"Optimal reinsurance programs for a portfolio of life annuities"

AUTHORS	Paolo De Angelis Nicolino E. D'Ortona Gabriella Marcarelli
ARTICLE INFO	Paolo De Angelis, Nicolino E. D'Ortona and Gabriella Marcarelli (2011). Optimal reinsurance programs for a portfolio of life annuities. *Insurance Markets and Companies*, *2*(3)
RELEASED ON	Thursday, 26 April 2012
JOURNAL	"Insurance Markets and Companies"
FOUNDER	LLC "Consulting Publishing Company "Business Perspectives"

NUMBER OF REFERENCES

0

NUMBER OF FIGURES

0

NUMBER OF TABLES

0

Paolo De Angelis (Italy), Nicolino E. D'Ortona (Italy), Gabriella Marcarelli (Italy)

Optimal reinsurance programs for a portfolio of life annuities

Abstract

The objective of this paper is to analyze – in a Risk-Based Capital framework – the equilibrium conditions between the Insurer (Cedant) and the Reinsurer with respect to linear and non-linear reinsurance strategies or appropriate combinations thereof. The analysis is conducted through a stochastic simulation of the management model of an insurance company managing a portfolio of life annuities.

Keywords: reinsurance programs, solvency requirements, probability of ruin, target capital, expected ROE, internal risk models.

Introduction

The insurance company is exposed to fluctuations affecting the evolution of the risks involved. These fluctuations have an impact on economic management, assets and liabilities, undermining the solvency of the company. In order to reduce these fluctuations to acceptable levels the insurance company relies on the reinsurance market.

To choose a reinsurance program is a typical economic problem of decision under uncertainty. The problem is the following: what is the share of risk transfer (to reinsurer) which realizes, the same cost, maximum risk reduction?

In literature, this problem has been analyzed mainly by the point of view of the cedant, sometimes by the point of view of the reinsurer and rarely simultaneously analyzing both sides of the reinsurer market. Milestones are the original works of de Finetti (1940), Borch (1974) and Buhlmann (1979) essentially based on the risk theory and expected utility theory arguments; some reinsurance programs based on combinations of traditional reinsurance treaties have been studied (Centeno and Simoes, 1989; Verlaak and Beirlant, 2003; Gajek and Zagrodny, 2004). Throughout this paper we want to analyze the point of view of both sides of the reinsurance market considering an Internal Risk Model based on a financial analysis of surplus dynamics as described in Olivieri (2005) and Cocozza, Di Lorenzo, Orlando and Sibillo (2010).

In this paper, effectiveness of some reinsurance programs on the control of the demographic risk of a life insurance company are measured by means of a criterion based on the return on equity under the ruin probability constraints; points of mutual advantage at the same level of confidence for both reinsurance market agents (the cedant and the reinsurer) are identified, using a stochastic simulation procedure drawing a management behavior of a life insurance company.

Numerical simulations show that, under suitable assumptions, the coverage period of reinsurance programs may match with the average residual life of the annuitants of the cedant life portfolio.

1. Managing a portfolio of life annuities

1.1. An internal actuarial model. An insurance portfolio is solvent on a predefined time horizon if and only if the value of assets is not less than the value of liabilities, with a high level of probability.

Let us consider a portfolio of non-deferred time-continuous life annuities (with single premium) sold to a cohort of males N_0 aged x at time $t = 0$. The value of assets allocated to the portfolio is partially funded by annuitants – through the single premium – and partially by the insurance company, through the shareholders' capital. Just a moment later the time 0 (i.e. at time 0_+), assets are defined as follows:

$$A_{0+} = (1 + \delta) \cdot V_{0+,n}^{[\Pi]} + M_0, \qquad (1)$$

where A_{0+} is the value of assets allocated to the portfolio; $V_{0+,n}^{[\Pi]}$ is the value of portfolio reserve and M_0 is the value of shareholders' capital set up by the insurer, δ is the safety loading on single premium, $\Pi_t = \{j : T_x^{(j)} > t, j = 1, 2, ..., N_t\}$ the portfolio in-force at time t $(t \geq 0)$.

The initial value of shareholders' capital plays the role of solvency margin. For simplicity, but without loss of generality (according to the approach adopted by the European Economic Community in the first life insurance Directive) we assume that the solvency margin is proportional to the initial value of the mathematical initial reserve:

$$M_{0+} = p \cdot V_{0+,n}^{[\Pi]}, \qquad (2)$$

where

$$p = \frac{M_0}{V_0}, \qquad (3)$$

measures the rate of capital required for the portfolio solvency. This parameter, following referred to

as relative target solvency capital, represents a solvency index and plays a key role in studying the efficiency of reinsurance programs.

The actuarial risk model adopted is a Discrete-Time Model based on a compact formulation of the company's financial surplus i.e. the difference between accrued assets and the present value of relevant liabilities. The value of assets at time t is clearly random. Whilst the individual reserve is deterministic, the portfolio reserve is random due to the uncertainty about the number of annuitants alive at time t. For $t > 0$, assets are defined by means of a recursive formula

$$A_t = \frac{A_{t-1}}{v(t-1,t)} - \sum_{j=1}^{N_t} b_t^{(j)}, \qquad (4)$$

where $b_t^{(j)}$ is the annual benefit at time t to annuitant j, N_t is the random number of annuitant alive at time t and $v(t-1, t)$ is the one-period deterministic discount factor.

1.2. Strategic variables in the decision problem. With reference to reinsurance strategies, we investigate effectiveness of some traditional reinsurance treaties or their combinations in controlling the demographic risk; the reinsurance arrangements efficiency is analyzed in terms of risk and performance. In order to analyze model's results, we have selected the following three indices:

♦ the Finite-Time Ruin probability over the horizon $(0, t)$, as a risk measure:

$$\varphi(M;T) =$$
$$= \Pr\{M_t < 0 \text{ for at least on } t = 1,2,...,T | M_0 = M\}; \; (5)$$

♦ the annual rate of expected return on equity, as a portfolio performance index:

$$i(0,t) = \left(1 + ROE(0,t)\right)^{\frac{1}{t}} - 1 \qquad (6)$$

where $ROE(0,t) = E\left(\dfrac{\widetilde{M}_t - M_0^*}{M_0^*}\right)$ measures the expected return on equity for the horizon $(0, t)$ and M_0^* is the target level of capital, as a risk measure for a given time horizon T and a given ruin probability[1]: $M_0^* = \inf\{M_0 \geq 0 \,|\, \Pr\{\bigcap_{s=0}^{T} M_s \geq 0\} \geq 1 - \varepsilon\}$;

♦ the annual rate of expected return on equity over the relative target solvency capital:

$$\mu = \frac{i(0,t)}{p} = \frac{i(0,t)}{M_0^*} \cdot V_0 . \qquad (7)$$

The index μ is meant as a synthetic performance index since it measures the annual return on equity per unit of capital allocated for solvency, with a fixed level of probability, within a given time horizon.

