“The fusion of insurance and financial structured products – a Monte Carlo valuation”

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The fusion of insurance and financial structured products – a Monte Carlo valuation

Abstract

The paper presents the valuation of contracts that combine financial structured products and insurance policies - pure endowment insurance and risk insurance contracts. The embedded options in these products promise, upon exercise, the higher of either the future value of the invested fund in risk-free interest rates (which is defined in the option contract), or the future value of the fund invested in a basket of risky assets. Whereas prior literature developed mathematical expressions for continuous processes, the study allows for jumps, admitting leptokurtic distributions of the risky assets stochastic processes. The authors solve the model numerically using Monte Carlo with parameters that are estimated via MLE from real market data and conclude with numerical examples.

Keywords: risk insurance, options, structured products, Monte Carlo, pure endowment insurance, risk insurance, jump diffusion process.

Introduction

Financial fund managers and insurance providers are constantly seeking new offerings and programs to attract new customers and retain their current clientele. Individuals, on the other hand, are becoming more market savvy and require more creative solutions to their life insurance and retirement savings, especially in light of the recent market crashes, elongated life expectancy, and changes in government regulations. This paper describes a hybrid – a solution that fuses together an insurance program with a financial structured product (SP). It presents two alternative versions, then describes their valuation model and provides numerical solutions of a few practical examples. Whereas prior literature on this topic was limited to analytically tractable distributions, this paper uses a Monte Carlo approach that can easily accommodate a wide spectrum of stochastic processes.

An ongoing academic interest and market activity is evident in hybrid insurance solutions (see for example, Broeders et al., 2011) and in growing demand for safer financial investments (e.g., Dichtl and Drobetz, 2011). Structured financial products are offered to investors interested in a relatively safe investment with a moderate upside. There is a large variety of SP alternatives that are issued by financial institutions worldwide. Hens and Rieger (2009), for example, describe the most popular SP types in the US, Germany and Switzerland. The most common SPs are those that promise investors a certain percentage of the returns on a portfolio of risky assets with a downside protection should that portfolio return be negative on the SP maturity date. The risky assets often include leading indexes (S&P500, DAX, NIKKEI 225, etc.), precious materials, commodities, etc. Retail investors' preferences and utility regarding SP investment are analyzed by Jessen and Jørgensen (2012), Hens and Rieger (2009) and others.

Yosef, Benzion and Gross (2004) describe the market and academic literature related to the evolution of integrated solutions bundling together financial type options and insurance programs. Motivated by the SP market, Yosef (2006) suggests an exotic option defined on SP and two types of life insurance contracts: pure endowment insurance and risk insurance. In this paper we present similar life insurance contracts, described below, and extend the prior work of Yosef (2006) by allowing for a floating exercise price and by admitting more complex underlying asset stochastic process, to exemplify the flexibility of our proposed Monte Carlo solution.

In the first case, where the option is defined on the SP and on pure endowment insurance, option holders buy the contract and deposit an amount of money, referred to as the invested fund. The seller of the contract invests the invested fund until a defined maturity date. In managing this investment, the goal of the option writer is to achieve a high excess return over the risk-free rate, which obviously could only be possible by investment in risky assets.

If the option holders survive through the exercise date of the option, they could exercise the option contract and receive the higher of either the future value of the invested fund in risk-free interest rate (which is defined in the option contract) or the future value of the invested fund which is invested in the basket of risky assets. The amount of money paid by the option writer upon exercise of the option contract is paid after deduction of a commission, which is a percentage of the difference between the future value of the invested fund in the risk-free

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interest rate and the future value of the invested fund in the risky assets. Note that option holders will only exercise the option contract on the maturity date if the value of the underlying asset (the invested fund in the risky assets) is, at that time, worth more than the future value of the invested fund earning the risk-free interest rate defined in the option contract. Also note that in the option contract suggested here, if option holders do not survive through the maturity date of the option, the value of the contract is zero and the beneficiaries of the option holders do not get back the invested fund. The contract described above may be extended to include additional features, for example, a similar contract that pays the beneficiaries the invested fund plus the accrued risk-free interest even if option holders do not survive through the exercise date.

A few pertinent issues: (a) this contract could only be exercised on the maturity date of the option; hence it is a European type option with a floor provision; (b) the option writer is motivated to maximize the returns on the investment as his commission is defined by a percentage of the difference between the invested fund gained on the basket of risky assets and the invested fund earning the risk-free rate; (c) if option holders decide to exercise the option on the maturity date, they "buy" the underlying asset from the option writer for an exercise price which equals the invested fund earning the risk-free interest rate plus a commission. Thus this contract is actually a version of a European call option settled in cash and contingent on the survival of the option holder; (d) we presume that individuals would prefer investing in such contracts instead of the underlying SP offered by banks, similar to their preference to invest in pure endowment insurance instead of investing in bank savings.

