical study, mentioned by Boyle (2010), summarizes many diffusions of innovation cases and concludes that the mean value of the coefficient of innovation, p, for a new product lies globally between and 0.03 and there are differences between markets. The mean value of the coefficient of innovation for a new product is 0.001 for developed countries and 0.0003 for developing countries The coefficient of innovation, q, is higher for European countries than for the United States The mean value of the coefficient of imitation for a new product lies globally between 0.38 and 0.53 and there are differences between developed countries and developing economies, as well as difference between industrial and consumer markets. Industrial/medical innovations have a higher coefficient of imitation than consumer durables and other innovations The mean value of the coefficient of imitation for a new product is 0.51 for developed countries and 0.56 for developing countries. In most of the empiric researches (i.e., Bass, 1969; Sultan et al., 1990; Chandrasekaran and Tellis, 2008), almost all products, usually durables, reside in region A. Ainslie et al. (2005) research new movies diffusion. They distinguish between Blockbuster-type movies with average $p = 0.38$, $q = 0.044$, which, according to our mapping, reside in region D, and Sleeper-type movies with average $p = 0.155$, $q = 0.474$, which, according to our mapping, reside in region B.

2.2. Can the innovation coefficient value be zero or negative? At a first look it seems impossible since diffusion cannot start. Srinivasan and Mason (1986) claim p must be positive and q must be non-negative. Rafi and Akthar (2011) allow also that $p = 0$. They explain that there are two special cases of the Bass diffusion model. (a) When $q = 0$, the Bass model reduces to the Exponential distribution, (b) when $p = 0$, the Bass model reduces to a special case of the Gamma/shifted Gompertz distribution. In fact, when using seeding, as in Jain et al. (1995), diffusion can start when $p + qF(0) > 0$. If p value is negative, but a marketer uses seeding strategy with seed size of $F(0) > |p|/q$, then, diffusion can mathematically start. The interpretation of a negative p value does not necessarily mean that the product is useless. There can be cases when there are price or effort barriers to adoption when very few others have already adopted. However, when others adopt the benefits from the product increase, due to externalities or uncertainty reduction, the product becomes more and more plausible for many potential customers. Katz and Shapiro (1985) mention that there are barriers to foreign automobile manufacturers, where sales are often retarded. These barriers are due to reluctance of customers and their concern regarding less experience and thinner service for new or less popular brands. As the brand gains popularity, the barriers are lowered and the drive to adopt it increases. Oren and Smith (1981) claim that a major problem facing a producer interested in introducing a good network is the ability to attain a critical mass. Once the critical mass is reached, the initial structural inertia is overcome and the network experiences growth. They propose a method for finding the optimal price (for profit maximization) and calculating the critical mass, or seed size, required to start the market. Their method is based on the demand curve, considering networks effect and cost structure. Farrell and Saloner (1986) and Economides (1996) applied and tested this method on several empiric cases. In such cases, seeding is used not only for accelerating diffusion, as in Jain et al. (1995), but is a precondition for starting the diffusion. Interestingly, Jain et al. (1995) do not re-

fer to such cases and check seeding effect only for positive values of p and q. The cases where seeding can overcome the barriers to adoption, expressed by the negative value of p, are when $q > |p|$. When this condition is not fulfilled, the barriers to adoption cannot be overcome by seeding. Note that, when the interest rate is low, diffusion acceleration does not justify the expenses and sales loss of seeding. With negative p that can be overcome with seeding, the justification for it is inherent. When interest rate is low and there is no incentive for acceleration, we will keep the seeding size close to its minimal limit ($F_0 \approx |p|/q$). Note that, when the ratio $|p|/q$ is higher, the seed that is required to start the market is larger.