1.3. A simulation procedure. For simulating the proper capital time evolution, a Monte Carlo method has been applied: numerical simulations are repeated several times in order to obtain multiple scenarios that give immediately an overview of the process, as well as provide valuable information on its distribution.

The simulation technique for defining insured annuities related to policies taken to the next maturities is based on the individual actual approach. Its key feature is to consider known, at the beginning of each year of portfolio management, the vector containing the actual level of insured annuities for each policy. The distribution of annuities is updated every time there is a release for death.

For each projection year, this approach provides the following steps:

Step 1:

♦ The simulation of the number of deaths (d). For this purpose we assume that the annual number of deaths follows a binomial distribution with the parameter fixed by the second order demographic basis $\{q_{x+t}^{(II)}\}$.

♦ The updating of the number of policies at time $t + 1$, N_{t+1}.

Step 2:

♦ Random extraction of (d) policies including those taken at the end of step 1 and update of the vector of the insured annuities, by canceling the policies randomly selected in step 1.

♦ The resulting distribution of annuities will be the initial one for the year after.

1.4. Portfolio characteristics and hypothesis. The model has been investigated on a portfolio of non-deferred time-continuous life annuities (with single premium) referred to a cohort of males currently aged 65. The number of annuitants at time 0 is $N_0 = 2500$; the maximum age is assumed to be: $\omega = 110$ years.

We suppose that the distribution of insurance annuities is not uniform, that is the amount of annual benefit is not the same for all annuitants.

In order to calculate premiums and reserves, the RG48 mortality table has been considered[2] as first

---

[1] Cfr., Olivieri A. (2005).

[2] The RG48 projected mortality tables are cohort mortality tables elaborated by the State General Accounting Department and refer to the cohort born in 1948.

order mortality basis; a second-order mortality basis has been introduced as a different demographic scenario, consisting of a reduction by 2,5% of all mortality rates. Let us summarize the assumptions:

♦ Demographic and financial hypothesis.
♦ First-order mortality basis:
  $\{q_x^{RG48}, x = 1,2,...,\omega - 1\}$.
♦ First-order financial basis: $i = 2,5\%$.
♦ Second-order mortality basis:
  $\{q_x = (1 - 0,025)q_x^{RG48}, x = 1,2,...,\omega - 1\}$.
♦ Second-order financial basis: at any time t the yield curve is deterministic $\{i_s = i\}$ with $s > t$.

Table 1. Initial distribution of insured annuity (expressed in any currency unit)

Initial amounts						
Annuity	5	10	15	20	25	Currency units
Frequency	35%	25%	20%	15%	5%	100%

Statistics	
Average insured amount	11,5
Total insured amount	28.750
S.q.m.	6
Skewness	0,55

Due to the above hypothesis the portfolio is only exposed to the demographic risk (given by the pooling component); randomness other than that coming from mortality evolution is disregarded (so our simulations do not consider investment risk).

Figure 1 shows a simulation of possible trajectories of shareholders' assets: there is a positive average trend of the capital but the variability increases with increasing t, so the ruin probability for the company increases with the time horizon (this probability is given by the number of trajectories with negative ordinate).
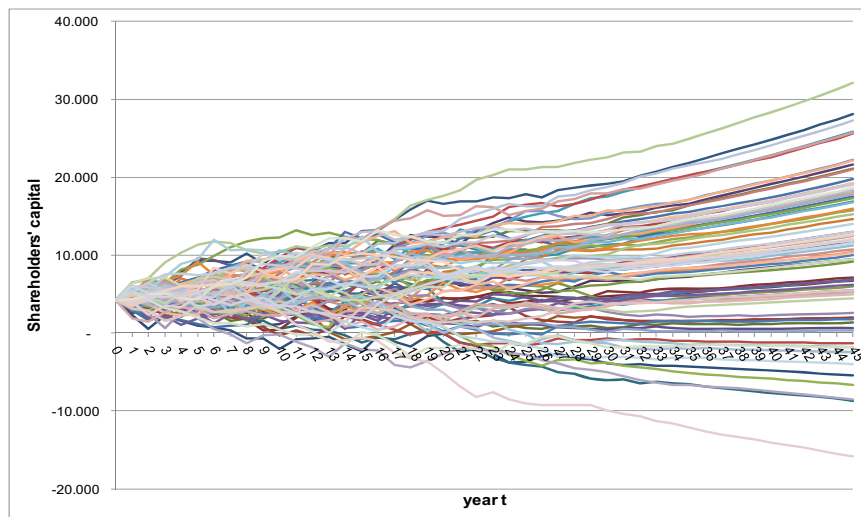

Fig. 1. Simulation of shareholders' equity, with $p = 1\%$

In order to control the ruin probability at the target level of 2.5%, it needs to increase the initial solvency ratio M_0^*.

Table 2 shows a comparison over time period between the capital target dynamics and the indices previously defined.

Table 2. Index with $\varphi(M; T) = \varepsilon = 2,5\%$

T:	5	10	15	20	25	30	35	40
$p \%$	0,491	0,872	1,140	1,329	1,430	1,460	1,463	1,463
$i(0,T)\%$	12,52	5,59	3,42	3,30	2,96	2,81	2,74	2,71
μ	25,50	6,41	3,00	2,48	2,07	1,92	1,88	1,85
M_0^*	2.059	3.657	4.781	5.573	5.997	6.123	6.135	6.135

From Table 2, we can observe that:

♦ as expected, the target capital M_0^* increases quickly with increasing T;
♦ as a consequence, there is a reduction of return on capital required for solvency, in reference to the given ruin probability and time horizon T.

The reduction on the equity return can be accounted as a cost of capital to maintain a specified level of solvency requirements;

♦ therefore, the return on equity results greater than the interest rate assumed as the free-risk interest rate ($\{i_s = i\}$).

2. Reinsurance programs and risk analysis

In this section we present results on the risk model discussed above, tested on some reinsurance programs. For each reinsurance program, the reinsurance premium and the asset and capital dynamics of the insurer and the reinsurer are formalized, then points of mutual advantage of the two market's agents are identified.

2.1. Quota share arrangement (QS). A quota share arrangement is structured in such a way that the reinsurer pays to the cedant the fixed share of the annuity. Let us remember the essential terms of a quota share agreement and specify liabilities, reinsurance premiums, assets and solvency margin.