In the second case we consider an option contract on an SP and a risk insurance contract. A risk insurance contract pays the beneficiaries a specified amount if the insured does not survive through the maturity date of the policy contract. Most of the above discussion regarding the first contract is also relevant for this second case. The option holder buys the contract for a fixed price and a specified sum of money is invested by the option writer in a basket of risky assets. Should the option holder die during the life of the option, his beneficiaries (who may be specified by name in the option contract) are entitled to exercise the option and receive from the option writer the higher of the risk-free interest rate or the returns on the investment in the basket of risky assets, after commission deduction. If the option holder survives through the maturity date, the option becomes worthless and expires[1]. Note that this exotic option resembles an American call in terms of the early exercise option; here the early exercise is triggered by the mortality of the option buyer. Prior studies in the actuarial literature integrating the mathematics of finance with the mathematics of insurance probably start with Brennan and Schwartz (1976) who identify the option structure of an equity-linked life insurance policy with an asset value guarantee, and analyze and price it. Briys and de Varenne (1994) deal with the bonus option of the policy holder and the bankruptcy option of the (owners of the) insurance company in terms of contingent claims analysis. Later studies of the bonus option include Miltersen and Persson (1998) and Grosen and Jørgensen (2000).

The study of options in other contexts, in which two or more stochastic processes govern the life of defaultable bonds or swaps is quite vast. In their seminal paper, Duffie and Singleton (1997) adjust the risk-free instantaneous interest rate by the default hazard of the firm issuing the bond or swap. Similar issues arise in the valuation of Asian options that are written on the exchange rate in a two-currency economy. In pricing these options both the stochastic nature of the foreign and domestic zero-coupon bond prices and the exchange rate process are modeled, see for example Nielsen and Sandmann (2001).

Both discrete and continuous-time stochastic models for interest rate processes have been presented in the actuarial literature, primarily Gaussian autoregressive processes. Examples include Panjer and Bellhouse (1980), Parker (1994) and references therein, Milevsky and Promislow (2001), and others. More recent studies combining call options on pension annuity insurance plans include works by Ballotta and Haberman (2003) and Yosef, Benzion, and Gross (2004).

This paper faces similar challenges, valuing contingent future payoffs that are derived from the stochastic process of risky asset returns and the mortality process of individuals. Unlike the bulk of prior research we base our solution on Monte Carlo simulations and thus it can be easily adapted to various stochastic processes as required. We present a solution where the risky assets model is a jump-diffusion process calibrated to actual market data. The same framework can be used for other processes such as stochastic volatility, stochastic volatility and jump diffusion, etc. Note that similar to prior

[1] Another version to this contract (with different price obviously) pays the surviving option holder the invested amount plus the risk-free rate on the maturity date.

research we ignore expenses, profits, and other administrative charges and thus present formulation and numerical results on a net basis, except for a performance based commission which is an integral part of the contracts in this paper.

The rest of the paper proceeds as follows. Section 1 presents the valuation models for the SP and pure endowment and SP with risk insurance contracts. Section 2 presents and discusses numerical results, and the final Section concludes the paper.

1. The valuation model

We start with a structured product with pure endowment insurance policy. We present its model in details and discuss key issues that are common to modeling other similar policies, such as the choice of the probability measure and the details of the underlying risky asset stochastic process. We then develop the valuation model for a structure product with risk insurance policy.

1.1. A structure product with pure endowment (SP-PE) insurance policy. SP-PE is a contract paying the insured a contingent amount at the maturity of the contract, in case he is alive at that time. It is a pure endowment insurance policy combined with an SP which pays the higher of a risk-free investment and the return on a portfolio of risky assets net of a commission. Assume that the risky asset portfolio gross returns are governed by the stochastic process $\tilde{R}(\tau)$:

$$c_{SP-PE}(0) = \mathrm{P}(T > \tau) B e^{-r\tau} \left\{ e^{r\tau} + (1-k) E\left[\tilde{R}(\tau) - e^{r\tau} \right]^{+} \right\}, \tag{1}$$

where τ is the time from 0 to the end of the policy contract; $\tilde{R}(\tau)$ is the gross return on the risky asset portfolio from time 0 to τ; T is the remaining lifetime of an individual (from time 0), a random variable; B is the value of the invested fund at time 0; k is a constant rate denoting the commission of the option writer; r is the risk free rate, assumed constant in this model[1].