Cumulative adoption and peak time with seeding, according to Jain et al. (1995), are applied (only when q value is positive) also for negative p and equal:

$$F(t) = \frac{1 - \frac{p(1-F_0)}{p+qF_0} e^{-(p+q)t}}{1 + \frac{q(1-F_0)}{p+qF_0} e^{-(p+q)t}};$$

$$t^* = \frac{1}{p+q} \cdot \ln\left[\frac{q(1-F_0)}{p+qF_0}\right]. \quad (10)$$

Without seeding, when $F_0 = 0$, equation (10) reduces back to (2, 8). With negative p and when the seed size is close to its minimal limit ($F_0 \approx |p|/q$), we always have a periodic sales peak. Furthermore, since (9) applies also for negative p, the condition for having two inflection points is:

$$\frac{q(1-F_0)}{p+qF_0} > 2 + \sqrt{3}. \quad (11)$$

Condition (11) is always fulfilled when seed size is close to its minimal limit $|p|/q$ for any q/p ratio.

Figure 2 presents two curves of negative p and two curves of their corresponding (absolute value) positive p. While the positive p curves may have a single or two inflection points, its corresponding curve has two inflection points in both cases. Note that, when p value is negative, seeding is essential and the diffusion is much slower. Due to the requirement that the seed $F_0 \approx |p|/q$ is less than 100% of the potential market, the corresponding positive p diffusion curve belongs to either region A or B and there is always a periodic sales peak.

Result 2: The assumption of positive p can be released. In such cases, seeding of at least $|p|/q$ is essential for starting the market. It represents a market where initial critical mass is required to start the diffusion.

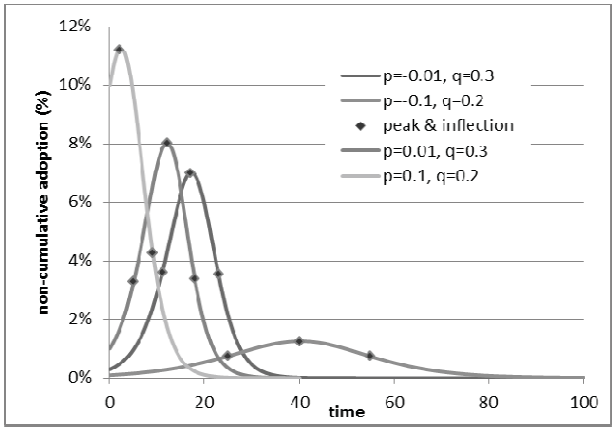
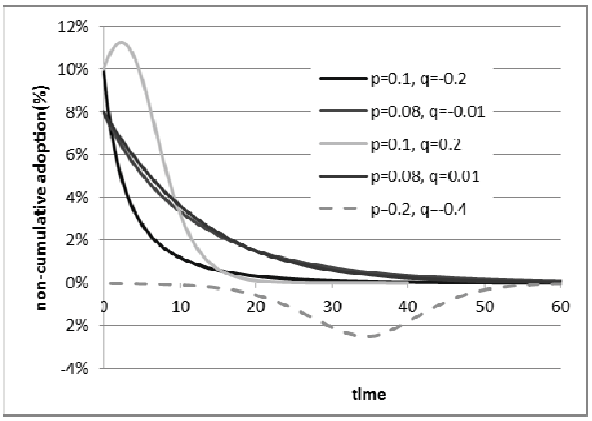


Fig. 2. Comparing positive and negative p values

Another question about the p, q parameters range is whether q can be negative. Moldovan and Goldenberg (2004) incorporate negative word-of-mouth (WOM) effect on diffusion models. However, they refer to a mix of both positive and negative WOM of adoption support and resistance with an overall positive effect. We claim that diffusion can progress even with overall negative q value. A negative q does not necessarily mean that adopters are disappointed and unpleased with their purchase. It can fit a case where the benefit from a product declines as more people adopt. For example, for a certain demand level for train commute, reserved tickets may be sold to those who would like to guarantee a seat. Those who do not buy a reserved ticked may have to commute while standing. As more reserved tickets are sold, the overload in the non-reserved train car is reduced, and the likelihood to find a free seat at the non-reserved train car increases, thus, reducing the incentive to buy a reserved ticket. Another example is taking a training course that qualifies PC technicians or mobile application programmers or a foreign language translation. When there are few PC technicians or other rare skill, their salaries are high. As knowledge becomes more common, the worth, and the salaries to those who have the required skill, decline. Negative externalities effects, where more users make a product less valuable, are mentioned by Naglerw (2011) who explains that they are usually caused by congestion that occurs due to overuse. While non-cumulative sales curve with negative q is similar to those with $q = 0$, the cumulative presents a more interesting feature. When $p > |q|$, the market will reach 100% of its potential at the long range, as for a regular positive value of q. However, if $p < |q|$, the long range forecast for cumulative sales, based on (2), is $F(t) \xrightarrow[t\to\infty]{} p/|q|$, which means that the market will saturate at an equilibrium level $p/|q|$ of its potential. Figure 3 presents both cumula-