Referring to an annuitant alive at time t, the individual random present value of benefits for the cedant and for the reinsurer are respectively defined as follows

$$Y_t^{QS(j)} = \alpha \sum_{s=1}^{T_{x+t}^{(j)}} b_{t+s}^{(j)} v(t, t+s) = \alpha Y_t^{(j)}, \qquad (9)$$

$$X_t^{QS(j)} = (1-\alpha) Y_t^{(j)}, \qquad (10)$$

where $\alpha \in [0,1]$ is the percentage of the risk retained by the cedant.

For the reinsurance premium (to be paid at time 0), we assume that the reinsurer adopts the following pricing principle:

$$R^{QS} = (1-\alpha)(1+\eta) V_{0,n}^{[\Pi]}, \qquad (11)$$

where η is the security rate on single reinsurance premium.

Under a QS treaty, the internal actuarial model at time t for the cedant and the reinsurer, respectively, holds:

♦ Shareholder's capital and assets dynamics for the insurer:

$$A_t^C = \begin{cases} M_0^C + (1+\delta) V_{0;n}^{[\Pi]} - R^{QS}, & t = 0^+ \\ A_{t-1}^C (1+i) - \alpha \cdot CF_t^{\Pi}, & t > 0 \end{cases},$$

$$M_t^C = \begin{cases} p \cdot \alpha \cdot V_{0;n}^{[\Pi]}, & t = 0^+ \\ A_t^C - \alpha \cdot V_{t;n-t}^{[\Pi]}, & t > 0 \end{cases},$$

where $CF_t^{\Pi} = \sum_{j=1}^{\Pi_t} b_t^{(j)}$ is the stochastic portfolio outflows.

♦ Shareholder's capital and assets dynamics for the reinsurer:

$$A_t^R = \begin{cases} M_0^R + R^{QS}, & t = 0^+ \\ A_{t-1}^R (1+i) - (1-\alpha) \cdot CF_t^{\Pi}, & t = 0^+ \end{cases},$$

$$M_t^R = \begin{cases} p \cdot (1-\alpha) \cdot V_{0,n}^{[\Pi]}, & t = 0^+ \\ A_t^R - (1-\alpha) \cdot V_{t;n-t}^{[\Pi]}, & t > 0 \end{cases}.$$

Considering three different assumptions on the ratio $\left(r = \frac{\eta}{\delta} \right)$, i.e. the security rate on single reinsurance premium (η) over the safety loading on single insurance premium (δ), different levels of the indices have been reproduced, as shown in the Figure 2. If the ratio is less than 1, it means that the insurer "buys security" by the reinsurer to "prices" lower than those charged (and therefore holds that income will accumulate, on average, shareholders' equity). However, if the ratio is greater than 1, it means that the insurer "buys security" by the reinsurer to "prices" higher than those charged (hence he gives income that reduces equity). Assuming the annual rate of expected return on equity $i(0, t)$ as the index of preferability, it results that:

1. For the cedant, reinsurance is profitable when the ratio is less than 1. The recourse to reinsurance does not improve the profitability (the indifference condition) if the ratio involved is equal to 1. If the ratio is greater than 1, it is better not to reinsure because the index is less than the rate of return that the cedant would obtain without reinsurance.

2. For the reinsurer, in all three cases, the rate of return is greater than the risk-free rate (2.5%); the most favorable situation, providing greater "risk premium", is when the ratio is greater than 1.
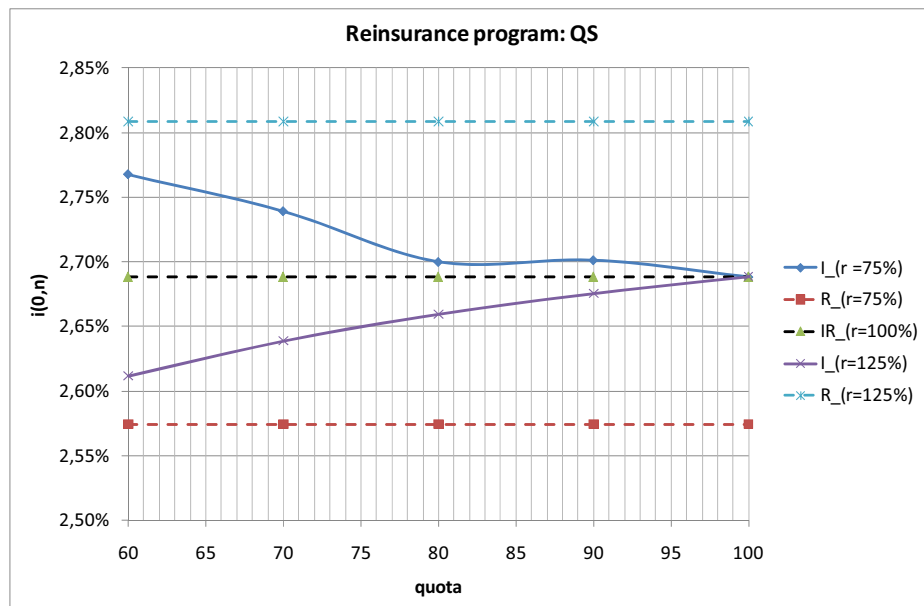

Fig. 2. Performance index $i(0, n)$ for QS treaty

2.2. Excess of loss arrangement (XLT). An excess of loss arrangement is structured in such a way that the reinsurer pays to the cedant the final part of the annuity when exceeding a given term.

Let us remark the essential conditions of this agreement in order to specify liabilities, reinsurance premiums, assets and solvency margin.

For this purpose, the following notation is adopted:

- T is the the maximum period of annuity payment such that the cedant does not receive any benefit from the reinsurer;
- $Y_t^{XL(j)}, Y_t^{XL[\Pi]}$ are the individual and portfolio random present value of (net) future benefits for the cedant, at time t;
- $X_t^{XL(j)}, X_t^{XL[\Pi]}$ are the individual and portfolio random present value of future benefits for the reinsurer, at time t;
- R_0^{XL} is the reinsurance premium to be paid at time 0.