P ($T > \tau$) is the probability that the individual survives through the life of the policy contract. This process is assumed to be independent of the financial market processes. The future payoffs at τ, conditional on $T > \tau$, for a unit of money invested at time 0, are written in the curly brackets. These payoffs are multiplied by B, the total investment fund value at time 0. These expected payoffs are discounted to calculate their present value. Generally there are three approaches to discount such uncertain future cash-flows: (1) the no-arbitrage pricing using the risk free rate to discount risk-neutral expected payoffs; (2) the equilibrium pricing using a risk-adjusted discount rate for the expectations under the physical measure; (3) the actuarial approach using a risk-free discount rate for the expectations under the physical measure.

Each of these approaches has its strengths and weaknesses. A complete discussion of this matter is beyond the scope of this paper. We present here only the key arguments guiding our choice of the solution[2]. The commonly used no-arbitrage approach and its risk-neutral measure are appropriate when the market is complete. In reality this is a rare situation. Even for simple vanilla options, when the underlying process admits jumps, the market is incomplete and the option cannot be fully hedged. Whereas no-arbitrage models price one instrument by relating it to the prices of other instruments, equilibrium models balance supply and demand and do not rely on market completeness. However they require knowledge of investor preferences and probabilities. Thus this approach, despite its theoretical appeal does not seem appropriate to our case. The actuarial approach, although often ignored in the financial engineering literature, has its solid theoretical and practical foundations in a long track of insurance mathematics and actual cumulative experience. It does not rely on market completeness or on the investor preferences. However, it seems inappropriate to a single occurrence; it relies on the central limit of many lotteries. This is the case of life insurance and policy contracts of this paper[3]. Hence, we use the risk free rate to discount expectations under the physical measure.

We adopt here the commonly accepted jump diffusion model of equation (2), which admits empirical leptokurtic distributions of many financial assets,

[1] The discount rate can be generalized to follow a stochastic process, under certain assumptions. However, to simplify the presentation we assume it is a constant value. Furthermore, the promised floor risk-free rate may be different than the market risk-free rate, without loss of generality we present a model where both are the market rate.

[2] A more detailed discussion of this matter is available (though usually scattered) in many prior publications, often biased by the disciplinary background of the authors. We embrace the critical approach of Wilmott who seems to combine strong theoretical foundations, real market familiarity, and a good sense of criticism. See mainly Wilmott (2006 and 2009).

[3] For an interesting perspective on actuarial and risk-neutral approaches see Wilmott (2009) and his blog: http://www.wilmott.com/blogs/paul/index.cfm/2008/11/17/Actuaries-Versus-Quants.

including equities, indexes, and foreign exchange rates[1]. Following Glasserman (2004) and others in the path laid out by Merton (1976) we assume[2]:

$$d\tilde{R}(t) = \mu dt + \sigma dW(t) + dJ(t), \qquad (2)$$

where μ is the deterministic drift rate; σ is the deterministic volatility of the process; $W(t)$ is the familiar Wiener process; $J(t)$ is the univariate jump process defined by:

$$J(t) = \sum_{j=1}^{N(t)} (Y_j - 1) \text{ or } dJ(t) = (Y_{N(t)} - 1) dN(t), \qquad (3)$$

where Y_1, Y_2,… are random variables independent of $N(t)$ which is a counting process. We also assume that $W(t)$ and $J(t)$ are independent. Restricting Y_i to positive values ensures that the gross returns (and thus the risky asset value) remain non-negative which is necessary for our model of long positions in the financial market. The solution of equation (2) is[3]:

$$\tilde{R}(t) = e^{\left(\mu - 0.5\sigma^2\right)t + \sigma W(t)} \prod_{j=1}^{N(t)} Y_j . \qquad (4)$$

To progress from equation (4) to a numerical solution we need to define the distributions governing the jump arrival and the jump size. Following the common practice, we assume that $N(t)$ is a homogeneous Poisson process with intensity λ, hence the jumps inter-arrival times $t_{j+1} - t_j$ are independent with exponential distribution $P(t_{j+1} - t_j \leq \Delta t) = 1 - e^{-\lambda \Delta t}$, $t \geq 0$, where t_j is the arrival time of jump j. Adopting the Merton (1976) assumption that the jump size is lognormally distributed, i.e. $Y_j \sim LN(\mu_y, \sigma_y^2)$ leads to a compact solution which facilitates practical calibration of the model and the simulation of the pension contracts in this paper using market compatible parameters. The assumption $Y_j \sim LN(\mu_y, \sigma_y^2)$, i.e. $ln(Y_j) \sim N(\mu_y, \sigma_y^2)$, results in the following unconditional distribution of the gross returns:

$$P\left[\tilde{R}(t) \leq x\right] = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F_{n,t}(x) , \qquad (5)$$

where $F_{n,t}(x) \sim LN\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \mu_y n,\ \sigma^2 t + \sigma_y^2 n\right]$.