tive and non-cumulative sales curve of cases with positive and negative q values. The effect of the $p/|q|$ ratio is presented too. The negative q creates a damping effect that contradicts the drive to adopt. The damping effect strengthens as cumulative adoption increases and, when $p = F(t) \cdot |q|$, the drive to adopt is balanced by the q damping effect. The saturation level is a stable equilibrium and if the cumulative adoption starts at a higher level, or even 100%, it will decline (i.e., negative non-cumulative adoption) to $p/|q|$. This is demonstrated by the broken line curves in Figure 3.

Non-cumulative adoption



Cumulative adoption

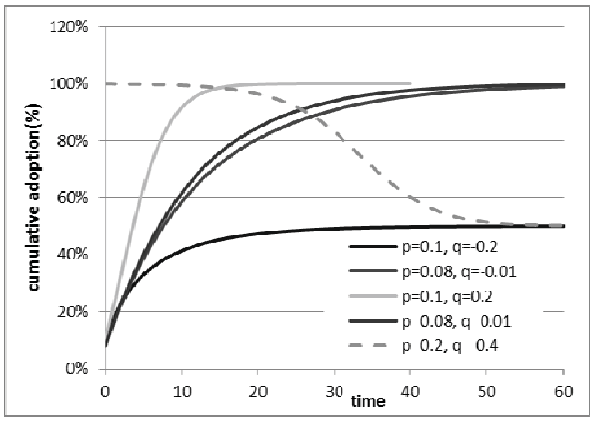


Fig. 3. Comparing cumulative and non-cumulative adoption with positive and negative *q*

Result 3: The assumption of positive *q* can be released when *p* is positive. At both, the periodic sales will experience steady decline, while, cumulative adoption asymptotically approaches its equilibrium. The equilibrium may be 100% of the potential market, as in a "regular" diffusion, when $p > |q|$. The equilibrium is $p/|q|$, which is a portion of the potential market, when $p < |q|$.

When $p + q = 0$, Bass solution (2) is undefined, although the discrete time (3) can still be applied. For this case, the Bass equation is:

$$f(t) = \frac{dF(t)}{dt} = p \cdot \left(1 - F(t)\right)^2. \tag{12}$$

The solution for this case is:

$$F(t) = 1 - \frac{1}{p \cdot t + 1}. \tag{13}$$

A summary map of the combinations we discussed is presented in Figure 4.

It presents the classic (positive p and q) Bass model $p - q$ space regions, but in a greater details (four region rather than two). The map also presents regions with negative p or q values that were considered to be out-of-scope in previous researches.

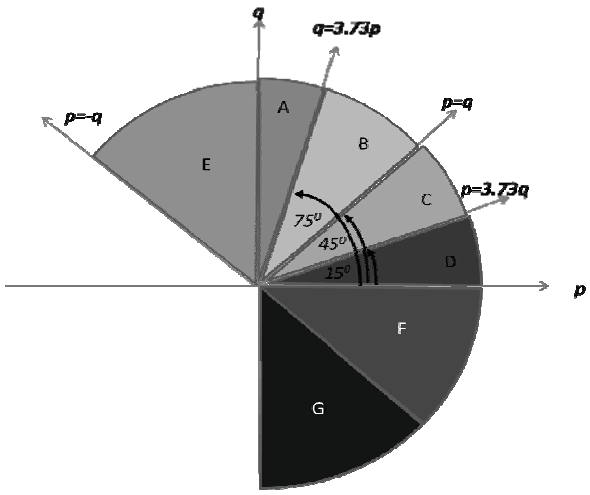


Fig. 4. Extended *p – q* space map

We summarize all seven regions of *q/p* that determine the existence of these points on the non-cumulative sales in Table 2.