Referring to an annuitant alive at time t, the individual random present value of benefits for the cedant and for the reinsurer is respectively defined as follows:

$$Y_t^{XL(j)} = \begin{cases} \sum_{s=0}^{\min\{T_{x+t}^{(j)}, T-t\}} b_{t+s}^{(j)} v(t,t+s) & t < T \\ 0 & t \geq T \end{cases}, \quad (12)$$

$$X_t^{XL(j)} = \begin{cases} \sum_{T-t}^{\max\{T-t, T_{x+t}^{(j)}\}} b_{t+s}^{(j)} v(t,t+s) & t < T \\ \sum_{s=0}^{T_{x+t}^{(j)}} b_{t+s}^{(j)} v(t,t+s) & t \geq T \end{cases}. \quad (13)$$

At the portfolio level we have:

$$Y_t^{XL[\Pi]} = \sum_{j=1}^{\Pi_t} Y_t^{XL(j)}, \quad (14)$$

$$X_t^{XL[\Pi]} = \sum_{j=1}^{\Pi_t} X_t^{XL(j)}. \quad (15)$$

Obviously:

$$Y_t^{XL} = Y_t^{XL[\Pi]} + X_t^{XL[\Pi]}. \quad (16)$$

As far as the reinsurance premium is concerned, we assume that the reinsurer adopts the percentile principle for pricing. Hence at time 0

$$R_0^{XL} = \inf\{u \geq 0 \mid \Pr\{X_t^{XL[\Pi]} > u\} \leq \gamma\}, \quad (17)$$

where γ represents the accepted probability of loss. We also disregard a spreading out of the single premium. Hence, $R_t^{XL} = 0$ for $t > 0$.

In the equations below, for each of reinsurance market's agents, recursive formulas of shareholder's capital and assets dynamics are shown; we point out that for the cedant the liability side is described by means of a technical provision referred to a temporary annuity with maturity T, while for the reinsurer the liability side is described by means of a technical provision referred to a deferred annuity with starting point the time T.

Shareholder's capital and assets dynamics for the insurer:

$$M_t^C = \begin{cases} p \cdot V_{0;T}^{[\Pi]}, & t = 0^+ \\ A_t^C - V_{t;T-t}^{[\Pi]}, & 0 < t \leq T, \\ A_t^C, & t > T \end{cases}$$

$$A_t^C = \begin{cases} M_0^C + (1+\delta) \cdot V_{0;n}^{[\Pi]} - R^{XLT}, & t = 0^+ \\ A_{t-1}^C(1+i) - CF_t^{[\Pi]}, & 0 < t \leq T. \\ A_{t-1}^C(1+i), & t > T \end{cases}$$

Shareholder's capital and assets dynamics for the reinsurer:

$$M_t^R = \begin{cases} p \cdot_{T/} V_{0;n-T}^{[\Pi]}, & t = 0^+ \\ A_t^R - _{T-t/} V_{t;n-T}^{[\Pi]}, & 0 < t \leq T, \\ A_t^R - V_{t;n-t}^{[\Pi]}, & t > T \end{cases}$$

$$A_t^R = \begin{cases} M_0^R + R^{XLT}, & t = 0^+ \\ A_{t-1}^R(1+i), & 0 < t \leq T. \\ A_{t-1}^R(1+i) - CF_t^{[\Pi]}, & t > T \end{cases}$$

Table 3 and 4 represent respectively the cedant and the reinsurer indexes in the case of a reinsurance premium calibrated in corrispondence to the 75 percentile.

Table 3. Insurer index. *Hp.: percentile γ = 75% and $\varphi(M; T)$ = 2,5%*

T:	5	10	15	20	25	30	35	40
p %	1,313	1,070	1,180	1,285	1,400	1,450	1,465	1,463
i(0,T)%	-12,585	-0,170	1,959	2,617	2,775	2,766	2,736	2,711
μ	-9,582	-0,159	1,660	2,037	1,982	1,907	1,867	1,853
μ (no reins.)	1,853	1,853	1,853	1,853	1,853	1,853	1,853	1,853

Table 4. Reinsurer index. *Hp.: percentile γ = 75% and $\varphi(M; T)$ = 2,5%*

T:	5	10	15	20	25	30	35	40
p %	1,72	2,50	3,50	5,45	10,70	25,00	72,00	490,0
i(0,T)%	10,170	6,070	4,875	4,073	3,361	3,018	2,898	2,686
μ	5,914	2,428	1,393	0,747	0,314	0,121	0,040	0,005

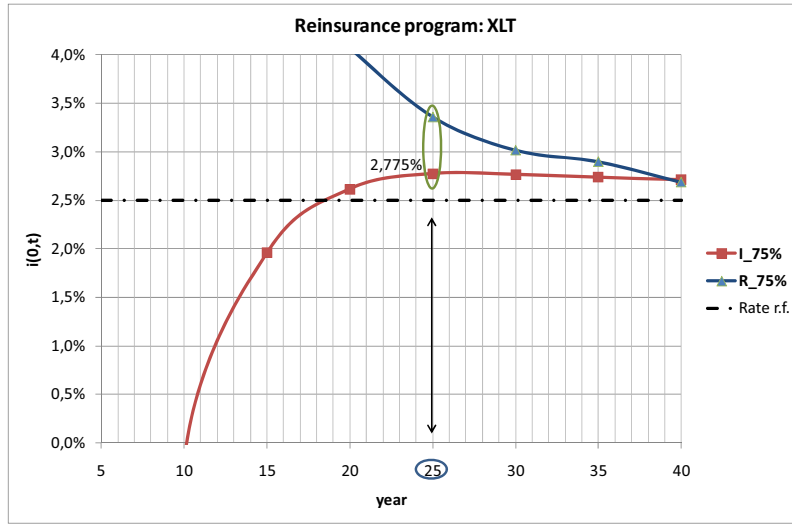**Fig. 3. Index *i*(0, *t*). Percentile γ = 75% and φ(*M*; *T*) = 2,5%**

From Figure 3 we can observe that:

♦ from the cedant's point of view the expected return on equity increases with increasing the maturity *T*;

♦ obviously, from the reinsurer's point of view the expected return on equity decreases rapidly with increasing *T*;

♦ the optimum level of the return on equity can be achieved at a maturity equal to 25 years, since none of the other possible solutions at the same time help to improve the position of both agents;

♦ both agents will have a better performance in respect to a free-risk investment.

2.3. Stop-loss arrangement on cash flows (SLCF). Let us now assume that at a given point in time demographic risk is perceived if the amount of benefits to be currently paid to annuitants is higher than expected. A transfer arrangement can then be designed so that the reinsurer takes charge of such extra-amount. Reinsurance conditions should concern the following items:

1. Stop loss priority: the minimum amount L_t of benefits (at time *t*), below which there is no payment by the reinsurer, is equal to the expected value of annuities to be paid increased, by a security rate (for the reinsurer) denoted by r. The formula is as follows:

$$L_t = E\left(\sum_{j=1}^{\Pi_t} b_t^{(j)}\right)(1+r), \ r \geq 0 \quad (18)$$

2. Reinsurance premium: let us assume that the reinsurance treaty is issued at time 0 and the single premium is based on the percentile principle;
3. Reserve: let us assume that the reserve is fully set up by the cedant.