To complete the modeling of equation (1) we need to assume a distribution for the probability that the individual survives through the life of the policy contract P $(T > \tau)$. We consider two cases of T, the random variable of the total lifetime for an individual. The first is an exponential lifetime distribution which is independent of individual age and is included here as a benchmark:

$$\mathrm{P}(T > \tau) = e^{-\varsigma\tau} , \qquad (6)$$

for positive constants ζ and τ (τ is defined in equation (1)). In the second case we model the individual lifespan by *Gompertz* law:

$$\mathrm{P}(T > \tau) = \exp\left[-\frac{\omega}{\ln c} c^{age}\left(c^{\tau} - 1\right)\right], \qquad (7)$$

where c, ω are positive constants, T, τ are defined above, and *age* is the present age of the individual.

1.2. A structure product with risk insurance (SP-RI) policy. Here we present a model for a contract paying the beneficiaries in case the option holder dies during the life of the option. It is a life insurance contract combined with an SP that, in the event of death, will pay the higher of either a risk-free investment or the (gross) return on a portfolio of risky asset, minus the commission. The price of such a contract is formulated in the following equation:

$$c_{SP-RI}(0) = Be^{-rT}\left\{e^{rT} 1_{T \leq \tau} + (1-k) E\left[\left(\tilde{R}(T) - e^{rT}\right)^+ 1_{T \leq \tau}\right]\right\}, \qquad (8)$$

where $1_{T \leq \tau}$ is the usual indicator function having a value of 1 if the condition $(T \leq \tau)$ is true and 0 otherwise. The other variables and parameters are defined in equation (1). Equations (1) and (8) are very similar, yet differ in a few significant matters. In equation (8) the time at which discounting and future values are calculated is the death of the insured T, while in equation it is the maturity of the contract τ. This leads to another matter – whereas in equation (1) the time variable is a constant allowing a simple separation between the stochastic life span T and the stochastic return $\tilde{R}(\tau)$, in equation (8) the return is $\tilde{R}(T)$ and the two random processes are combined, this increases the complexity of the solution.

The modeling of the risky asset returns are discussed in detail in the prior section on an SP and a pure endowment contract (SP-PE). In the case of an SP and a risk insurance (SP-RI), as the time of death T, a random variable, affects the stochastic return

[1] A more comprehensive model would include also stochastic volatility which captures volatility clustering effects; however, this would require the estimation of additional market variables and calibration which may add rigor to the model, yet might introduce additional uncertainty to the results. See, for example, Wilmot (2009) on model calibration.

[2] See also Brigo et al. (2009).

[3] For a detailed development of the following expressions see for example Glasserman (2004).

$\tilde{R}(T)$, we need the appropriate distribution for the event $T \le \tau$. Here again we consider two cases of T, the first is an exponential lifetime:

$$\mathrm{P}(T \le \tau) = 1 - e^{-\varsigma\tau}, \tag{9}$$

for positive constants ζ and τ as defined for equation (6). In the second case we model the individual lifespan by *Gompertz* law:

$$P(T \le \tau) = 1 - \exp\left[-\frac{\omega}{\ln c} c^{age}\left(c^{\tau} - 1\right)\right], \tag{10}$$

where the parameters are defined in equation (7).

Given the above model and the assumption that individual death events are independent of the risky portfolio returns we can rewrite equation (8) using conditional expectations as follows:

$$c_{SP-RI}(0) = BE_T\left\{1_{T\le\tau} + (1-k)E_{\tilde{R}(T)}\left[\left(e^{-rT}\tilde{R}(T) - 1\right)^{+} 1_{T\le\tau} \mid T\right]\right\}, \tag{11}$$

where the expectations $E_T[\cdot]$ and $E_{\tilde{R}(T)}[\cdot]$ are respective to the random variables T and $\tilde{R}(T)$ respectively. Equation (11) can be straightforwardly solved numerically using Monte Carlo (MC) simulations. Effectively it is a double integral where the inner integral is with respect to $\tilde{R}(T)$ given T and the outer integral is with respect to T.