Table 3 presents examples for the values of p and q at each region and whether there are peak and inflection points, as well as a need for seeding and saturation level.

Table 2. The seven regions of *q/p*, boundaries, peak, inflection points and equilibrium

Region	Lower bound of *q/p*	Upper bound of *q/p*	Lower phase of *q/p*	Upper phase of *q/p*	Number of inflection points	Peak exists	Equilibrium (saturation)
A	$2+\sqrt{3}$		75º	90º	2	Yes	100%
B	1	$2+\sqrt{3}$	45º	75º	1	Yes	100%
C	$2-\sqrt{3}$	1	15º	45º	1	No	100%
D	0	$2-\sqrt{3}$	0º	15º	0	No	100%

Table 2 (cont.). The seven regions of *q/p*, boundaries, peak, inflection points and equilibrium

Region	Lower bound of *q/p*	Upper bound of *q/p*	Lower phase of *q/p*	Upper phase of *q/p*	Number of inflection points	Peak exists	Equilibrium (Saturation)
E		-1	90º	135º	2	Yes	100%
F	-1	0	-45º	0º	0	No	100%
G		-1	-90º	-45º	0	No	*p/|q|*

Table 3. Examples of different parameters sets that belong to different regions of $p - q$ space and their properties

Region	A	B	C	D	E	F	G
$p - q$ phase	75°-90°	5°- °	5°- °	°- °	°- °	-45°-0°	-90°-45°
Peak	yes	yes	no	no	yes	no	no
Inflection points	2	1	1	0	2	0	0
Minimal seed	0	0	0	0	|p|/q	0	0
Saturation	100%	100%	100%	100%	100%	100%	*p/|q|*
p					-0.1		0.1
q					0.2	-	-0.2
q/p					-2	-	-2

Note that most empiric studies (Sultan et al., 1990; Lilien et al., 2000; Chandrasekaran and Tellis, 2008) found that most successful product $p - q$ values reside in region A. Still, with proper marketing actions, products at region E can succeed as well. Products in region G can succeed as well, but will be adopted only by a portion of the potential market.

3. Mapping the discrete time Bass model parameters space

When referring to the discrete time diffusion equation (3), formulated by Lilien et al. (2000), and to the likelihood of purchased (5), mentioned by Bass (1969), it adds some boundaries not only to the *p/q* ratio, as presented above, but also to parameters absolute values.

When referring to regions A to D, where p and q are non-negative, and seeding in optional, we can assume that at launch $F(0) = 0$, thus, the probability of purchase at time $n = 0$, according to (5), is $P(0) = p_d$. Since $P(n)$ represents a probability, one can imply that the value of p_d is bounded between [0:1]. After a long time the whole market adopts $F_d(n)_{n\to\infty} = 1$, thus, $P(n)_{n\to\infty} = p_d + q_d$. We can imply that the value of p_d+q_d is also bounded in the range [0:1]. Indeed, Noratikah and Ismail (2013) do mention that the values of p_d and q_d are bounded in the range of [0:1]. While the parameters p and q, of the continuous Bass model represent a rate and, thus, can vary to any positive number, the parameters of the discrete time Bass model (p_d and q_d) are bounded in the range of [0:1] and also their sum. For regions F and G, where p is positive, but q is negative, the likelihood probability at launch $P(0) = p_d$ and with no seeding $F_d(1) = p_d$, purchase likelihood , see (5), during the following period is:

$$P(1) = p_d + q_d \cdot F_d(1) = p_d + q_d \cdot p_d = p_d \cdot (1 + q_d). \qquad (14)$$

It means that, since p_d is positive and $P(1)$ is a probability, limited to [0:1], the value of q_d must be higher than –1. For region E, where p_d is negative, seeding is essential. The values of q_d, as well as $p_d + q_d$, must be positive. For region E, we do not have further limitations regarding the individual values of p_d and q_d. There may be cases where the absolute values of p_d and q_d are higher than 1, but their sum $p_d + q_d$ is between 0 and 1.