According to the reinsurance conditions, the outflows of the cedant and the reinsurer at time *t* are:

$$CF_t^Y = \begin{cases} \sum_{j=1}^{\Pi_t} b_t^{(j)} \ if \ \sum_{j=1}^{\Pi_t} b_t^{(j)} < L_t \\ L_t \quad if \ L_t \leq \sum_{j=1}^{\Pi_t} b_t^{(j)} \end{cases}. \quad (19)$$

Trivially:

$$CF_t^X = \begin{cases} 0 \quad if \ \sum_{j=1}^{\Pi_t} b_t^{(j)} < L_t \\ \sum_{j=1}^{\Pi_t} b_t^{(j)} - L_t \ if \ L_t \leq \sum_{j=1}^{\Pi_t} b_t^{(j)} \end{cases}. \quad (20)$$

As far as the reinsurance premium is concerned, we assume that the reinsurer adopts the percentile principle for pricing. Hence at time 0

$$R_0^{SLCF} = \sum_{t=1}^{+\infty} \inf\{u_t \geq 0 \,|\, \Pr\{CF_t^X > u_t\} \leq \gamma\} \cdot v(0,t) \quad (21)$$

where γ represents the accepted probability of loss. Then the shareholder's capital and assets dynamics are respectively:

♦ For the insurer:

$$A_t^C = \begin{cases} M_0^C + (1+\delta) \cdot V_{0;n}^{[\Pi]} - R_0^{SLCF}, t = 0^+ \\ A_{t-1}^C(1+i) - CF_t^{[Y]}, \quad\quad t > 0 \end{cases},$$

$$M_t^C = \begin{cases} p \cdot V_{0;n}^{[\Pi]}, \quad t = 0^+ \\ A_t^C - V_{t;n-t}^{[Y]}, t > 0 \end{cases},$$

where $CF_t^{[Y]} = \min\{CF_t^{[\Pi]}, L_t\}$ is the stochastic outflows of the insurer.

♦ For the reinsurer:

$$A_t^R = \begin{cases} M_0^R + R_0^{SLCF}, & t = 0^+ \\ A_{t-1}^R (1+i) - CF_t^{[X]}, & t > 0 \end{cases},$$

$$M_t^R = \begin{cases} p \cdot \beta \cdot V_{0;n}^{[\Pi]}, & t = 0^+ \\ A_t^R - V_{t;n-t}^{[X]}, & t > 0 \end{cases},$$

where $CF_t^{[X]} = \max\{CF_t^{[\Pi]} - L_t; 0\}$ is the stochastic outflows of the reinsurer and $\beta = 1 - \gamma$ is the percentage of portfolio reserve.

Tables 5 and 6 reproduce the levels of the indices for the cedant and the reinsurer, having fixed the security rate equal to 1,5% and the ruin probability on the time horizon equal to 2,5%.

Table 5. Insurer index. *Hp.: r = 75% and*
$\varphi(M; T) = 2,5\%$

Percentile γ	0,5	0,65	0,75	0,85	0,95
$p\%$	0,843	0,849	0,890	1,006	1,333
$i(0, T)\%$	3,226	3,208	3,087	2,772	2,051
μ	3,826	3,777	3,469	2,756	1,539

Table 6. Reinsurer index. *Hp.: r = 75% and*
$\varphi(M; T) = 2,5\%$

Percentile γ	0,5	0,65	0,75	0,85	0,95
$p\%$	1,555	2,210	2,930	4,100	5,750
$i(0, T)\%$	2,015	2,033	2,156	2,548	4,294
μ	1,296	0,920	0,736	0,621	0,747

Let us note that:

♦ the reinsurance premium (that increases by increasing the percentile) has opposite effects on the profitability of the agents involved;

♦ the point of mutual advantage is reached for a reinsurance premium corresponding to a percentile of just above 85%. In this point the cedant regains the level of expected return without reinsurance (approximately 2,69%).

3. The effects of some reinsurance programs

The above results confirm what is reported in literature about the efficacy of traditional reinsurance treaties. Now, some programs based on combinations of traditional reinsurance treaties are analyzed. The effectiveness of the programs is measured in terms of ability to produce a positive expected return and not less than that achieved without reinsurance for the cedant, and higher than the risk-free rate for the reinsurer, providing the parties a level of solvency, within the horizon of reference, with an assigned probability.

We will assume that the reinsurance program adopted by the cedant is formed, at time 0, as a combination of two treaties signed with the same reinsurer.

3.1. The quota share-excess of loss (QS-XLT) program and excess of loss-quota share (XLT-QS) program. The Quota share-excess of loss (QS-XLT) is a program given by the combination of the following two treaties:

1. A proportional coverage (the quota share, QS treaty) for the entire duration of the contract, with the relative reinsurance premium depending on the retention rate (share of reinsurance).

2. A nonproportional coverage (the excess of loss, XLT treaty) for the range $(0, T)$ with relative reinsurance premium depending on the percentile and the duration of the contract.

The reinsurance premium (to be paid at time 0) is a sum of two terms: a QS reinsurance premium and an alpha – proportion XLT reinsurance premium:

$$R^{QS-XLT} = R^{QS} + \alpha \cdot R^{XLT}. \quad (22)$$

Then, the excess of loss-quota share (XLT-QS) is a program given by the combination of the following two treaties:

1. A nonproportional coverage (the excess of loss) with relative reinsurance premium depending on the percentile and the duration of the contract.

2. A proportional coverage (the quota share treaty) for charges of the cedant for its coverage period specified by the XLT treaty, with the relative reinsurance premium depending on the retention rate (share of reinsurance).

In this case, the reinsurance premium (to be paid at time 0) is a sum of two terms: XLT reinsurance premium and a QS reinsurance premium:

$$R^{XLT-QS} = R^{XLT} + (1-\alpha)(1+\eta)V_{0+;T}^{[\Pi]}. \quad (23)$$

Comparing reinsurance premiums of the two programs: the QS-XLT program is preferable to the program XLT-QS because of the percentile approach used in pricing XLT-component.
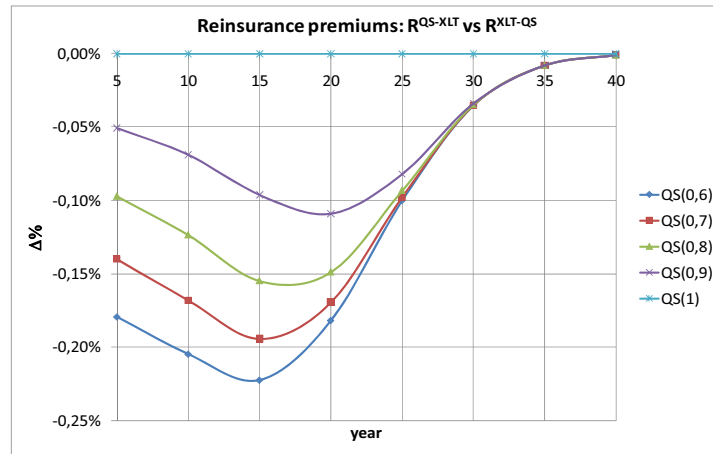
Fig. 4. Comparison of reinsurance premiums. Hp.: Percentile XLT = 75% and $\varphi(M; T) = 2{,}5\%$

However, we can easily demonstrate that in the program QS-XLT, respectively, the payoffs of two agents are equal to those determined by the program XLT-QS that is: changing the coverage order do not modify the quality of reinsurance coverage.