2. Numerical examples and discussion

The valuation of the options in equations (1) and (8) can be easily implemented using Monte Carlo simulations. We use here the fixed interval solution similar to Glasserman (2004) and Brigo et al. (2009)[1]. As an example for the risky asset we arbitrary selected the German index DAX. It can be naturally replaced by any other index or portfolio of indexes. Following Brigo et al. (2009), we use a maximum likelihood estimation (MLE) to estimate the risky asset parameters: $\mu^* = 0.0488$, $\sigma = 0.215$, $\lambda = 2.122$, $\mu_y = 0.0531$, $\sigma_y = 0.00527$, where $\mu^* = \mu - 0.5\sigma^2 + \lambda\mu_y$. For our estimation we use daily closing prices of the DAX in the period November 1983 to December 2011[2]. The DAX price path is presented in Figure 1a and its returns QQ plot in Figure 1b which clearly shows the fat-tails of its empirical distribution (see Appendix).

Since the maturity τ is a constant specified in the contract we can rewrite equation (1) to simplify the numerical calculations as follows:

$$c_{SP\text{-}PE}(0) =$$
$$= P(T > \tau)\, B\left\{1 + (1-k)\, E\left[e^{-r\tau}\tilde{R}(\tau) - 1\right]^{+}\right\}, \tag{12}$$

To exemplify the numerical valuation of the contracts in this paper we look at young males aged 30 and 40 years. The risk-free rate (r) is assumed either 0.03 or 0.05. We assume the commission is k = 10% and that the invested amount is B = \$1. For the survival distribution we use ζ = 0.01 and 0.015 for the exponential distribution and c = 1.1, $\omega = 10^{-4}$ for Gomprtz[3]. The survival probabilities are depicted in Figure 2 (see Appendix).

The results for the structured product with pure endowment (SP-PE), using equation (12) for τ = 5, 15, and 30 years, are presented in Table 1. The numbers, in \$ amount for B = \$1, are each the mean of 100 MC runs, each run includes 1000 price paths of the risky asset portfolio. The *italic* number below each mean is the standard deviation of the MC simulation results. Hence, an estimate for the standard deviation of the mean is 10% of the italicized number ($1/\sqrt{100}$). This dispersion, measure for the error of the calculated values, seems well below 1%, mostly a few tenths of a percent.

As expected, the value of the SP-PE depends on the time to contract expiration (maturity). Take for example the case of Gompertz law and a 30 year old insured. Even for a long contract of 30 years, his survival probability is 74% (see Figure 2b) which immediately sets the floor value of his contract at 0.74B. To this value we need to add the possible upside of the investment in the risky asset. This upside is essentially a call option (less commission) which increases with the length of its duration. A 30 year call option on a volatile asset is quite valuable. Hence no wonder the value of this contract is 2.75B and 1.67B (see Table 1) for a risk-free rate of 0.03 and 0.05 respectively. For a vanilla call option the value increases as the risk-free rate increases. However, in the option embedded in this contract the relevant exercise price is a risk-free gross return. A higher risk-free rate reduces the moneyness of this exotic call option. Increasing the age of the insured

[1] This can be replaced by other models such as variable interval simulations, see for example Glasserman (2004), or asymmetric up and down jumps model, see for example Chacko and Viceira (2003), etc.

[2] This period may of course be replaced by another. It was selected to include the significant booms and busts of recent decades. A much longer period, starting in October 1959 results in slightly different numbers: $\mu^* = 0.0542$, $\sigma = 0.186$, $\lambda = 1.841$, $\mu_y = 0.0496$, $\sigma_y = 0.00218$. This confirms that recent decades are more volatile than earlier periods (mainly by the increase in the jump intensity λ and σ, σ_y parameters). Bloomberg is the source of the DAX closing price data in our example.

[3] Similar to Yosef (2006).

to 40 years old reduces the survival probability to less than 46% (see Figure 2b). Hence, ceteris paribus, the value of the same contract for the older person reduces to 1.71B and 1.03B for a risk-free rate of 0.03 and 0.05 respectively. Therefore, time to maturity affects the value of the contract in a complex manner where the two main effects are the survival probability function and the time value of the option. Whereas in the examples of Table 1 for the case of the exponential distribution the value monotonically increases with maturity, in the case of the Gompertz distribution it increases initially (for short maturities), reaches a peak, and then decreases (for long maturities).