Figure 5 presents the $p_d - q_d$ space mapping of the discrete model (3). Note that it includes a sub-area of the continuous model (1) map (see Figure 4).

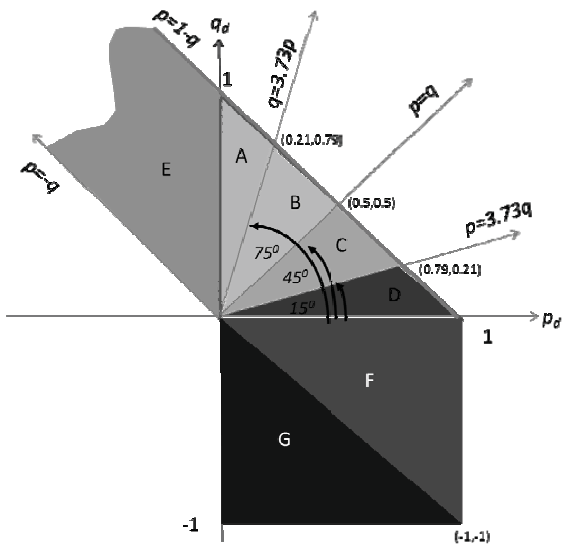


Fig. 5. Discrete time p_d - q_d space map

Result 4. Discrete time model parameters $p_d - q_d$ space is more constrained compared with the $p - q$ space of the continuous time map. Its valid areas reside in a sub-area of the continuous time map.

4. Mapping the Satoh discrete time analytic solution parameters space

The Satoh (2001) model is a discrete solution for the continuous diffusion model (1). Its solution (6) matches perfectly the Bass solution (2) when time is an integer but it still follows the probability (5) restrictions. When referring to regions A to D, where p and q are non-negative, we can directly imply from (7) that both p_s and q_s are within the range of [0:1]. For regions E to G, where p and q have opposite signs, there may be cases where the absolute values of p_s and q_s are higher than 1. Still, Satoh parameters still follow the $-1 < p_s + q_s < 1$ constraint. Figure 6 presents the relation between Satoh (2001) k and $p_s + q_s$ parameters to the Bass (1969) $p + q$. From (7) we see that the $p - q$ phase of Bass and Satoh are identical $q_s / p_s = p / q$ so it maintains the same region.

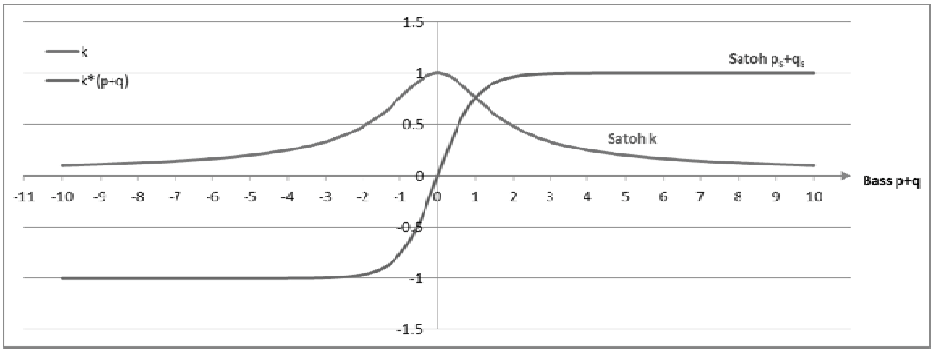


Fig. 6. Relation of Satoh discrete parameters and continuous Bass parameters

Note that there is singularity when $p_s + q_s = 0$, which applies also for $p + q = 0$ and for $p_d + q_d = 0$, and also that, for regions E to G, the individual absolute values of both p_s and q_s may be much higher than 1, while their sum is bounded by –1 and 1. Table 4 presents such three examples, where both p_s and q_s have high absolute values while their sum is 1. Note that, for region E also, the parameters of the discrete model (3) can have individual absolute value higher than 1.