For both reinsurance programs, recursive formulas of assets and shareholder's capital dynamics are shown, respectively for each of reinsurance market's agents.

Shareholder's capital and assets dynamics for the insurer:

$$A_t^C = \begin{cases} M_0^C + (1+\delta) \cdot V_{0;n}^{[\Pi]} - R^{(.)}, & t = 0^+ \\ A_{t-1}^C(1+i) - \alpha \cdot CF_t^{[\Pi]}, & 0 < t \le T, \\ A_{t-1}^C(1+i), & t > T \end{cases}$$

$$M_t^C = \begin{cases} p \cdot \alpha \cdot V_{0;T}^{[\Pi]}, & t = 0^+ \\ A_t^C - \alpha \cdot V_{t;T-t}^{[\Pi]}, & 0 < t \le T, \\ A_t^C, & t > T \end{cases}$$

Shareholder's capital and assets dynamics for the reinsurer:

$$A_t^C = \begin{cases} M_0^R + R^{(.)}, & t = 0^+ \\ A_{t-1}^R(1+i) - (1-\alpha) \cdot CF_t^{[\Pi]}, & 0 < t \le T, \\ A_{t-1}^R(1+i) - CF_t^{[\Pi]}, & t > T \end{cases}$$

$$M_t^R = \begin{cases} p \cdot (1-\alpha) \cdot V_{0;T}^{[T]} + p \cdot_{T/} V_{0;n-T}^{[\Pi]}, & t = 0^+ \\ A_t^R - [(1-\alpha) \cdot V_{t;T-t}^{[\Pi]} +_{T-t/} V_{t;n-T}^{[\Pi]}], & 0 < t \le T. \\ A_t^R - V_{t;n-t}^{[\Pi]}, & t > T \end{cases}$$

where $R^{(.)} = R^{QS-XLT}$ for the QS-XLT program and $R^{(.)} = R^{XLT-QS}$ for the XLT-QS program.

We remark that for the cedant the liability is described by means of an alpha-proportion of the technical provision of a temporary annuity with maturity T, while for the reinsurer the liability is described by means of a sum of two reserve components respectively referred to an one-minus-alpha proportion of a temporary annuity and a deferred annuity with starting point in time T.

For the QS-XLT program, Figure 5 represents respectively the cedant and the reinsurer indices in the case of a reinsurance premium calibrated in corrispondence to the 75° percentile, for different levels of retention rate.
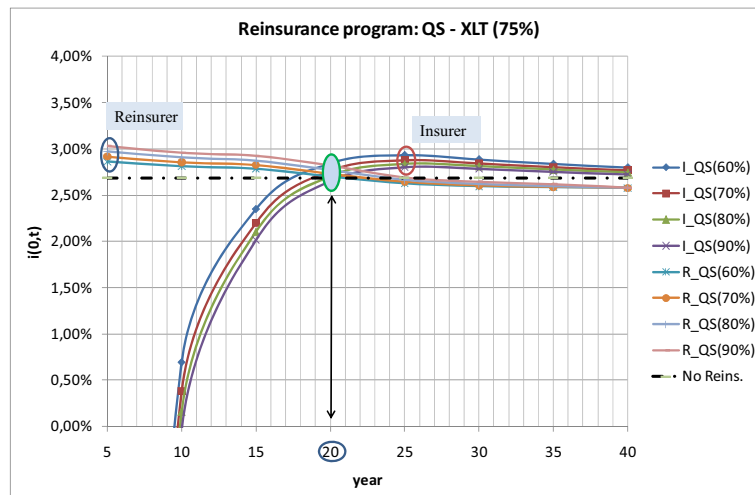

Fig. 5. Index $i(0, t)$. Hp.: Percentile $\gamma = 75\%$ and $\varphi(M; T) = 2{,}5\%$

We can observe that:

♦ from the cedant's point of view the expected return on equity increases when the retention rate decreases; the maximum return can be reached at a coverage period of 25 years;

♦ obviously, from the reinsurer's point of view the expected return on equity increases with increasing the retention rate;

♦ the optimum level of the return on equity can be

achieved at a maturity equal to 20 years, since none of the other possible solutions at the same time help to improve the position of both agents.

For the XLT-QS program, the next figure represents the cedant and the reinsurer indexes for different levels of retention rate respectively in the case of a reinsurance premium calibrated in corrispondence to the 75° percentile.
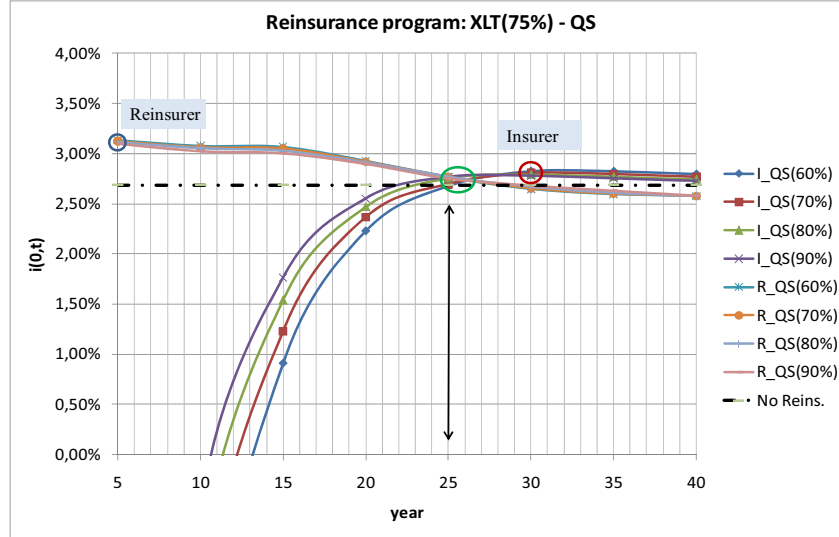

Fig. 6. Index $i(0, t)$. Hp.: Percentile $\gamma = 75\%$ and $\varphi(M; T) = 2,5\%$

We can observe that:

♦ for the cedant, it is better to reinsure with a decreasing proportional share (for $t < 28$ years), preserving an optimal coverage period equal to 30 years;

♦ for the reinsurer, it is preferable that the cedant reinsure with an increasing proportional share, preserving a period of no coverage;

♦ for both agents, mutual advantage points correspond to a period of coverage retained by the cedant approximately 25 years (within the range bounded by the optimal durations for the reinsurer and the only cedant). Optimal points provide a good performance at least equal to what the cedant would achieve without reinsurance (about 2,69%).