For the numerical solution of equation (11), valuing the SP with risk insurance contract (SP-RI), we need to draw random numbers from the appropriate distribution of *T*. We use the common procedure, see for example Glasserman (2004), drawing uniformly distributed numbers $u \sim U\ (0,1)$ and finding the random draws for *T* via the appropriate inverse cumulative distribution function (CDF). For the exponential distribution the inverse CDF is:

$$T = -\tfrac{1}{\zeta}\ln\left(1-u\right)\ , \tag{13}$$

and for the Gompertz law it is:

$$T = \tfrac{1}{\ln c}\ln\left[1-\tfrac{\ln c}{\varpi c^{age}}\ln\left(1-u\right)\right]\ . \tag{14}$$

For each draw of *T* one needs to draw (N_{itr}) sample paths of $\tilde{R}\left(T\right)$, to calculate an average for the sampled payoffs of *T*, approximating $E_{\tilde{R}(T)}\left[\cdot\right]$ given *T* (assigning zero value to payoffs if $T > \tau$). This process is repeated N_T times averaging these estimated conditional expectations to approximate $E_T\left[\cdot\right]$. Such a procedure becomes computationally long and tedious. We have chosen to alleviate it by performing a simple numerical integral over $T \leq \tau$ using the appropriate life distribution, using 50 subdivisions of the time to maturity. Finally, to further reduce the computation load, for each contract maturity, in each MC run, we generated a single set of N_{itr} sample paths of daily $\tilde{R}\left(t\right)$, instead of generating N_{itr} samples of $\tilde{R}\left(T\right)$ for each *T* value. The three computation alternatives have been compared and yield similar results.

The results for the structured product with risk insurance (SP-RI), using equation (11) for τ = 5, 15, and 30 years, of the efficient procedure, are presented in Table 2. As described above, the numbers, in $ amount for B = $1, are each the mean of 100 MC runs, where each run includes 1000 price paths of the risky asset portfolio and integrated over 50 subdivisions of the time to maturity. The *italic* number below each mean is the standard deviation of the MC simulation results. Here again, an estimate for the standard deviation of the mean is 10% of the italic number ($1/\sqrt{100}$) and it is well below 1%, mostly a few tenths of a percent.

Obviously the maturity of the contract affects an SP-RI differently than it affects an SP-PE contract. As explained in the prior section SP-RI value depends on $\tilde{R}\left(T\right)$, where *T* is a random variable, while SP-PE depends on $\tilde{R}\left(\tau\right)$ where τ is a constant. Secondly, the two contract types depend on complimentary events. An SP-PE is worthless if the insured dies prior to the maturity of the contact and an SP-RI is worthless if the insured survives through the maturity date. Lastly SP-PE resembles a European option which can be exercised only at its expiration date, whereas SP-RI resembles an American option that can be exercised at any date, upon the death of the insured, until the expiration date. Therefore, unlike the SP-PE, the SP-RI value increases monotonically with the time to maturity of the contract since both the embedded option value and the likelihood of death increase with the duration of the contract. Similarly, under Gompertz law, ceteris paribus, the older the insured, the higher the value is of the contract. This is valid not only for the three maturities' results shown in the table, it is supported by the non-crossing curves in Figure 2b. Lastly, in our example we assume constant parameters for the process of the risky assets, hence the value of the contract for the higher risk-free rate is smaller than that for the lower risk-free rate. For example, the value of an SP-RI contract maturing in 30 years, for an insured who is 30 years old presently, is 0.685B and 0.482B (see Table 2) for a risk-free rate of 0.03 and 0.05 respectively, under the specified Gompertz law. We use Tables 1 and 2 to demonstrate the sensitivity of the contract value to its parameters and variables, however a dependence of the risky asset process on the risk-free rate would affect the calculated results. We expect that professional practitioners, after conducting a sensitivity analysis for their specific scenario, would add to the contract price a margin, not just for profit and operational expenses, but also for model and parameters uncertainties.

Conclusion

This paper presents the theoretical valuation models and practical numerical solution procedures for two contracts, each combining a financial product with an insurance policy – a pure endowment and a risk insurance policy. It extends prior work by presenting an adaptable solution to more complex contracts and richer sets of underlying asset stochastic processes.

In recent decades insured individuals become more financially savvy; the financial markets' volatility and risk seem to be rising; more insurance policies are of a defined contribution type and the share of defined benefit policies decreases. These trends are combined with a diminishing trust of the public in the ability of professional investors to time the market and select winning investments on one hand, and increasing doubts in the fair conduct of their investment manager on the other hand, irrespective of whether this is caused by these managers' incentive compensation or the inherent moral hazard of the system. The two contracts presented in this paper combine together an insurance package and an investment in a risky asset portfolio, where the insured is protected by a guaranteed floor motivating the insurer to properly manage the investment risk. On the other hand, the insurer benefits from an increased commission upon achieving higher returns on the risky assets. Hence these contracts include embedded risk control and success incentives that benefit both the insured and the insurer and thus may help to remedy some of the prevailing distortions in the market and enhance a safer competition among the insurers.