Table 4. Examples of p_s, p_d and q_s, q_d that have absolute values higher than 1

Region	E	G	F
Bass p	-190	190	200
Bass q	200	-200	-190
Satoh k	0.1	0.1	0.1
Satoh p_s	-19	19	20
Satoh q_s	20	-20	-19
Lillien p_d	-2	0.95	0.99
Lillien q_d	3	-0.99	-0.95

The $p - q$ space map of the Satoh (2001) is presented in Figure 7.

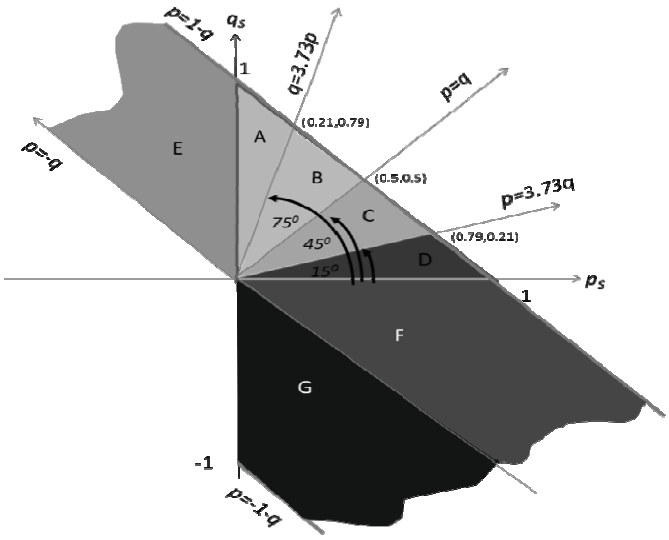


Fig. 7. The $p - q$ space map of the Satoh (2001)

Result 5. The Satoh p_S, q_S space is similar to the discrete time model parameters $p_d - q_d$ space but is less constrained compared in regions F and G.

5. Switching between continuous time and discrete time forms

Many researchers, including Bass (1969), switch between the continuous form (1) and the discrete time form (3) without an explicit notice. While, in many cases, the discrete time parameters and forecast seem to be very close to those of the continuous time, in some cases, the difference between the forecasts is significant. The major issue for maintaining the properties of the curve, in the transition from continuous time to discrete time form, is the requirement to have enough data points per time. The theory of how many data points are required was outlined by Shannon (1948) who provided a theoretic basis for the sampling rate boundaries found by Nyquist. Nyquist (1928) determines that, when we are interested in maintaining the information of the harmonies as a signal, and being able to restore the original continuous time signal from its discrete time samples with little distortion, we need at least two data points between successive curve peaks. In the context of the Bass model curve, we need to consider how many data points are between start (t = 0) and first inflection point (when we have one), and between inflection points and peak. These times are determined by p, q values. The distortion, or the RMSE (Root Mean Square Error) of the continuous and discrete time forecasts difference, is calculated for each p, q coordinate as:

$$RMSE = \sqrt{\frac{1}{n} \cdot \sum_{i=0}^{n-1} \left(F(i) - F_d(i) \right)^2}. \qquad (15)$$

Where n is the number of periods from launch to saturation and F, F_d are the forecasts according to the continuous (2, 10, 13) and discrete (3) forms correspondingly, when the same parameters values are used ($p = p_d ; q = q_d$).

Figure 8 presents how the distortion (*RMSE*) varies across the p, q space. The distortion at the green areas is between 0 and 0.015. At the yellow areas, it is between 0.015 and 0.07. The distortion at red areas is more than 0.07 and up to a maximum of 0.154. The map was generated with 0.01 steps of p and q values.