3.2. Stop loss on cash flows-excess of loss (SLCF-XLT) program and excess of loss-stop loss on cash flows (XLT-SLCF) program. The SLCF-XLT program is given by the combination of the following two treaties:

1. A nonproportional coverage with a fixed priority level that limits the amount of annual benefits to be paid by the cedant.
2. A second treaty which reduces the period covered by the cedant.

In this way the cedant reduces the annual benefits and its coverage time horizon. The reinsurance premium (to be paid at time 0) is:

$$R^{SLCF-XLT} = R^{SLCF} + R^{XLT|SLCF}. \qquad (24)$$

The XLT-SLCF program is a combination of the following two treaties:

1. A treaty that reduces the period covered by the cedant.
2. A nonproportional treaty with a fixed priority level.

Essentially, the cedant reduces its coverage period and its annual benefits. The reinsurance premium (to be paid at time 0) is a sum of two terms: a XLT reinsurance premium and a SLCF reinsurance premium, both calculated in reference to a percentile approach:

$$R^{XLT-SLCF} = R^{XLT} + R^{SLCF|XLT}. \qquad (25)$$

The SLCF-XLT and XLT-SLCF reinsurance programs achieve the same configuration of risk transfer. But the composition of the treaties leads to a different global reinsurance premium.

The figure below shows the percentage differences between the reinsurance premiums of the two programs.
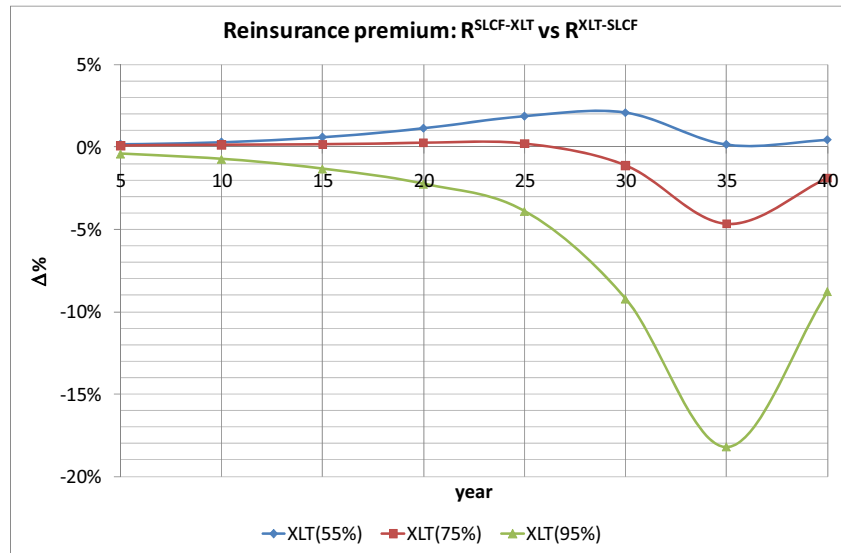
Fig. 7. Reinsurance premiums comparison. Hp.: Percentile SLCF $\gamma = 75\%$ and $\varphi(M; T) = 2,5\%$

The percentage differences change sign depending on the level of the percentile of the XLT treaty. These differences are enhanced if the retention period is high (greater than 30). From the relationship between prices, one might conclude that, for percentiles above 75% in the XLT, the SLCF-XLT program is preferable to the XLT-SLCF program for any length of time conservation.

For both reinsurance programs and two reinsurance market's agents, recursive formulas of assets and shareholder's capital dynamics hold:

Shareholder's capital and assets dynamics for the insurer:

$$A_t^C = \begin{cases} M_{0+} + (1+\delta) \cdot V_{0+;n}^{[\Pi]} - R^{(\cdot)}, t = 0^+ \\ A_{t-1}(1+i) - CF_t^{[Y]}, & 0 < t \leq T \\ A_{t-1}(1+i), & t > T \end{cases}$$

$$M_t^C = \begin{cases} p \cdot V_{0+;T}^{[\Pi]}, & t = 0^+ \\ A_t - V_{t;T.-t}^{[Y]}, 0 < t \leq T \\ A_t, & t > T \end{cases}.$$

Shareholder's capital and assets dynamics for the reinsurer:

$$M_t^R = \begin{cases} p \cdot \beta \cdot V_{0+;T}^{[\Pi]} + p \cdot {}_T/V_{0+;n-T}^{[\Pi]}, t = 0^+ \\ A_t - [V_{t;T.-t}^{[X]} + {}_{T-t/}V_{t;n.-T}^{[\Pi]}], \ 0 < t \leq T, \\ A_t - V_{t;n.-t}^{[\Pi]}, & t > T \end{cases}$$

$$A_t^R = \begin{cases} M_{0+} + R^{(\cdot)}, & t = 0^+ \\ A_{t-1}(1+i) - CF_t^{[X]}, 0 < t \leq T, \\ A_{t-1}(1+i) - CF_t^{[\Pi]}, \ t > T \end{cases}$$

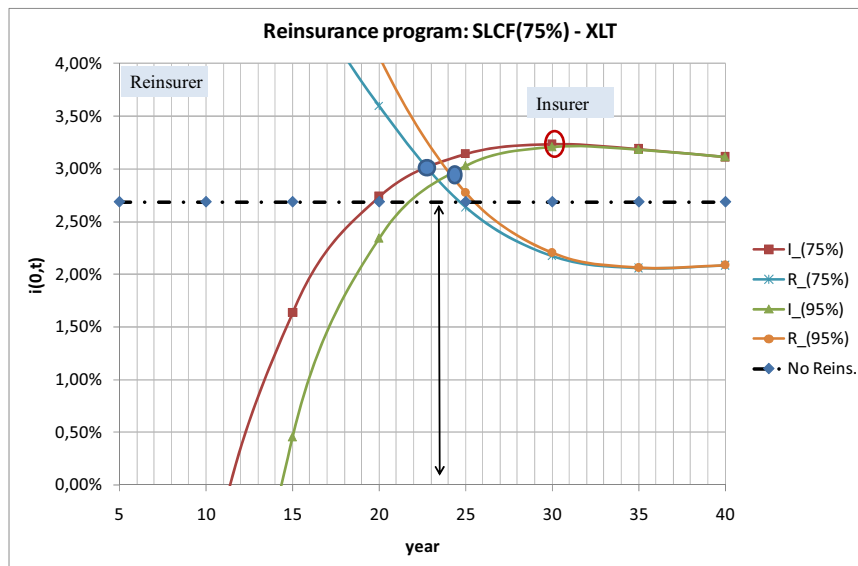where $R^{(\cdot)} = R^{SLCF-XLT}$ for the SLCF-XLT program and $R^{(\cdot)} = R^{XLT-SLCF}$ for the XLT-SLCF program.