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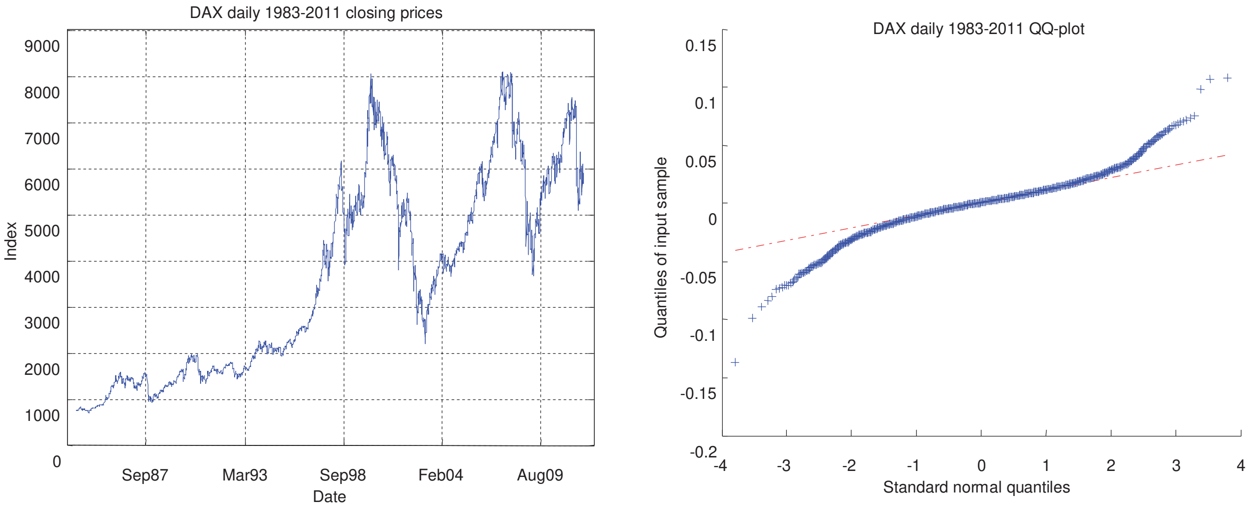
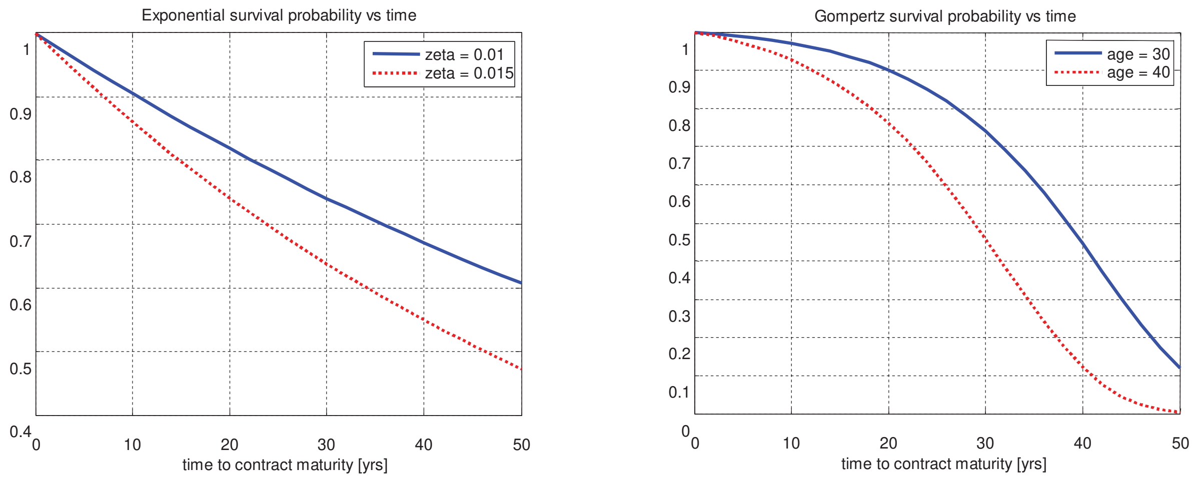
Appendix



1a: index price path

1b: index returns QQ plot

Fig. 1. DAX daily closing prices and returns QQ plot for the period November 11, 1983 to December 30, 2011



2a: exponential distribution

2b: Gompertz distribution

Notes: Life span probabilities $P(T > \tau)$ for the exponential and Gompertz distributions, defined in equations (6) and (7) respectively, used for the valuation in equation (1). The parameters are: $\zeta = 0.01$ and 0.015 for the exponential distribution and $c = 1.1$, $\omega = 10^{-4}$, *age* = 30 and 40 years old for Gompertz. The valuation of equation (8) requires the complimentary event, i.e. the CDF defined by $P(T \leq \tau)$.