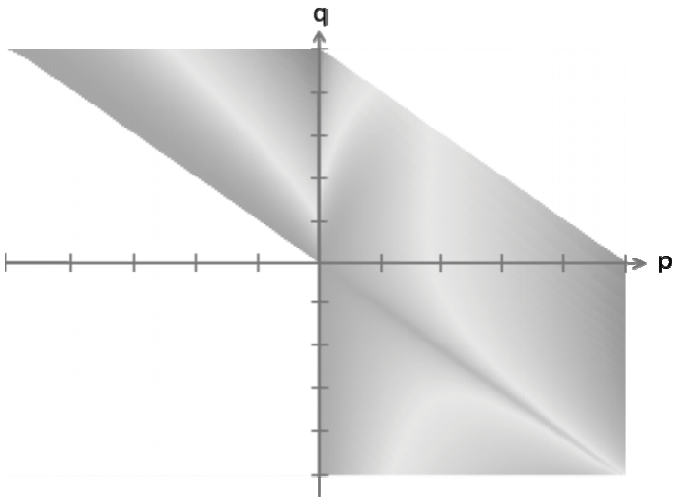


Fig. 8. Distortion (*RMSE*) $p - q$ space map

In almost all empiric cases, the values of p, q reside in the low distortion (green) areas where a transition between the continuous and discrete time forms is seamless and causes insignificant distortion. There are some other cases, where the values of p, q reside in the yellow areas with minor impact. In the rare cases, where the values of p, q reside in the red areas, a transition between the continuous and discrete time causes a significant distortion and requires careful adjustments.

Result 6: The transition between the continuous time and discrete time form is justified, as in most existing empiric researches, where p, q values reside in regions with low distortion. In cases where p, q values reside in high distortion regions, such a transition would have a significant impact on the forecasts.

Conclusion

In this paper, we explored the properties of diffusion curve patterns and how they depend on the p, q parameters values and p/q ratio. Rather than distinguishing between two regions, with or without peak, as in previous research, we refer to four different regions {A, B, C, D} that are categorized also by the number of the inflection points. We also develop an analytic formula for the inflection points' times. We also extend the common $p - q$ space to include regions {E, F, G} with negative values of p or q and provide marketing intuition or insight to the meaning of the negative values and to the market beha-

vior. In region E, we define a new motivation for seeding which, unlike previous research, is not used only for accelerating the diffusion, but, in certain conditions is essential for starting the market. Another contribution of this paper is defining the conditions to saturation below 100% (unlike previous concept that any product will finally cover the entire market). We also highlight some differences between discrete time and continuous time flavors of diffusion models and the map of the regions, where a switch between them is appropriate. Future research may propose an intuition for the regions that are still white in the $p - q$ space maps and explore their properties. Another direction may be developing an analytic solution for the discrete time diffusion difference equation.

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Appendix A: Inflection points calculations

Bass solution for the non-cumulative adoption rate is: $f(t)=\frac{(p+q)^2}{p}\left[\frac{e^{-(p+q)\cdot t}}{\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)^2}\right]$

For finding the inflection points, we need to find where the derivative equals 0.

The derivative is calculated using the formula:

$$f(t)=\frac{h(t)}{g(t)}\Rightarrow\frac{df(t)}{dt}=\frac{\frac{dh(t)}{dt}\cdot g(t)+\frac{dg(t)}{dt}\cdot h(t)}{g^2(t)}$$

$$f'(t)=\frac{(p+q)^2}{p}\left[\frac{-(p+q)e^{-(p+q)\cdot t}\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)^2-e^{-(p+q)\cdot t}\cdot 2\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)\cdot\left(\frac{q}{p}\right)\cdot(-(p+q))e^{-(p+q)\cdot t}}{\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)^4}\right]=$$

$$=\frac{(p+q)^2}{p}\left[\frac{-(p+q)e^{-(p+q)\cdot t}\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)-e^{-(p+q)\cdot t}\cdot 2\cdot\left(\frac{q}{p}\right)\cdot(-(p+q))e^{-(p+q)\cdot t}}{\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)^3}\right]=$$

$$=\frac{(p+q)^3}{p}\left[\frac{-e^{-(p+q)\cdot t}\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)+e^{-(p+q)\cdot t}\cdot 2\cdot\left(\frac{q}{p}\right)\cdot e^{-(p+q)\cdot t}}{\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)^3}\right]=\frac{(p+q)^3}{p}\left[\frac{-e^{-(p+q)\cdot t}\left(1-\frac{q}{p}e^{-(p+q)\cdot t}\right)}{\left(1+\frac{q}{p}e^{-(p+q)\cdot t}\right)^3}\right]$$

When comparing the numerator to 0, since denominator is always positive, as Bass (1969) we can find the periodic sales peak time t^*.

$$t^*=\left[\frac{1}{p+q}\right]\ln\left(\frac{q}{p}\right)$$

For finding the inflection points, we need to calculate when the second derivative equals 0.