Fig. 8. Index $i(0, t)$. Hp.: Percentile SLCF $\gamma = 75\%$ and $\varphi(M; T) = 2,5\%$

For the SLCF-XLT program, Figure 8 represents the cedant and the reinsurer indexes in the case of a reinsurance premium calibrated in corrispondence to the 75 percentile.

For both agents, mutual advantage points correspond to a coverage period retained by the cedant ranging between 23-24 years (within the range bounded by the optimal durations only for the reinsurer and the cedant). For the two cases shown, the profitability indic-

es corresponding to the mutual advantage points increase with decreasing the percentile of the XLT treaty. In addition, in these cases optimal points ensure a higher return than what the cedant would achieve without reinsurance (about 2.69%).

For the SLCF-XLT program, Figure 9 represents respectively the cedant and the reinsurer indices in the case of a reinsurance premium calibrated in corrispondence to the 75° percentile.
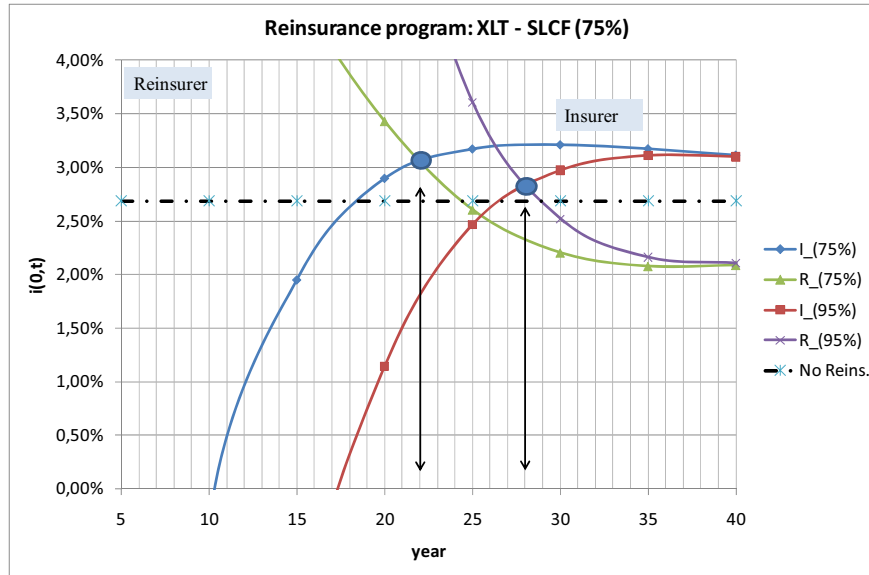

Fig. 9. Index *i*(0, *t*). Hp.: Percentile SLCF *γ* = 75% and *φ*(*M*; *T*) = 2,5%

We can observe that:

♦ from the cedant's point of view the expected return on equity increases when the reinsurance premium decreases that is when the parameter gamma (accepted probability of loss) decreases (comparison between red and blue line); the maximum return can be realized in correspondence to a coverage period ranging from 30 to 35 years;

♦ obviously, from the reinsurer's point of view the expected return on equity increases with increasing parameter gamma (i.e. reinsurance premium);

♦ for both agents, the optimum level of the return on equity can be achieved in reference to a cedant's retention period ranging from 22 to 28 years; a reduction of the reinsurance premium (i.e. gamma coefficient) produces an improvement in the level of profitability of both agents, with a reduction of the coverage period.

Conclusions

In conclusion, it is evident that the risk of default for a portfolio of life annuities is heavily affected by, among others, the demographic risk. A natural choice to reduce risk and to get an efficient capital allocation is to transfer a portion of the risks to reinsurers,

possibly with a favorable pricing. With reference to the risk model adopted and the criteria analyzed, we can remark that:

♦ for the cedant, the SLCF represents the optimal strategy, while the XLT is the best for the reinsurer;

♦ programs that combine non-proportional treaties provide excellent bilateral performance levels, preferable to those reached matching nonproportional and proportional ones.

As expected, results of simulations show that the reinsurance policy can reduce not only the insolvency risk but also the expected profitability of the company. In some extreme cases, notwithstanding reinsurance, the insolvency risk may result larger because of an extremely expensive cost of the reinsurance coverage: that happens when the reinsurance premium is incoherent with the structure of the transferred risk.

It is possible to define an efficient frontier based on the trade-off insolvency risk/shareholders return on equity, according to different reinsurance treaties and different retentions, taking in account not only the demographic non systematic risk, but also the systematic one and the financial risk.

References

1. Berliner B. (1999). "The strategic reinsurance program (SRP)", Proceedings Astin.
2. Borck K. (1974). *The mathematical theory of insurance*, Lexington Books.
3. Buhlmann H., Jewell W.S. (1979). Optimal risk exchanges, *Astin Bulletin*, 10.
4. Centeno L., Simoes O. (1989). "Combining quota-share and excess of loss treaties on the reinsurance of n-indipedent risks", *Astin Bulletin*, Vol. 21.
5. Cocozza R., Di Lorenzo E., Orlando A., Sibillo M. (2010). "A financial analysis of surplus dynamics for deferred life schemes", in Corazza M., Pizzi C. Eds., *Mathematical and Statistical Methods for Actuarial Sciences and Finance*, Springer-Verlag.
6. de Finetti B. (1940). "Il problema dei pieni", Giornale dell'Istituto Italiano degli Attuari, XI.
7. D'Ortona N.E., Marcarelli G. (2011). "Insurer's optimal reinsurance programs", *International Journal of Dyses*, Vol. 3, No. 1, pp. 1-18.
8. Gajek L., Zagrodny D. (2004). "Optimal reinsurance under general risk measures", *Insurance: Mathematics and Economics*, No. 34.
9. Olivieri A. (2005). "Designing longevity risk transfers the point of view of the cedant", *Giornale dell'Istituto Italiano degli Attuari*, No. 1-2.
10. Olivieri A., Pitacco E. (2003). "Solvency requirements for pension annuities", *Journal of Pension Economics and Finance*, No. 2.
11. Verlaak R., Beirlant J. (2003). "Optimal reinsurance programs: an optimal combination of several reinsurance protections on a heterogeneous insurance portfolio", *Insurance: Mathematics and Economics*, No. 33.