Fig. 2. Life span probabilities

Table 1. The valuation of SP and pure endowment contracts (SP-PE)

The valuation of SP and pure endowment contracts (SP-PE) maturing in 5, 15, and 30 years for a notional amount of B = $1 invested today, promising, if the insured doesn't die prior to the maturity date, to pay the higher of the risk free rate or the returns on a portfolio of risky asset, with a commission of k = 10% on the gains above the risk free rate. If the insured dies until the maturity date, the contract expires worthless. The value is calculated using equation (12) for risk-free rates of 0,03 and 0,05 and for two types of life duration models: Gompertz ($c = 1.1$, $\omega = 10^{-4}$, age = 30 and 40 years) and exponential ($\zeta = 0.01$ and 0.015). The numbers, in $ amount, are the mean of 100 MC runs, each run includes 1000 price paths of the risky asset portfolio. The *italic* number below each mean is the standard deviation of the MC simulation results. Hence, an estimate for the standard deviation of the mean is 10% of the italic numbers ($1/\sqrt{100}$).

Life model →		Gompertz		Exponential	
τ [yrs]	r [%]	age = 30	age = 40	ζ = 0.01	ζ = 0.015
5	3	1.32 *0.016*	1.3 *0.015*	1.27 *0.016*	1.239 *0.015*
15	3	1.88 *0.06*	1.714 *0.043*	1.726 *0.056*	1.596 *0.046*

Table 1 (cont.). The valuation of SP and pure endowment contracts (SP-PE)

Life model →		Gompertz		Exponential	
τ [yrs]	r [%]	age = 30	age = 40	ζ = 0.01	ζ = 0.015
30	3	2.75 *0.17*	1.711 *0.092*	2.749 *0.16*	2.386 *0.14*
5	5	1.24 *0.016*	1.22 *0.011*	1.196 *0.014*	1.17 *0.015*
15	5	1.52 *0.04*	1.38 *0.032*	1.389 *0.036*	1.291 *0.034*
30	5	1.67 *0.08*	1.03 *0.05*	1.675 *0.081*	1.437 *0.077*

Table 2. The valuation of SP and risk insurance contracts (SP-RI)

The valuation of SP and risk insurance contracts (SP-RI) maturing in 5, 15, and 30 years for a notional amount of B = $1 invested today, promising, if the insured dies prior to the maturity date, to pay the higher of the risk free rate or the returns on a portfolio of risky assets, with a commission of k = 10% on the gains above the risk free rate. If the insured doesn't die by the maturity date, the contract expires and is worthless. The value is calculated using equation (11) for risk free rates of 3 and 5% and for two types of life duration models: Gompertz (c = 1.1, ω = 10-4 ages 30 and 40 years) and exponential (ζ = 0.01 and 0.015). The numbers, in $ amount, are the mean of 100 MC runs, each run includes 1000 price paths of the risky asset portfolio (and 50 subdivision over the contract period, for the external expectations regarding the death probability). The italic number below each mean is the standard deviation of the MC simulation results. Hence, an estimate for the standard deviation of the mean is 10% of the italic numbers ($1/\sqrt{100}$).

Life model →		Gompertz		Exponential	
τ [yrs]	r [%]	age = 30	age = 40	ζ = 0.01	ζ = 0.015
5	3	0.0134 *0.0001*	0.0344 *0.0002*	0.0582 *0.0004*	0.086 *0.0006*
15	3	0.0904 *0.0016*	0.223 *0.0041*	0.2069 *0.0026*	0.2979 *0.0049*
30	3	0.685 *0.026*	1.376 *0.048*	0.5358 *0.0131*	0.7388 *0.0211*
5	5	0.01298 *0.0001*	0.0333 *0.0002*	0.0564 *0.0003*	0.0836 *0.0005*
15	5	0.0793 *0.0013*	0.196 *0.0032*	0.186 *0.0024*	0.2683 *0.0034*
30	5	0.4823 *0.016*	0.984 *0.024*	0.4158 *0.0093*	0.5763 *0.0133*