Changing variables for convenience

$$X(t)\equiv e^{-(p+q)\cdot t}\qquad \text{-}\rightarrow\ f'(t)=\frac{(p+q)^3}{p}\left[\frac{-X(t)\left(1-\frac{q}{p}X(t)\right)}{\left(1+\frac{q}{p}X(t)\right)^3}\right]$$

$$f''(t)=\frac{(p+q)^3}{p}\left[\frac{\left(-1+\frac{2q}{p}X(t)\right)\cdot X`(t)\cdot\left(1+\frac{q}{p}X(t)\right)^3-3\left(1+\frac{q}{p}X(t)\right)^2\cdot\left(\frac{q}{p}\right)X`(t)\cdot\left(-X(t)+\frac{q}{p}X^2(t)\right)}{\left(1+\frac{q}{p}X(t)\right)^6}\right]$$

Comparing the numerator to 0

$$\left(-1+\frac{2q}{p}X(t)\right)\cdot X`(t)\cdot\left(1+\frac{q}{p}X(t)\right)^3-3\left(1+\frac{q}{p}X(t)\right)^2\cdot\left(\frac{q}{p}\right)X`(t)\cdot\left(-X(t)+\frac{q}{p}X^2(t)\right)=0$$

$$\left(-1+\frac{2q}{p}X(t)\right)\cdot\left(1+\frac{q}{p}X(t)\right)^3-3X(t)\left(1+\frac{q}{p}X(t)\right)^2\cdot\left(\frac{q}{p}\right)\cdot\left(-1+\frac{q}{p}X(t)\right)=0$$

$$\left(-1+\frac{2q}{p}X(t)\right)\cdot\left(1+\frac{q}{p}X(t)\right)=3X(t)\cdot\left(\frac{q}{p}\right)\cdot\left(-1+\frac{q}{p}X(t)\right)$$

$$\left(\frac{q}{p}\right)^2\cdot X^2(t)-4\cdot\left(\frac{q}{p}\right)\cdot X(t)+1=0$$

$$X(t)=\frac{4\frac{q}{p}\pm\sqrt{16\left(\frac{q}{p}\right)^2-4\left(\frac{q}{p}\right)^2}}{2\left(\frac{q}{p}\right)^2}=\frac{4\frac{q}{p}\pm 2\left(\frac{q}{p}\right)\sqrt{3}}{2\left(\frac{q}{p}\right)^2}=\frac{2\pm\sqrt{3}}{\frac{q}{p}}=\frac{p}{q}\cdot\left(2\pm\sqrt{3}\right)$$

$$X(t)=e^{-(p+q)t}=\frac{p}{q}\cdot\left(2\pm\sqrt{3}\right)\Rightarrow -(p+q)t=\ln(2\pm\sqrt{3})-\ln\left(\frac{q}{p}\right)=>t=\frac{\ln\left(\frac{q}{p}\right)-\ln(2\pm\sqrt{3})}{p+q}$$

Since $\ln(2+\sqrt{3})+\ln(2-\sqrt{3})=\ln(2^2-3)=0\Rightarrow\ln(2\pm\sqrt{3})=\pm\ln(2+\sqrt{3})$

The times of the inflection points are:

$$t^{**}=\frac{\ln\left(\frac{q}{p}\right)-\ln(2\pm\sqrt{3})}{p+q}=t^{*}\pm\frac{\ln(2+\sqrt{3})}{p+q}$$

Note that the inflection points are at equal distance from the peak.